ORTHOGONAL BUNDLES, THETA CHARACTERISTICS AND THE SYMPLECTIC STRANGE DUALITY

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ABSTRACT. A basis for the space of generalized theta functions of level one for the spin groups, parameterized by the theta characteristics on a curve, is shown to be projectively flat over the moduli space of curves (for Hitchin's connection). The symplectic strange duality conjecture, conjectured by Beauville is shown to hold for all curves of genus ≥ 2, by using Abe's proof of the conjecture for generic curves, and the above monodromy result.

1. INTRODUCTION

Consider the moduli stacks $\mathcal{M}_{\mathrm{Spin}(r)}(X)$ and $\mathcal{M}_{\mathrm{SO}(r)}(X)$ of principal $\mathrm{Spin}(r)$, and $\mathrm{SO}(r)$-bundles, $r \geq 3$ on a smooth connected projective curve X of genus $g \geq 2$ over $\mathbb{C}$. Let $\mathcal{M}_{\mathrm{SO}(r)}(0)$ be the connected component of $\mathcal{M}_{\mathrm{SO}(r)}(X)$, which contains the trivial $\mathrm{SO}(r)$-bundle. There is a natural map

$$p : \mathcal{M}_{\mathrm{Spin}(r)} \to \mathcal{M}_{\mathrm{SO}(r)}(0).$$

A line bundle κ on X is said to be a theta characteristic if $\kappa^{\otimes 2}$ is isomorphic to the canonical bundle K_X. The set of theta characteristics $\theta(X)$ forms a torsor for the 2-torsion $J_2(X)$ in the Jacobian of X, and hence $|\theta(X)| = 2^{2g}$. Recall that a theta characteristic κ is said to be even (resp. odd) if $h^0(\kappa)$ is even (resp. odd).

For each theta-characteristic κ on X there is a line bundle $\mathcal{P}_k$ on $\mathcal{M}_{\mathrm{SO}(r)}$ with a canonical section s_κ (see the pfaffian construction in [LS, BLS]). On $\mathcal{M}_{\mathrm{SO}(r)}(0)$, $s_\kappa = 0$ if and only if both κ and r are odd.

For theta characteristics κ and κ', the line bundle $p^*\mathcal{P}_\kappa$ is isomorphic to $p^*\mathcal{P}_{\kappa'}$ (see [LS]). Set $\mathcal{P} = p^*\mathcal{P}_\kappa$ which is well defined upto isomorphism. The line bundle $\mathcal{P}$ is the positive generator of the Picard group of the stack $\mathcal{M}_{\mathrm{Spin}(r)}$. It is known that $\mathcal{P}$ does not descend to the moduli-space $M_{\mathrm{Spin}(r)}$, (similarly $\mathcal{P}_\kappa$ does not descend to the moduli-space $M_{\mathrm{SO}(r)}$). Clearly, $\mathcal{P}$ comes equipped with sections s_κ for each theta characteristic κ, coming from the identification $p^*\mathcal{P}_\kappa \xrightarrow{\sim} \mathcal{P}$ (s_κ are well defined up to scalars).

Let $\pi : \mathcal{X} \to S$ be a smooth projective relative curve of genus g. Assume by passing to an étale cover that the sheaf of theta-characteristics on the fibers of π is trivialized. For $s \in S$, let $X_s = \pi^{-1}(s)$. It is known that the spaces $H^0(\mathcal{M}_{\mathrm{Spin}(r)}(X_s), \mathcal{P})$ form the fibers of a vector bundle on S, which is equipped with a projectively flat connection (WZW or equivalently Hitchin's connection).

Theorem 1.1. *For even r, each section $s_\kappa \in H^0(\mathcal{M}_{\mathrm{Spin}(r)}(X_s), \mathcal{P})$, for $\kappa \in \theta(X_s)$ is projectively flat.*

Theorem 1.2. *For odd r, each section $s_\kappa \in H^0(\mathcal{M}_{\mathrm{Spin}(r)}(X_s), \mathcal{P})$, for even $\kappa \in \theta(X_s)$ is projectively flat.*

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It is known (see [O]) that the dimension of the space $H^0(\mathcal{M}_{\mathrm{Spin}(r)}(X_s),\mathcal{P})$ is equal to the number of theta characteristics (if r is odd, the number of even theta characteristics). It has been proved by Pauly and Ramanan (see Proposition 8.2 in [PR]), that in Theorems 1.1 and 1.2, the sections are linearly independent, and hence form a basis. Our methods give a new proof of this result of Pauly and Ramanan.

We use Theorem 1.1 to show that the symplectic strange duality formulated by Beauville [B] is, in a suitable sense, projectively flat: Hence it is an isomorphism for all curves of a given genus if it is an isomorphism for some curve of that genus (see Corollary 5.1). Takeshi Abe [A2, A3] has recently formulated a very interesting parabolic generalization of Beauville's conjecture, and has proved this conjecture for generic curves by using powerful degeneration arguments. His results imply Beauville's conjecture for generic curves[1]. Therefore Abe's results (together with Corollary 5.1) imply that the symplectic strange duality conjecture of Beauville holds for all curves. It should be pointed out that Abe's parabolic symplectic duality conjecture has not yet been shown to hold for all curves.

We would like to point out that Theorem 1.1 and 1.2 *do not* imply that the global projective Hitchin monodromy on the vector spaces $H^0(\mathcal{M}_{\mathrm{Spin}(r)}(X_s),\mathcal{P})$ is finite. The analogous question for the symplectic group is also not known (but see Section 5.1).

The proofs of Theorem 1.1 and Theorem 1.2 have the following main ingredients.

(1) The map p can perhaps be interpreted as a "stacky" torsor for $J_2(X_s)$. We will instead work over the regularly stable locus in $M_{\mathrm{SO}(r)}(0)$, over which p is a torsor (using results in [BLS]).
(2) By Proposition 5.2 in [BLS], for different theta characteristics κ and κ', the bundles $\mathcal{P}_k$ on and $\mathcal{P}_{\kappa'}$ on $\mathcal{M}_{\mathrm{SO}(r)}(X_s)$ are not isomorphic. The isomorphism class of $\mathcal{P}_\kappa\otimes\mathcal{P}_{\kappa'}^{-1}$ is explicitly computed in [BLS], and this computation constitutes the heart of the matter in the proofs of Theorems 1.1 and 1.2.

Avoiding technicalities, it is easy to summarize the proof of Theorem 1.1: Fix a theta characteristic κ. There is an action of $J_2(X_s)$ on $(\mathcal{M}_{\mathrm{Spin}(r)}(X_s),\mathcal{P})$ which lies over a trivial action on the pair $(\mathcal{M}_{\mathrm{SO}(r)}(X_s)(0),\mathcal{P}_\kappa)$. Since this action preserves the so-called geometric Segal-Sugawara tensor (see Section 3), it preserves Hitchin's connection on the spaces $H^0(\mathcal{M}_{\mathrm{Spin}(r)}(X_s),\mathcal{P})$. Therefore the connection preserves each $J_2(X_s)$-isotypical subspace of $H^0(\mathcal{M}_{\mathrm{Spin}(r)}(X_s),\mathcal{P})$. Each isotypical subspace will be shown to contain a pfaffian section $s_{\kappa'}$. Counting dimensions, we are then able to conclude the proof.

We will use the language of moduli-spaces and not of stacks (except in recalling some results from [BLS]). The main technique is to work over the regularly stable locus in $M_{\mathrm{SO}(r)}$, and to use results of Beauville, Laszlo and Sorger [BL, LS, BLS].

2. Reformulation in terms of Moduli spaces

We will use the notation, setup and results from Section 13 of [BLS], which we recall for the benefit of the reader. Let G be a simple (not necessarily simply connected) algebraic group. Let M_G denote Ramanathan's moduli space of principal semistable $G-$bundles on a smooth projective and connected curve X of genus $g\geq 2$. Let us assume that G does not map to PGL_2, or that $g>2$.

[1]Abe's parabolic generalization is essential in his proof of Beauville's conjecture for generic curves by degeneration.

Definition 2.1. *A $G-$bundle on X is regularly stable if it is stable and its automorphism group is equal to the center $Z(G)$ of G.*

The open subset $M_G^{\mathrm{reg}} \subset M_G$ is smooth, and as pointed out in [BLS], the method of proof of a theorem of Faltings (Theorem II.6 in [F]) implies that the complement of M_G^{reg} in M_G is of codimension ≥ 2.

Let A be the group of principal A'-bundles where A' is the kernel of $\mathrm{Spin}(r) \to \mathrm{SO}(r)$ (clearly A is isomorphic to J_2). Denote as usual the group of one dimensional characters of A by $\hat{A}$.

Let $M_{\mathrm{SO}(r)}(0)$ denote the connected component of $M_{\mathrm{SO}(r)}$ which contains the trivial $\mathrm{SO}(r)$-bundle. By a result of Beauville-Laszlo-Sorger (see the proof of Proposition 13.5 in [BLS]), the natural finite Galois covering with Galois group A

$$p : M_{\mathrm{Spin}(r)} \to M_{\mathrm{SO}(r)}(0)$$

is étale over $Y = M^{\mathrm{reg}}_{\mathrm{SO}(r)}(0)$. Set $\widetilde{Y} = p^{-1}(Y)$. It follows from the proof of Proposition 13.5 in [BLS], that $\widetilde{Y} \subseteq M^{\mathrm{reg}}_{\mathrm{Spin}_r}$.

Since $M_{\mathrm{SO}(r)}(0) - Y$ has codimension ≥ 2 and $p : M_{\mathrm{Spin}(r)} \to M_{\mathrm{SO}(r)}(0)$ is finite and dominant, $M_{\mathrm{Spin}(r)} - \widetilde{Y}$ has codimension ≥ 2. Therefore

$$H^0(Y, \mathcal{O}_Y) = H^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}}) = \mathbb{C}.$$

It is easy to see that there is a decomposition as sheaves of A-modules:

$$(p \mid \widetilde{Y})_* \mathcal{O}_{\widetilde{Y}} = \oplus_{\chi \in \hat{A}} L_\chi,\ L_\chi \in \mathrm{Pic}(Y).$$

where as a sheaf,

$$L_\chi(U) = \{s \in p_* \mathcal{O}(U) \mid gs = \chi(g)s, \forall g \in A\}.$$

It is easy to verify that

- $H^0(Y, L_\chi) = 0$ unless $\chi = 1$, $H^0(Y, L_1) = \mathbb{C}$.
- $p^* L_\chi = \mathcal{O}_{\widetilde{Y}}$,
- $L_\chi \otimes L_{\chi'} = L_{\chi\chi'}$
- L_χ is not isomorphic to $L_{\chi'}$ for $\chi \neq \chi'$.

According to Proposition 9.5 in [LS], the line bundle $\mathcal{P}_\kappa$ on $\mathcal{M}_{\mathrm{SO}(r)}(0)$ descends to $M^{\mathrm{reg}}_{\mathrm{SO}(r)}(0)$, to a line bundle which we will denote by P_κ similarly the line bundle $p^*\mathcal{P}_\kappa$ on the moduli stack $\mathcal{M}_{\mathrm{Spin}(r)}$ descends to the moduli space $M^{\mathrm{reg}}_{\mathrm{Spin}(r)}$.

The Weil pairing (the cup product in cohomology) $J_2 \times J_2 \to \mu_2$, where $\mu_2 = \{+1, -1\} \subseteq \mathbb{C}^*$ induces an isomorphism of groups $W : J_2 \to \hat{A}$. The following proposition follows from results of [BLS] (see Section 4).

Proposition 2.2. *For $\alpha \in J_2 = A$,*

$$P_{\kappa \otimes \alpha} = P_\kappa \otimes L_{W(\alpha)} \in \mathrm{Pic}(Y).$$

Notation: Fix an even theta characteristic κ for the rest of this paper. Let $P = p^*(P_\kappa) \in \mathrm{Pic}(\widetilde{Y})$, $\mathcal{P} = p^* P_\kappa \in \mathrm{Pic}(\mathcal{M}_{\mathrm{Spin}(r)})$. Denote the descent of $\mathcal{P}$ to $M^{\mathrm{reg}}_{\mathrm{Spin}(r)}$ again by P. Note that the two definitions of P are canonically identified under the inclusion $\widetilde{Y} \subseteq M^{\mathrm{reg}}_{\mathrm{Spin}(r)}$ (using descent theory). Also note that $p^* P_{\kappa'}$ is isomorphic to P, for any theta characteristic κ', the isomorphism is unique upto scalars.

We have a decomposition as A-modules:

$$H^0(\widetilde{Y}, P) = \oplus_{\chi \in \hat{A}} H^0(Y, P_\kappa \otimes L_\chi)$$

Proposition 2.3. *For even r,*

1. *$H^0(\widetilde{Y}, P)$ is 2^{2g} dimensional.*
2. *Each $H^0(Y, P_\kappa \otimes L_\chi)$ is one dimensional and spanned by the pfaffian section of $P_{\kappa \otimes W^{-1}(\chi)}$ corresponding to the isomorphism in Proposition 2.2.*
3. *The elements $s_{\kappa'}$ in $H^0(\widetilde{Y}, P)$ for $\kappa' \in \theta(X)$ form a basis.*
4. *The element $s_{\kappa'}$ for $\kappa' \in \theta(X)$ spans the $\chi = W(\kappa' \otimes \kappa^{-1})$ isotypical subspace of $H^0(\widetilde{Y}, P)$.*

Proof. $M_{\mathrm{Spin}(r)} - \widetilde{Y}$ has codimension ≥ 2. Using results in [BLS],

$$H^0(\widetilde{Y}, P) = H^0(M^{\mathrm{reg}}_{\mathrm{Spin}(r)}, P) = H^0(\mathcal{M}_{\mathrm{Spin}(r)}, \mathcal{P}) = 2^{2g}. \tag{2.1}$$

(for the last equality see [O].)

Clearly, the vector space in (2) has at least the (non-zero) pfaffian section. Since the number of theta-characteristics is 2^{2g}, (2) follows from (1). Finally, (3) and (4) are restatements of (2). □

For r odd, we have the following result, whose proof is similar to that of Proposition 2.4 (recall that our fixed theta characteristic κ is assumed to be even).

Proposition 2.4. *For odd r,*

1. *$H^0(\widetilde{Y}, P)$ is $2^{g-1}(2^g + 1)$ dimensional.*
2. *The elements $s_{\kappa'}$ in $H^0(\widetilde{Y}, P)$ for even $\kappa' \in \theta(X)$ form a basis.*
3. *The element $s_{\kappa'}$ for even $\kappa' \in \theta(X)$ spans the $\chi = W(\kappa' \otimes \kappa^{-1})$ isotypical subspace of $H^0(\widetilde{Y}, P)$.*

3. Hitchin's connection and the geometric Segal-Sugawara tensor

Let $\widetilde{G}$ be a simple, simply connected group and $M = M^{\mathrm{reg}}_{\widetilde{G}}(X)$ the smooth open subvariety of $M_{\widetilde{G}}(X)$ parameterizing regularly stable bundles E. Let $\mathcal{M}_g$ denote the moduli stack of smooth and connected projective algebraic curves of genus g, and $X \in \mathcal{M}_g$ as before. The cup product

$$H^1(X, T_X) \otimes H^0(X, \mathrm{ad}(E) \otimes K_X) \to H^1(X, \mathrm{ad}(E))$$

and the identification $T_E M = H^1(X, \mathrm{ad}(E))$ defines a ("geometric Segal-Sugawara") morphism

$$S : T_X \mathcal{M}_g \to H^0(M, S^2 TM)$$

The group B of principal $Z(\widetilde{G})$-bundles acts on M and the functoriality of the cup product implies that the morphism S has its image in the subspace of invariants $H^0(M, S^2 TM)^B$. *We will assume by passing to étale covers, that in any family of curves, the group scheme of principal B-bundles (which sits inside the torsion in a product of Jacobians) has been trivialized.*

Let $\mathcal{L}$ be the generating line bundle of the Picard stack of $\mathcal{M}_{\widetilde{G}}(X)$, which descends to M. The action of B on M may not lift to the pair $(M, \mathcal{L})$. For $b \in B$, $b^*\mathcal{L}$ is isomorphic to $\mathcal{L}$ and hence we can form a Mumford-theta group $\mathcal{G}(X)$, a central extension of B by $\mathbb{C}^*$, which does act on the pair $(M, \mathcal{L})$.

In the case $\widetilde{G} = \mathrm{SL}(n)$, it is possible to identify this Mumford-theta group (the author learned this from M. Popa, and appears in [Be2]). In the case of the odd spin groups $\mathrm{Spin}(r)$, r odd, the group extension $\mathcal{G}(X)$ splits, because replacing $M^{\mathrm{reg}}_{\mathrm{Spin}_r}(X_s)$ by $\widetilde{Y}_s$ (which does not change Picard groups, and isomorphisms of line bundles over $\widetilde{Y}_s$ extend to $M^{\mathrm{reg}}_{\mathrm{Spin}_r}(X_s)$) the pfaffian line bundle is pulled back from the (regularly stable) moduli of odd orthogonal bundles of rank r. Here we note that the center of the odd spin groups is $\mathbb{Z}/2$.

We do not know in general how to identify $\mathcal{G}(X)$. However suppose we are given a lifting of the B action on M to an action of a subgroup $A \subseteq B$ on $(M, \mathcal{L})$, where A is the group of principal A'-bundles for some subgroup $A' \subset Z(\widetilde{G})$. In this setting, it is easy to see by an obvious generalization of Corollary 5.2 and Lemma 4.1 in [Be2], that

Lemma 3.1. *The action of A on $H^0(M, \mathcal{L}^k)$ preserves the Hitchin connection as X varies in a family.*

Remark 3.2. *Hitchin's connection is given by "projective heat operators". By averaging over A one can find heat operators invariant under the action of A (as in* [vGdJ]*). Lemma 3.1 follows immediately.*

We will now carry out the proof of Theorem 1.1. The proof of Theorem 1.2 is similar and hence omitted.

Let us by passing to an open cover in the étale topology, assume that the sheaf of theta characteristics and also the two-torsion in the Jacobian of the fibers are trivial over S. We can form relative versions of the spaces $\widetilde{Y}$, Y from the previous discussion. There is an action of A on $(M^{\mathrm{reg}}_{\mathrm{Spin}(r)}, P)$, which restricts to the action on $(\widetilde{Y}_s, P)$ (because of the codimension estimates).

Clearly, by the fiberwise equality (2.1),

$$H^0(\mathcal{M}_{\mathrm{Spin}(r)}(X_s), \mathcal{P}) = H^0(\widetilde{Y}_s, P).$$

We have an action of the (trivial group scheme) $A = J_2$, corresponding to $A' = \ker(\mathrm{Spin}(r) \to \mathrm{SO}(r))$ on the right hand side. This action preserves the Hitchin connection (by (2.1) and Lemma 3.1). Given this it is easy to finish the argument. The isotypical components of the action of A are preserved by the Hitchin connection (this is obvious if we choose a A-invariant heat operator). In particular each of the isotypical spaces, each spanned by some $s_{\kappa'}$ is preserved by the Hitchin connection.

4. The proof of Proposition 2.2

The arguments in this section are taken from [BLS] and Section 5.3 of [L2]. Fix a point $x \in X$ and a formal coordinate z at x. For ease of notation let $\widetilde{G} = \mathrm{Spin}(r)$ and $G = \mathrm{SO}(r)$, $LG = G(\mathbb{C}((z)))$, $L_X G = G(\mathcal{O}(X - x))$, $L^+G = G(\mathbb{C}[[z]])$ (similarly define $L\widetilde{G}$, $L_X G$ and $L^+\widetilde{G}$). We have two "infinite" Grassmannians

$$\mathcal{Q}_G = LG/L^+G,\ \mathcal{Q}_{\widetilde{G}} = L\widetilde{G}/L^+\widetilde{G}$$

The space $\mathcal{Q}_G$ (similarly $\mathcal{Q}_{\widetilde{G}}$) parameterizes isomorphism classes of principal G-bundles equipped with a trivialization on $X - \{x\}$.

It is known from [BLS], that the neutral component $\mathcal{Q}^o_G$ of $\mathcal{Q}_G$ is canonically isomorphic to $\mathcal{Q}_{\widetilde{G}}$. Hence a G-bundle (in the neutral component of $\mathcal{M}_G$) trivialized on the complement of x has a canonical $\widetilde{G}$-structure. It is also known that $L_X G$ is contained in the neutral component of LG.

Finally, one has the stack-theoretic uniformization theorems [BL, LS, BLS]

$$\mathcal{M}_G = L_X G \backslash \mathcal{Q}_G, \mathcal{M}_{\widetilde{G}} = L_X \widetilde{G} \backslash \mathcal{Q}_{\widetilde{G}}.$$

Let us show that $L_X G$ acts on $\widetilde{Y}$. Let $P \in \widetilde{Y}$ and $\beta \in L_X G$. Represent P as the image of a point $q \in \mathcal{Q}_{\widetilde{G}}$ and hence as a point of $\mathcal{Q}_G$. Clearly $L_X G$ acts on $\mathcal{Q}_G$ preserving the connected components. Therefore βq gives a new point of $\mathcal{Q}_{\widetilde{G}}$, and hence a new point of $\widetilde{Y}$. In fact this action of $L_X G$ factors through the quotient by image of $L_X \widetilde{G}$. The quotient $L_X G / i(L_X \widetilde{G})$ is naturally isomorphic to $J_2 = A$ (see Lemma 1.2 in [BLS]), and this action of A on $\widetilde{Y}$ coincides with the natural Galois action of A on $\widetilde{Y}$ (see Section 6). There is another way to describe this action. There is a natural map $L_X G \to L\widetilde{G}/\pi_1(G)$ (both sides sit naturally in LG). Through this map $L_X G$ acts on $\widetilde{Y}$, and it is easy to see that it coincides with the action above (the natural map $\mathcal{Q}_{\widetilde{G}} \to \mathcal{Q}_G$ is equivariant for the map of groups $L\widetilde{G} \to LG$).

In Section 5 of [BLS], an injective homomorphism $\lambda : \hat{A} \to \mathrm{Pic}(\mathcal{M}_G)$ is constructed and it is shown that as line bundles on $\mathcal{M}_G$, $\mathcal{P}_{\kappa \otimes \alpha} \otimes \mathcal{P}_{\kappa}^{-1}$ equals $\lambda(W(\alpha))$ (see the proof of Proposition 5.2 in [BLS]). We claim that the descent of $\lambda(\chi)$ to Y equals L_χ for $\chi \in \hat{A}$. This would prove Proposition 2.2.

For simplicity, we will work in the classical topology over Y (which is sufficient for our purposes, because of the codimension conditions). In fact, it is easy to replace the argument by an analogous argument in the étale topology, and prove Proposition 2.2 in the algebraic category. Let us first recall our construction of L_χ. Cover Y by (analytic) open subsets U_i and choose a lifting $U_i \to \widetilde{Y}$. On overlaps $U_i \cap U_j$, the two maps differ by a section of A. Hence a character χ of A gives the patching functions for a line bundle on Y (note that $\chi = \chi^{-1}$).

We will now realize this construction by making loop group choices. This is then easily seen to be the construction in [BLS]: Refine the cover U_i and on each U_i choose a local universal bundle Q_i (this is possible using $Y \subseteq M^{\mathrm{reg}}_{\mathrm{SO}_r}(X)$) and a trivialization of Q_i on the complement of x. This gives Q_i a $\widetilde{G}$-structure, and hence we obtain liftings $U_i \to \widetilde{Y}$. On overlaps $U_i \cap U_j$, the different trivializations give a class in $L_X G/Z(G)$. Therefore any character χ of $L_X G/Z(G)$, produces a line bundle on Y. Any such character is necessarily trivial on the image of $L_X \widetilde{G}$, and factors through the quotient $L_X G/i(L_X \widetilde{G}) = A$ where $i : L_X \widetilde{G} \to L_X G$ (note that the center of $\widetilde{G}$ surjects on to the center of G). By the basic compatibility verification in Section 6, the proof of our claim is complete.

5. Application to the Symplectic strange duality

Let us first recall the set up of the symplectic strange duality from [B]. Let $M_{\mathrm{Sp}(2n)}$ denote the moduli space of vector bundles on X of rank $2n$, equipped with a non-degenerate symplectic form (with values in $\mathcal{O}_X$). In fact, $M_{\mathrm{Sp}(2n)}$ is the moduli space of principal $\mathrm{Sp}(2n)$-bundles on X. Let $\mathcal{L}$ be the positive generator of the Picard group of $M_{\mathrm{Sp}(2n)}$. We can take $\mathcal{L}$ to the determinant of cohomology of the tautological bundle tensored with a degree $g-1$ line bundle on X (this makes good sense on the moduli stack, and descends to the moduli space).

Similarly let $M'_{\mathrm{Sp}(2m)}$ denote the moduli space of vector bundles on X of rank $2m$, equipped with a non-degenerate symplectic form with values in K_X (therefore the underlying degree of the vector bundles is $2m(g-1)$). A choice of a theta characteristic κ gives an isomorphism $M_{\mathrm{Sp}(2m)} \to M'_{\mathrm{Sp}(2m)}$. Let $\mathcal{L}$ again denote the positive generator of the Picard group of $M'_{\mathrm{Sp}(2m)}$. Note that for both $M_{\mathrm{Sp}(2n)}$ and $M'_{\mathrm{Sp}(2m)}$, the global sections of powers of $\mathcal{L}$ over the corresponding moduli stack, coincides with the global sections over the moduli space.

On $M_{\mathrm{Sp}(2n)} \times M'_{\mathrm{Sp}(2m)}$, there is a natural Cartier divisor Δ of the line bundle $\mathcal{L}^m \boxtimes \mathcal{L}^n$, such that 2Δ is the theta section of the determinant of cohomology of the tensor product. The non-zeroness of this divisor has been shown by Beauville [B]. Therefore one finds a non-zero homomorphism, conjectured by Beauville to be an isomorphism

$$H^0(M'_{\mathrm{Sp}(2m)}(X), \mathcal{L}^n)^* \to H^0(M_{\mathrm{Sp}(2n)}(X), \mathcal{L}^m) \tag{5.1}$$

Said in a different way, the divisor on the product of the moduli-stacks $\mathcal{M}_{\mathrm{Sp}(2n)} \times \mathcal{M}_{\mathrm{Sp}(2m)}$ is the pull back of the pfaffian section s_κ on $\mathcal{M}_{\mathrm{Spin}(4mn)}$ of the line bundle $\mathcal{P}$. That is, the image of s_κ under the map

$$H^0(\mathcal{M}_{\mathrm{Spin}(4mn)}(X), \mathcal{P}) \to H^0(\mathcal{M}_{\mathrm{Sp}(2m)}(X), \mathcal{L}^n) \times H^0(\mathcal{M}_{\mathrm{Sp}(2n)}(X), \mathcal{L}^m) \tag{5.2}$$

It is known that the map (5.2) is projectively flat (see [NT] and [Be2]). Therefore, by Theorem 1.1, we see that the map (5.1) is a projectively flat map after making the identification $M_{\mathrm{Sp}(2m)} \to M'_{\mathrm{Sp}(2m)}$. We therefore obtain the following corollary to Theorem 1.1:

Corollary 5.1. *The homomorphism* (5.1) *has constant rank as X varies over the moduli space of curves M_g.*

5.1. **Relations to the strange duality for vector bundles.** Consider the case $n = 1$ and (for technical reasons) $g > 2$. By the above results, the local system on the moduli of curves with a choice of theta characteristic, given by $H^0(M_{\mathrm{Sp}(2m)}(X), \mathcal{L})$ is naturally (projectively) dual to the local system with fibers

$$H^0(M_{\mathrm{Sp}(2)}, \mathcal{L}^m) = H^0(M_{\mathrm{SL}(2)}, \mathcal{L}^m).$$

Using the $\mathrm{SL}(2)$-$\mathrm{GL}(m)$ strange duality, and its flatness [L1, A1, Be1, MO, Be2] we find that the latter space is naturally dual, preserving connections to $H^0(M_{\mathrm{GL}(m)}(0), \mathcal{L}^2)$, where $M_{\mathrm{GL}(m)}(0)$ is the moduli space of semi-stable degree 0 and rank m vector bundles[2] on X. Actually, there is a natural embedding $M_{\mathrm{GL}(m)}(0) \subseteq M_{\mathrm{Sp}(2m)}$ which pulls back $\mathcal{L}$ to $\mathcal{L}^2$, and consistent with the above identifications. Therefore the natural map $H^0(M_{\mathrm{Sp}(2m)}, \mathcal{L}) \to H^0(M_{\mathrm{GL}(m)}(0), \mathcal{L}^2)$ is an isomorphism, preserving connections.

Note that $\mathrm{GL}(m) \subseteq \mathrm{Sp}(2m)$ appears as a conformal embedding in the tables of conformal embeddings, but the author does not know how to use this to directly prove that the natural map from $H^0(M_{\mathrm{Sp}(2m)}, \mathcal{L})$ to $H^0(M_{\mathrm{GL}(m)}(0), \mathcal{L}^2)$ preserves connections (the problem is the non-semisimplicity of GL).

6. A VERIFICATION OF COMPATIBILITY

Let G be a semisimple algebraic group, with universal cover $\widetilde{G}$. We have a basic central extension

$$1 \to \pi_1(G) \to \widetilde{G} \xrightarrow{\pi} G \to 1$$

[2]Note that we have made a choice of a theta characteristic on X, and the line bundle on $\mathcal{L}$ on $M_{\mathrm{GL}(m)}(0)$ is the determinant of cohomology of the tautological bundle $\otimes\kappa$. In fact $\mathcal{L}^2$ does not depend upon κ but the strange duality with $H^0(M_{\mathrm{SL}(2)}, \mathcal{L}^m)$ does depend on κ.

Let X be a smooth projective curve as before and x a point on it. Set $X^* = X - \{x\}$, and consider a map $\phi : X^* \to G$, or an element $\phi \in L_X G$ using our earlier notation. We find by base change a cover $\widetilde{X}^*$ of X^* which fits into a cartesian diagram

(6.1)
$$\begin{array}{ccc} \widetilde{X}^* & \xrightarrow{\tilde{\phi}} & \widetilde{G} \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ X^* & \xrightarrow{\phi} & G \end{array}$$

Now unramified abelian covers of X^* extend to unramified abelian covers of X. Therefore we can extend π' to a cover $\pi' : \widetilde{X} \to X$ and thus obtain a principal $\pi_1(G)$-bundle α on X in the étale topology. Given a principal $\widetilde{G}$-bundle Q on X we can obtain a new bundle Q_1 on X whose sheaf of sections is for an open subset U of X, sections of the pull back of Q over the inverse image of U which twist by the image of $\pi_1(G)$ in $\widetilde{G}$ upon the action of the covering group $\pi_1(G)$. It is easy to see that Q_1 is the same as $Q \times_{\pi_1(G)} \alpha$ (this leads to the Galois action of α on $M_{\widetilde{G}}$).

On the other hand, given Q we have another construction of a principal $\widetilde{G}$-bundle on X. There is a natural map $L_X G \to L\widetilde{G}/\pi_1(G)$. To do this pick a point $\tilde{x}$ over x and a coordinate z on X at x. Since π' is étale, z lifts to a coordinate on $\widetilde{X}$ near $\tilde{x}$. The map $\tilde{\phi}$ therefore gives us an element $\psi \in L\widetilde{G}$, which normalizes $L_X\widetilde{G}$ (by descent theory) and hence left multiplication by ψ gives a principal $\widetilde{G}$-bundle Q_2 on X. We contend that Q_1 and Q_2 are isomorphic.

Let s be a section of Q over X^*, clearly $\tilde{\phi}s$ gives a section of Q_1 over X^*. Also a section of Q_1 over D a formal neighborhood of x and the choice of $\tilde{x}$ over x, determines a section of Q_1 over D. The new patching function for Q_1 (in the punctured disc around x) is given by the image ψ of $\tilde{\phi}$ times the patching function of Q, hence Q_1 is isomorphic to Q_2.

The following diagram (easily seen to be commutative) is useful in studying the various maps, where the vertical arrow is the map $\phi \to \tilde{\phi}$ as above:

(6.2)
$$\begin{array}{ccc} L\widetilde{G}/\pi_1(G) & \longrightarrow & LG \\ \uparrow & \nearrow & \\ L_X G & & \end{array}$$

6.1. **Action of the center.** The above discussion has the following interesting consequence (take $G = \widetilde{G}/Z(\widetilde{G})$): The action of a principal $Z(\tilde{G})-$bundle on the set of isomorphism classes of principal $\widetilde{G}$-bundles on X, lifts to left multiplication by an element of $L\widetilde{G}$ on $\mathcal{Q}_{\widetilde{G}}$.

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