Weak disorder in the stochastic mean-field model of distance II.

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Abstract

In this paper, we study the complete graph K_n with n vertices, where we attach an i.i.d. weight to each of the $n(n-1)/2$ edges. We focus on the weight W_n and the number of edges H_n of the minimal weight path between vertex 1 and vertex n.

It is shown in [6] that when the weights on the edges are independent and identically distributed (i.i.d.) with distribution equal to E^s, where $s > 0$ is some parameter and E has an exponential distribution with mean 1, then H_n is asymptotically normal with asymptotic mean $s \log n$ and asymptotic variance $s^2 \log n$. In this paper, we analyze the situation when the weights have distribution E^{-s}, $s > 0$, where the behavior of H_n is markedly different as H_n is a tight sequence of random variables. More precisely, we use Stein's method for Poisson approximation to show that, for almost all $s > 0$, the hopcount H_n converges in probability to the nearest integer of $s+1$ greater than or equal to 2, and identify the limiting distribution of the recentered and rescaled minimal weight. For a countable set of special s values denoted by $\mathcal{S} = \{s_j\}_{j\geq 2}$, the hopcount H_n takes on the values j and $j+1$ each with *positive* probability.

Key words: First passage percolation, complete graph, extreme value theory, hopcount, minimal path weight, Stein's method, Poisson approximation, weak disorder, stochastic mean-field model.

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1 Introduction

One of the central themes of modern discrete probability has been the study of the effect of random edge disorder on various properties of the underlying network. The base network itself could be *deterministic*, e.g. a large finite box in the lattice or the complete graph on n vertices, or *random*, e.g. the giant component of the Erdős-Rényi random graph or the configuration model. Each edge is assigned a random edge weight, whose interpretation varies depending on the context. One can think of this weight as the *cost* in traversing the edge, yielding first passage percolation type models. Alternatively, one could think of the underlying graph as a set

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of resistors connected together and the assigned weights as resistors, yielding a random resistor network, or as capacities on edges and the underlying graph as a flow carrying network, entrusted with carrying flow (commodities, information etc) between various parts of the network.

One graph model that has resulted in many problems of fundamental interest is the complete graph K_n on n vertices with random edge weights. In the various contexts mentioned above, this model both gives rise to very interesting conjectures as well as generates many new techniques and insights in probability theory that can then be applied in a number of other contexts. While providing a complete list of references of the various models that have been studied in this context would be impractical, we direct the interested reader to [14] for one of the first refined results in first passage percolation in this context, [10] for an analysis of the cost of the minimal spanning tree, [11] for an analysis of random electrical networks on the complete graph, [2] for a study of the random assignment problem, [4] for an analysis of the multicommodity flow problem, and the survey paper [1] where a number of other examples are analyzed via the powerful *local weak convergence* method.

Let us now focus on the particular problem dealt with in this study and the motivations behind the problem. Suppose we start with a connected graph $\mathcal{G}_n$ (deterministic such as K_n or random) on n vertices. Suppose each edge is assigned a random positive edge weight u_e. We shall assume that the weights are independent and identically distributed over edges with some distribution F, with density f. Fix two vertices (say chosen uniformly at random from $\mathcal{G}_n$) and let us denote them by 1 and n say. For any path $\mathcal{P}$ between the two vertices, let the weight of the path $w(\mathcal{P})$ be defined by

$$w(\mathcal{P}) := \sum_{e\in\mathcal{P}} u_e,$$

i.e. the sum of weights of the edges in the path. The *optimal* or *minimal weight* path (which is unique since the edge weights have a density) is the path that minimizes the above weight function. In the study of random systems, this regime is often called the **weak disorder** regime, while probabilists know this problem as "first passage percolation". The mental picture one can have is that the network is entrusted with carrying flow between various parts of the network, and the way it performs this duty is via routing flow through optimal paths. We shall defer a more extensive discussion of the relevant literature to Section 3.

Another regime which is of tremendous interest is the **strong disorder regime**. Here the weight of a path which we shall denote by $w_{\mathtt{st}}$ as the *maximum* or *minimum* weight of all edges in the path. In these two cases we shall denote the weight functions as

$$w_{\mathtt{max}}(\mathcal{P}) := \max_{e\in\mathcal{P}} u_e \tag{1.1}$$

and

$$w_{\mathtt{min}}(\mathcal{P}) := \min_{e\in\mathcal{P}} u_e. \tag{1.2}$$

In both situations, one is the interested in properties of the path which minimizes the above weight function. One is also interested in formulating a model, depending on a real valued parameter, the "inverse parameter", (in this study denoted by $s\in\mathbb{R}$) which interpolates between these two models. One can then study questions such as phase transitions, where these is a change in the behavior of the system from the weak disorder regime to the strong disorder regime. Given the set of edge weights u_e, one method of doing this is as follows: assign each edge a cost u_e^s where $s\in\mathbb{R}$ is a real valued parameter. With these edge weights, suppose that, as before, we consider the weak disorder regime, so that the weight of a path $\mathcal{P}$ is

$$w_s(\mathcal{P}) := \sum_{e\in\mathcal{P}} u_e^s.$$

Then
(a) **Original model:** $s = 1$ is our original model.
(b) **Graph distance:** $s = 0$ gives us the graph distance between the chosen vertices in the graph $\mathcal{G}_n$.
(c) **Strong disorder, max edge weight:** As $s \to +\infty$ gives us the strong disorder regime where the weight of a path is given by (1.1). This is also called the *minimal spanning tree* regime as the optimal path between the two vertices is the same as the path in the minimal spanning tree on $\mathcal{G}_n$ with edge weights u_e.
(d) **Strong disorder, min edge weight:** $s \to -\infty$ gives us the strong disorder model where the weight of a path is given by (1.2).

Thus, this model allows us to interpolate between various regimes of interest. We shall denote the optimal path by $\mathcal{P}_{\mathtt{opt}}(s)$. Given a particular base network $\mathcal{G}_n$ and edge weight distribution u_e, two statistics are of paramount interest:
(i)*Minimal weight:* This is the actual weight of the optimal path, namely $W_n = \sum_{e\in\mathcal{P}_{\mathtt{opt}}(s)} u_e^s$.
(ii)*Hopcount:* This is defined as the number of edges in the optimal path $\mathcal{P}_{\mathtt{opt}}(s)$. We shall denote this random variable by $H_n(s)$.

Aim of this paper: In this paper, we shall specialize to the case where the graph $\mathcal{G}_n$ is the complete graph K_n and each edge originally has edge weight $u_e \sim \text{Exp}(1)$. We shall study the case where $s < 0$. The case where $s > 0$ has been solved in [6], where it was proved that, for $s > 0$,

$$\frac{H_n(s) - s\log n}{\sqrt{s^2 \log n}} \xrightarrow{d} Z \tag{1.3}$$

where $Z \sim N(0,1)$ denotes a standard normal distribution, while there exists a constant $\lambda = \lambda(s) > 0$ and a non-degenerate real-valued random variable $\Xi(s)$ such that

$$W_n - \frac{1}{\lambda}\log n \xrightarrow{d} \Xi(s) \tag{1.4}$$

In this study, we shall derive asymptotics for the two random variables of interest namely $W_n(s)$ and $H_n(s)$ as $n \to \infty$, and see that the behavior in the case when $s < 0$ is markedly different. Without further ado let us now state our main results and defer a further discussion to Section 3.

2 Results

Before stating the main result, we need some notation. We will study the complete graph K_n with i.i.d. edge weights $E_{\{i,j\}}^{-s}$, $1 \le i \le j \le n$, on the edges of K_n. Thus, compared to the discussion in the previous section, we have replaced s there by $-s$ here, and we shall study the $s > 0$ regime. For fixed $s > 0$, define the function

$$g_s(x) = \frac{x^{s+1}}{(x-1)^s}, \qquad x \ge 2. \tag{2.1}$$

Observe that, for $0 < s \le 1$, the function $g_s(x)$, $x \ge 2$, is increasing, while for $s > 1$, the function is strictly convex with unique minimum at $x = s + 1$. We shall be interested in minimizing this function only on the space $\mathbb{Z}_+$, the set of positive integers. Then, there is a sequence of values $s = s_j$, $j \ge 2$, for which the minimum integer of g_s is *not* unique. From the equation

$g_s(j) = g_s(j+1)$ and the bounds $j-1 < s < j$, it is not hard to verify that

$$s_j = \frac{\log(1+j^{-1})}{\log(1+(j^2-1)^{-1})} \in (j-1, j), \qquad j = 2, 3, \ldots. \tag{2.2}$$

We will need to deal with these special points separately. When $s \notin \{s_2, \ldots\}$ then there is a unique integer which minimizes the function $g_s(x)$ on $\mathbb{Z}_+$. Let $\mathcal{S} = \{s_2, \ldots\}$.

Let us now state the main theorems. Below and the rest of the paper, for notational simplicity, we define $p = 1/s$, for $s > 0$, and, for $x \in \mathbb{R}$, we denote by $\lfloor x \rfloor$ the largest integer smaller than or equal to x and by $\lceil x \rceil$ the smallest integer larger than or equal to x.

Theorem 2.1 (Hopcount and weight asymptotics) *For any fixed $s > 0$ with $s \notin \mathcal{S}$, let $k^*(s) \in \{\lfloor s+1 \rfloor, \lceil s+1 \rceil\}$ denote the unique integer that minimizes the function defined in (2.1). Then,*
(a) the hopcount $H_n = H_n(s)$ converges in probability to $k^*(s)$, i.e., for $n \to \infty$

$$\mathbb{P}(H_n = k^*(s)) \to 1;$$

(b) the optimal weight W_n, properly normalized converges in distribution as $n \to \infty$,

$$\mathbb{P}\left(\frac{k-1}{sg_s(k)}(\log n)^{s+1}\left(W_n - \frac{g_s(k)}{(\log n)^s}\right) + \frac{k-1}{2}\log\log n - \frac{p(k-1)\log g_s(k)}{2} > t\right) \\ \to \exp(-a_k \mathrm{e}^t/(k-1)!),$$

where $k = k^*(s)$, and the sequence of constants $(a_k)_{k\geq 1}$ satisfies the recursion $a_1 = 1$,

$$a_k = a_{k-1} \cdot \frac{p\sqrt{2\pi}}{\sqrt{p(p+1)}}\Big(\frac{k-1}{k}\Big)^{(k-2)p/2} k^{(p-1)/2}(k-1)^{1/2}, \quad k \geq 2. \tag{2.3}$$

Theorem 2.1 states that the hopcount H_n converges to the optimal value of the function $x \mapsto g_s(x)$ defined in (2.1), while the rescaled and recentered minimal weight W_n converges is distribution to a Gumbel distribution. We can intuitively understand this as follows. For fixed k, the minimal path of length k is similar to an *independent* minimum of copies of sums of k random variables E^{-s}. The number of independent copies is equal to the number of disjoint paths between vertices 1 and 2, which is close to n^{k-1}. While on K_n, the appearing paths do *not* have independent weights, the paths that are particularly short are almost independent. Now, the independent problem can be handled in two steps. First, we analyse the behavior of the random variable $Z_k = E_1^{-s} + \cdots + E_k^{-s}$. In this analysis, the function g_s appears in the lower tails of the distribution. Secondly, we study the asymptotics of the minimum of n^{k-1} of such random variables, which can be seen to be of order $g_s(k)/(\log n^s)$. This explains why the minimal integer value of g_s is the crucial value for the hopcount, while the minimum of a large number of independent random variables with distribution Y, properly rescaled and recentered, converges to a Gumbel distribution by standard extreme value arguments. This intuitively explains Theorem 2.1. The main difficulty in the proof is to handle the fact that the weights of paths in the complete graph are actually *not* independent, and we use Stein's method for the Poisson approximation to deal with the available dependence.

Let us now deal with the case where $s \in \mathcal{S}$.

Theorem 2.2 (The special set $\mathcal{S}$) *Suppose $s \in \mathcal{S}$, so that both $\lceil s+1 \rceil$ and $\lfloor s+1 \rfloor$ minimize $g_s(\cdot)$ over $\mathbb{Z}_+$. Define a sequence of independent random variables $(\Xi_k)_{k\geq 2}$, where, for any $k \geq 2$, Ξ_k has the Gumbel distribution*

$$\mathbb{P}(\Xi_k > t) = \exp\left(-\frac{a_k}{(k-1)!}\mathrm{e}^t\right) \qquad t \in \mathbb{R}, \tag{2.4}$$

with $(a_k)_{k\geq 2}$ is defined in (2.3). Then,
(a) the weight W_n of the optimal path satisfies that, as $n \to \infty$,

$$\frac{(\log n)^{s+1}}{sg^*}\left(W_n - \frac{g^*}{(\log n)^s}\right) + \frac{1}{2}\log\log n - \frac{p\log g^*}{2} \xrightarrow{d} \min\left(\frac{\Xi_{\lfloor s+1\rfloor}}{\lfloor s+1\rfloor - 1}, \frac{\Xi_{\lceil s+1\rceil}}{\lceil s+1\rceil - 1}\right),$$

where $g^* = g_s(\lfloor s+1 \rfloor) = g_s(\lceil s+1 \rceil)$.
(b) the hopcount $H_n(s)$ satisfies that, as $n \to \infty$,

$$H_n(s) \xrightarrow{d} H^*,$$

where

$$H^* = \arg\min\{\Xi_k/(k-1) : k \in \{\lfloor s+1\rfloor, \lceil s+1\rceil\}\}.$$

Another quantity of interest is the distribution of optimal paths between vertex 1 and a *set* of vertices. In telecom, this is called multicast, since one source sends to a multiple number of users. This also follows from the analysis in the paper. We shall give a brief idea of the proof in Section 4.6. This result is stated for $s \notin \mathcal{S}$, but one could state an equivalent result for $s \in \mathcal{S}$ as well. Before we state the result we need some notation. Recall that we used $k^*(s)$ to denote the unique minimizer of $g_s(\cdot)$ over $\mathbb{Z}_+$. For fixed $m \geq 1$, let $\{\eta_i\}_{1\leq i\leq m}$ denote independent copies of the Gumbel random variable defined in (2.4) with $k = k^*(s)$.

Corollary 2.3 (Multipoint weights) *Fix m distinct vertices say $2, 3, \ldots m+1$ in K_n. Suppose $s \notin \mathcal{S}$ and let $\{W_n^j\}_{2\leq j\leq m+1}$ denote the weight of the optimal path from 1 to these vertices. Write*

$$\tilde{W}_n^j = (\log n)^{s+1}\left(W_n^j - \frac{g_s(k)}{(\log n)^s}\right) + \frac{k-1}{2}\log\log n - \frac{p(k-1)\log g_s(k)}{2},$$

where $k = k^*(s)$. Then, as $n \to \infty$,

$$(\tilde{W}_n^j)_{2\leq j\leq m+1} \xrightarrow{d} (\eta_j)_{1\leq i\leq m}.$$

Organization of the paper: The paper is organized as follows. We first discuss the relevance of our results and techniques in Section 3. We shall then continue to prove the main results in Section 4.

3 Discussion

We now provide a discussion of the various concepts used in this paper and the relevance of the results.
(a) **Stochastic mean-field model of distance:** This notion refers to the complete graph with exponential mean 1 edge weights. The model gives a simpler but mathematically more tractable

model of distances between random points in high dimensions. While one can consider other edge distributions, the lack of memory property of the exponential random variable allows one to give clean proofs in a number of different contexts, including first passage percolation, see [14] where this property is used to great effect to derive refined asymptotics. We also refer to [1] where many other computations are derived in this context with the help of a powerful infinite random structure called the Poisson weighted infinite tree.

(b) **Weak and strong disorder:** The last few years, with the availability of an enormous amount of data on real-world networks, has also witnessed an explosion in network models for these real-world networks as well as dynamics on them. Physicists have been highly interested in understanding the effect of random disorder on the various flow carrying properties of these network models. Via simulations, they have predicted a number of fascinating phenomena in these networks. Regarding the notions of weak and strong disorder mentioned in this paper, we refer the interested reader to [12], [8], [9] and [17] and the references therein.

(c) **First passage percolation:** First passage percolation problems have been of great interest to probabilists for quite a while now, not just because of their origin from physical motivations of modeling disordered random flow systems, but also because this process and it's variants (e.g. oriented first passage percolation and last passage percolation) arise as basic constructing blocks for more complicated problems such as the contact process. There has been an intensive study of this model on the d-dimensional lattice (see e.g. [15], [16] and [13]). The case of the complete graph with exponential edge weights was analyzed in [14], where in particular it was proved that the weight and hopcount of the optimal path satisfy:

$$nW_n - \log n \to \Xi,$$

and

$$\frac{H_n}{\log n} \to 1,$$

as $n \to \infty$, where $\Xi \in \mathbb{R}$ is a non-degenerate random variable.

In the last few years, due to the connections to real-world networks described above, these questions have taken on an added significance and a number of studies both at the non-rigorous level ([8]) and rigorous level (see e.g. [7]) have been undertaken to study such questions in many other random graph models.

(d) **Proof techniques:** A number of different techniques have been used in the analysis of first passage percolation asymptotics in various contexts, ranging from subadditive methods in the context of the lattice, to continuous time branching process embeddings and renewal theory in the context of various random graph models. In particular, for $s > 0$, [6] used embeddings into a particular continuous-time branching process theory to derive the results in (1.3) and (1.4). As far as we know, this is the first paper that uses Stein's method for Poisson approximation to derive refined asymptotics in the first passage percolation context. In particular the results here complete the program started in [6] and show that for $s \leq 0$, $H_n = O_{\mathbb{P}}(1)$ while for $s > 0$, $H_n \sim s \log n$. Clearly, this further shows that there are at least *two* universality classes for first-passage percolation on the complete graph in terms of the edge weight distribution. When $s \leq 0$, the weights E^s are in the same universality class as the weight 1, in the sense that the hopcount remains bounded, while for $s > 0$, they are in the same universality class as the exponential distribution arising for $s = 1$. As discussed in more detail in [6], this raises the question what the universality classes for first passage percolation on K_n are. In particular, does H_n always satisfy a central limit theorem whenever $H_n \to \infty$? Or are there classes of edge weight distributions where the behavior is even different? For example, is there a class of random edge weights where the behavior is similar as for the minimal spanning tree, where H_n is of the order $n^{1/3}$.

(e) **Multi-point distances and exchangeability:** The classical probability theory of exchangeability has in the last few years been used to analyze various complex random structures, see [3] for a nice modern survey. In the context of Corollary 2.3, one can analyze such questions in a number of different contexts (such as the stochastic mean-field model of distance or your favorite random graph model with your favorite random edge weights). For the stochastic mean-field model, the multi-point optimal path weights converge (after proper rescaling and recentering) to an *exchangeable* sequence of random variables. In the present model, we can once again show convergence but to an independent sequence of random variables.

4 Proofs

This section contains the proofs of the main results. Throughout the paper, we make use of the following standard notation. We let $\xrightarrow{d}$ denote convergence in distribution, and $\xrightarrow{\mathbb{P}}$ convergence in probability. For a sequence of random variables $(X_n)_{n\geq 1}$, we write $X_n = O_{\mathbb{P}}(1)$ when $|X_n|$ is a tight sequence of random variables as $n \to \infty$, and $X_n = o_{\mathbb{P}}(1)$ when $|X_n| \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$. For a non-negative function $n \mapsto g(n)$, we write $f(n) = O(g(n))$ when $|f(n)|/g(n)$ is uniformly bounded, and $f(n) = o(g(n))$ when $\lim_{n\to\infty} f(n)/g(n) = 0$. We let $\mathrm{Exp}(\lambda)$ denote an exponential random variable with rate λ and $\mathrm{Poi}(\lambda)$ a Poisson random variable with mean λ. Finally, we write that a sequence of events $(\mathcal{E}_n)_{n\geq 1}$ occurs *with high probability* (**whp**) when $\mathbb{P}(\mathcal{E}_n) \to 1$.

Let us first give an outline of the proof. In Section 4.1, we shall show that the hopcount H_n is a tight sequence of random variables as $n \to \infty$. In Section 4.2, we shall derive the asymptotic behavior, for $z \downarrow 0$, of the distribution function $F_k(z)$, where

$$F_k(z) = \mathbb{P}(E_1^{-s} + E_2^{-s} + \ldots + E_k^{-s} \leq z), \tag{4.1}$$

and where $E_1, E_2, \ldots$ is an independent and identically distributed (i.i.d.) sequence of $\mathrm{Exp}(1)$ random variables. Denoting the weight of the minimal path with exactly k edges between vertex 1 and vertex n by $W_k(n)$, we then show that, uniformly in $k \geq 2$ and any $0 < \varepsilon < 1$, the following statement holds **whp**,

$$(\log n)^s W_k(n) \geq (1-\varepsilon) g_s(k), \tag{4.2}$$

where $g_s(x)$ is defined in (2.1).

The inequality (4.2) yields a first order lower bound for all values of $k \geq 2$. We will show in the paper that the function $g_s(x)$ determines the first order asymptotics of the weights $W_k(n)$. The behavior of g_s near its minimum value determines the asymptotic behavior of the hopcount H_n. Roughly speaking, the hopcount H_n will converge in probability to the integer $k = k^*(s)$ that minimizes the function $g_s(x)$, $x \geq 2$. The above statement about the convergence of H_n is true for every $s > 0$ for which the minimizing integer of $g_s(x)$ for $x \geq 2$ is unique, i.e., $s \notin \mathcal{S}$. In Section 4.3, we study the minimum of an independent number of sums of n^{k-1} random variables with distribution E^{-s}, and use it to complete the proof of Theorem 2.1 when $0 < s \leq 1$. In Section 4.4, we extend the analysis to $s > 1$ and complete the proof of Theorem 2.1 by studying the second order asymptotics of the minimal weight of paths of length k in the complete graph K_n.

For $s \in \mathcal{S}$, to decide whether the hopcount H_n converges in probability either to $\lfloor s_j + 1 \rfloor$ or to $\lceil s_j + 1 \rceil$ we need the second second order asymptotics of $W_k(n)$, which is carried out in detail in Section 4.5. In Section 4.6, we sketch the proof of Corollary 2.3.

4.1 Tightness of the hopcount

Note that the minimal weight W_n satisfies the following inequality:

$$W_n \geq H_n \cdot \min_{1 \leq i \leq n(n-1)/2} E_i^{-s}, \tag{4.3}$$

where $E_i \sim \text{Exp}(1)$, are independent. Since the maximum of n independent exponentials scales like $(1 + o_{\mathbb{P}}(1)) \log n$, we obtain from (4.3) that **whp**

$$W_n \geq H_n \log^{-s}(n(n-1)/2)(1 + o_{\mathbb{P}}(1)). \tag{4.4}$$

On the other hand W_n is at most equal to the minimal weight of all two-edge paths between 1 and n. Here, a two-edge path is a path of the form $1 \to i \to n$, $i = 2, 3, \dots n-1$, so that:

$$W_n \leq \min_{2 \leq j \leq n-1} ((E_j')^{-s} + (E_j'')^{-s}), \tag{4.5}$$

where E_j', $2 \leq j \leq n-1$, and E_j'', $2 \leq j \leq n-2$, are independent $\text{Exp}(1)$ random variables. It is not hard to verify, see Lemma 4.1 in Section 4.2, that (4.5) implies that **whp**,

$$W_n \leq \frac{C}{(\log n)^s}. \tag{4.6}$$

The inequalities (4.4) and (4.6) together imply that **whp**,

$$H_n = O_{\mathbb{P}}(1). \tag{4.7}$$

We conclude that H_n is a tight sequence of random variables. The remainder of the proof will reveal that in fact H_n converges in distribution, either to a constant $k^*(s)$ when $s \notin \mathcal{S}$, or to a random variable giving positive mass to two values when $s \in \mathcal{S}$.

4.2 The first-order lower bound

We start with an investigation of the distribution function F_k of an *independent* sum of inverse powers of exponentials, i.e.,

$$Z_k = E_1^{-s} + \dots + E_k^{-s}. \tag{4.8}$$

Lemma 4.1 (Sums of inverse powers of exponentials) *Fix $s > 0$ and put $p = 1/s$. Then, for $z \to 0$,*

$$F_k(z) \sim a_k z^{-(k-1)p/2} \mathrm{e}^{-k^{p+1} z^{-p}}, \tag{4.9}$$

where $a_1 = 1$ and $(a_k)_{k \geq 2}$ is defined in (2.3) and where $f(z) \sim g(z)$, $z \to 0$, means that $\lim_{z \to 0} f(z)/g(z) = 1$.

Proof. The result for $k = 1$ is immediate from $F_1(z) = \mathrm{e}^{-z^{-p}}$, $z > 0$. We proceed by induction. Suppose that (4.9) holds for some $k \geq 1$, then

$$\begin{aligned} F_{k+1}(z) &\sim \int_0^z a_k (z-y)^{-(k-1)p/2} \mathrm{e}^{-k^{p+1}(z-y)^{-p}} d\Big(\mathrm{e}^{-y^{-p}}\Big) \\ &= p a_k z^{-(k-1)p/2} z^{-p} \int_0^1 x^{-p-1} (1-x)^{-(k-1)p/2} \mathrm{e}^{-z^{-p} h_k(x)} \, dx, \end{aligned}$$

where $h_k(x) = x^{-p} + k^{p+1}(1-x)^{-p}$. The function h_k has a saddle point at $x = 1/(k+1)$, since $h_k(1/(k+1)) = (k+1)^{p+1}$, $h_k'(1/(k+1)) = 0$, and

$$h_k''(1/(k+1)) = p(p+1)(k+1)^{p+2}\Big(1 + \frac{1}{k}\Big).$$

Hence, from a standard saddle point approximation we obtain

$$\begin{aligned} F_{k+1}(z) &\sim pa_k z^{-(k-1)p/2} z^{-p} \int_0^1 x^{-p-1}(1-x)^{-(k-1)p/2} \mathrm{e}^{-z^{-p}h_k(x)}\,dx \\ &\sim pa_k z^{-(k-1)p/2} z^{-p}(k+1)^{p+1}(k/(k+1))^{-(k-1)p/2} \mathrm{e}^{-z^{-p}h_k(1/(k+1))} \sqrt{\frac{2\pi}{z^{-p}h_k''(1/(k+1))}}, \end{aligned}$$

which yields the result. ∎

Using the above lemma, we obtain the following first-order lower bound for $W_k(n)$:

Theorem 4.2 (First-order lower bound) *Fix $s > 0$. For each $0 < \varepsilon < 1$, with the function g_s defined in (2.1),* **whp** *and uniformly in $k \in \{2, 3, \ldots, n-1\}$,*

$$(\log n)^s W_k(n) \geq (1-\varepsilon) g_s(k).$$

Proof. Fix $s > 0$ and $2 \leq k < n$, and define, for $0 < \varepsilon < 1$,

$$x_{k,n} = x_{k,n}(\varepsilon) = (1-\varepsilon)\frac{g_s(k)}{(\log n)^s}. \tag{4.10}$$

Let $N_k^{(n)}(x)$, $x > 0$, be the number of paths between 1 and n with exactly k edges and weight at most x. Then, according to Lemma 4.1, for $x \downarrow 0$,

$$\mathbb{E}[N_k^{(n)}(x)] = \binom{n-2}{k-1} F_k(x) \sim \binom{n-2}{k-1} a_k x^{-(k-1)p/2} \mathrm{e}^{-k^{p+1}x^{-p}}. \tag{4.11}$$

Note that $x_{k,n} \downarrow 0$, and that the term $x_{k,n}^{-(k-1)p/2}$ blows up only polynomially fast, while $\exp\{-k^{p+1}x_{k,n}^{-p}\}$ tends to 0 exponentially fast. Using that $\binom{n-2}{k-1} < n^{k-1}$ and abbreviating $N_k^{(n)} = N_k^{(n)}(x_{k,n})$, we reach to the conclusion that

$$\mathbb{E}[N_k^{(n)}] \leq n^{k-1}\exp\{-k^{p+1}x_{k,n}^{-p}\} = \exp\Big\{-\Big(\frac{1}{(1-\varepsilon)^p} - 1\Big)(k-1)\log n\Big\}.$$

Boole's inequality and the Markov inequality together yield

$$\begin{aligned} &\mathbb{P}\Big(\bigcup_{k=2}^{n-1}\{(\log n)^s W_k(n) < (1-\varepsilon)g_s(k)\}\Big) \leq \sum_{k=2}^{n-1}\mathbb{P}\big((\log n)^s W_k(n) < (1-\varepsilon)g_s(k)\big) \\ &\leq \sum_{k=2}^{n-1}\mathbb{P}\big(N_k^{(n)} \geq 1\big) \leq \sum_{k=2}^{n-1}\mathbb{E}[N_k^{(n)}] \leq \sum_{k=1}^{\infty}\exp\Big\{-\Big(\frac{1}{(1-\varepsilon)^p} - 1\Big)k\log n\Big\}. \end{aligned}$$

Since the summand on the right-hand side is of order $n^{-p\varepsilon k}$, we may conclude that the probability that $(\log n)^s W_k(n) < (1-\varepsilon)g_s(k)$, for some $2 \leq k \leq n-1$, tends to 0 as $n \to \infty$. ∎

4.3 Second order asymptotics

In this section we shall identify the second order asymptotics for the *independent* minimum of n^{k-1} random variables, where each of these random variables has distribution function $F_k(z)$, $z > 0$. The proof of Theorem 2.1 for $0 < s \le 1$ follows quite easily from this and the lower bound (4.2). The proof of Theorem 2.1 for $s > 1$ is postponed to the next section.

We denote by

$$W_k^{(\mathrm{ind})} = \min_{1 \le j \le n^{k-1}} Y_{k,j}, \tag{4.12}$$

where $Y_{k,1}, \dots, Y_{k,n^{k-1}}$ are i.i.d. with distribution function F_k. The following theorem derives the asymptotics of $W_k^{(\mathrm{ind})}$:

Theorem 4.3 (Minimum for independent r.v.'s) *The minimal weight $W_k^{(\mathrm{ind})}$ defined in (4.12), satisfies*

$$\mathbb{P}\Big(\frac{k-1}{sg_s(k)}(\log n)^{s+1}\Big(W_k^{(\mathrm{ind})} - \frac{g_s(k)}{(\log n)^s}\Big) + \frac{k-1}{2}\log\log n - \frac{(k-1)p\log g_s(k)}{2} > t\Big) \to \mathrm{e}^{-a_k \mathrm{e}^t}, \tag{4.13}$$

where $(a_k)_{k\ge 2}$ is defined in (2.3).

Proof. Denote by $\bar{F}_k$ the survival function of the distribution function F_k, defined in (4.1), i.e., $\bar{F}_k(z) = 1 - F_k(z)$. We compute $z_n = z_n(t)$, such that

$$\bar{F}_k^{n^{k-1}}(z_n) \xrightarrow{d} \exp\{-a_k \mathrm{e}^t\}.$$

This is equivalent to

$$n^{k-1}\log\{1 - F_k(z_n)\} \to -a_k\mathrm{e}^t.$$

Since $\log\{1 - F_k(z_n)\} \sim -F_k(z_n)$ for every k fixed and for $z_n \to 0$, we find in turn that

$$n^{k-1}F_k(z_n) \to a_k\mathrm{e}^t, \tag{4.14}$$

or

$$(k-1)\log n + \log\{F_k(z_n)\} \to t + \log a_k. \tag{4.15}$$

Put $z_n = \kappa(\log n)^{-s} + \zeta_n(t)$, where $\kappa = g_s(k)$. From Lemma 4.1, we find that (4.15) is equivalent to

$$(k-1)\log n - (k-1)p/2\log\big(\kappa(\log n)^{-s} + \zeta_n(t)\big) - k^{p+1}\big(\kappa(\log n)^{-s} + \zeta_n(t)\big)^{-p} \to t, \tag{4.16}$$

Writing

$$\kappa(\log n)^{-s} + \zeta_n(t) = \kappa(\log n)^{-s}\big(1 + \zeta_n(t)(\log n)^s/\kappa\big),$$

yields

$$\begin{aligned}&(k-1)\log n - (k-1)p/2\log\Big(\kappa(\log n)^{-s}\big(1 + \zeta_n(t)(\log n)^s/\kappa\big)\Big)\\ &-k^{p+1}\kappa^{-p}\log n \cdot \big(1 + \zeta_n(t)(\log n)^s/\kappa\big)^{-p} \to t.\end{aligned}$$

Using that $k^{p+1}\kappa^{-p} = k-1$ and $ps = 1$, we arrive at

$$\begin{aligned}&(k-1)\log n + (k-1)/2\log(\log n) - (k-1)p/2\log\big(1 + \zeta_n(t)(\log n)^s/\kappa\big)\\ &-(k-1)\log n \cdot \big(1 + \zeta_n(t)(\log n)^s/\kappa\big)^{-p} \to t + (k-1)p/2\log\kappa.\end{aligned}$$

Now we choose

$$\zeta_n(t) = (\log n)^{-s-1} \cdot (\zeta \log\log n + h(t)) \quad \text{or} \quad \zeta_n(t)(\log n)^s = \frac{(\zeta \log\log n + h(t))}{\log n}. \tag{4.17}$$

Then,

$$(k-1)p/2 \log\left(1+\zeta_n(t)(\log n)^s/\kappa\right) = O(\frac{\log\log n}{\log n}) \to 0,$$

and

$$\begin{aligned} -(k-1)\log n \cdot \left(1+\zeta_n(t)(\log n)^s/\kappa\right)^{-p} &\sim -(k-1)\log n \cdot \left(1 - p\zeta_n(t)(\log n)^s/\kappa\right) \\ &= -(k-1)\log n + \frac{(k-1)p}{\kappa}(\zeta \log\log n + h(t)), \end{aligned}$$

resulting in

$$\zeta = -\kappa/2p \quad \text{and} \quad \frac{(k-1)ph(t)}{\kappa} = t + \frac{1}{2}(k-1)p\log\kappa. \tag{4.18}$$

Hence,

$$\begin{aligned} z_n(t) &= g_s(k)(\log n)^{-s} + \zeta_n(t) = g_s(k)(\log n)^{-s} + (\log n)^{-s-1} \cdot (\zeta \log\log n + h(t)) \\ &= \frac{g_s(k)}{(\log n)^s} + \frac{g_s(k)}{(\log n)^{s+1}}\left[-\frac{\log\log n}{2p} + \frac{t}{(k-1)p} + \frac{\log g_s(k)}{2}\right]. \end{aligned} \tag{4.19}$$

■

We now turn to the proof of Theorem 2.1 in the case where $0 < s \le 1$:

Proof of Theorem 2.1 in case $0 < s \le 1$. Observe from Theorem 4.2 that, **whp** and uniform in $k \ge 3$,

$$(\log n)^s W_k(n) \ge (1-\varepsilon) g_s(3) > g_s(2), \tag{4.20}$$

where the latter inequality follows since for the indicated values of s, the function g_s is increasing on $[2,\infty)$, and where we can take $\varepsilon < \min_{0<s\le 1}[1 - g_s(2)/g_s(3)] = 1/9$. On the complete graph with n vertices the paths of length 2 have *independent* total weight, since they are disjoint. The number of paths of length 2 is equal to $n-2 \sim n$, so that we can conclude from the previous theorem that for, any $\varepsilon > 0$ and **whp**,

$$(\log n)^s W_n(2) \in (g_s(2) - \varepsilon, g_s(2) + \varepsilon). \tag{4.21}$$

From (4.20) and (4.21) it is immediate that $W_n = W_n(2)$, **whp**, and hence

$$\lim_{n\to\infty} H_n(s) \xrightarrow{\mathbb{P}} 2,$$

for $0 < s \le 1$. Similarly, since $W_n = W_n(2)$, **whp**, statement (b) of Theorem 2.1 follows from (4.13) for $0 < s \le 1$ and $k = 2$. ■

4.4 The case $s > 1$

In this section we treat the case $s > 1$. The number of paths with k edges between the vertices 1 and n is equal to $\binom{n-2}{k-1} \sim n^{k-1}/(k-1)!$. Let $\mathcal{S}_k(n)$ denote the set of all such paths. As before, we let F_k denote the distribution function of the sum of k independent random variables each with distribution equal to the distribution of E^{-s}, and by $N_k^{(n)}(z)$, $z > 0$, the number of paths with k

edges which have total weight $w_s(\mathcal{P}) = \sum_{e\in\mathcal{P}} E_e^{-s}$ less than z. Recall the definition of $z_n(t)$ in (4.19) From Theorem 4.3 and its proof (compare (4.14)), we conclude that, as $n \to \infty$,

$$\lambda_k^{(n)}(t) := \mathbb{E}[N_k^{(n)}(z_n(t))] \sim n^{k-1} F_k(z_n(t))/(k-1)! \to a_k \mathrm{e}^t/(k-1)!. \tag{4.22}$$

We shall prove the following proposition:

Proposition 4.4 (Poisson approximation for small weight paths) *Fix $s > 1$ and let $Z_k^{(n)}(t)$ be a Poisson random variable with mean $\lambda_k^{(n)}(t)$. Then, both for $k = \lfloor s+1 \rfloor$ and $k = \lceil s+1 \rceil$, as $n \to \infty$,*

$$d_{\mathrm{TV}}(N_k^{(n)}(z_n(t)), Z_k^{(n)}(t)) \to 0,$$

where d_{TV} denotes the total variation distance.

Assuming the proposition let us first show how to complete the proof of Theorem 2.1.

Proof of Theorem 2.1 in case $s > 1$ and $s \notin \mathcal{S}$. Observe that

$$\mathbb{P}(W_k(n) > z_n(t)) = \mathbb{P}(N_k^{(n)}(z_n(t)) = 0). \tag{4.23}$$

Now Proposition 4.4 together with (4.22) implies that, for $k = \lfloor s+1 \rfloor$ and $k = \lceil s+1 \rceil$, as $n \to \infty$,

$$\mathbb{P}(W_k(n) > z_n(t)) \to \exp(-a_k \mathrm{e}^t/(k-1)!). \tag{4.24}$$

Note that the weak convergence shows in particular that $(\log n)^s W_k(n)$ converges in probability to $g_s(k)$ for the two indicated values of k. This together with the lower bound proven in Theorem 4.2, and an argument similar to the case $0 < s \le 1$ then completes the proof of Theorem 2.1, in case the integer that minimizes $g_s(x)$ is unique, i.e., in case $s \notin \mathcal{S}$. ∎

Proof of Proposition 4.4. We apply [5, Theorem 1.A] with index set $\mathcal{S}_k(n)$, the set of paths between 1 and n having precisely k edges. For $\alpha \in \mathcal{S}_k(n)$, we denote by

$$I_\alpha = I_\alpha(z_n(t)) = \mathbb{1}\{w_s(\alpha) < z_n(t)\}, \tag{4.25}$$

where as before $w_s(\alpha) = \sum_{e\in\alpha} E_e^{-s}$, denotes the weight of the path α, and and where $\mathbb{1}\{A\}$ denotes the indicator of event A. Furthermore, let $p_k^{(n)}(t)$ denote the expectation of $I_\alpha(z_n(t))$, i.e.,

$$p_k^{(n)}(t) = \mathbb{P}(w_s(\alpha) < z_n(t)) = F_k(z_n(t)). \tag{4.26}$$

Let $\mathcal{I}^*(\alpha) \subseteq \mathcal{S}_k(n)$ denote the set of paths (not including α) which have at least one edge in common to α (*i.e., $\mathcal{I}^*(\alpha)$ is the set of paths β for which I_β is 'strongly' dependent on I_α*), and let $\mathcal{S}^*(\alpha) \subseteq \mathcal{S}_k(n)$ denote the set of paths that do not overlap on any edge with α. Note that the random variable $w_s(\alpha)$ is independent of $\{w_s(\beta)\colon \beta \in \mathcal{S}^*(\alpha)\}$. Finally, let

$$Z_\alpha = \sum_{\beta\in\mathcal{I}^*(\alpha)} \mathbb{1}\{w_s(\beta) < z_n(t)\}.$$

By the independence $w_s(\alpha)$ and $\{w_s(\beta)\colon \beta \in \mathcal{S}^*(\alpha)\}$, [5, Theorem 1.A] implies that

$$\begin{aligned} d_{\mathrm{TV}}(N_k^{(n)}(z_n(t)), Z_k^{(n)}(t)) &\le \frac{\sum_{\alpha\in\mathcal{S}_k(n)} \left[(p_k^{(n)}(t))^2 + p_k^{(n)}(t)\mathbb{E}[Z_\alpha] + \mathbb{E}[I_\alpha Z_\alpha]\right]}{\lambda_k^{(n)}(t)} \\ &= p_k^{(n)}(t) + \mathbb{E}[Z_\alpha] + \frac{\mathbb{E}[I_\alpha Z_\alpha]}{p_k^{(n)}(t)}, \end{aligned} \tag{4.27}$$

where the last equality follows since $\lambda_k^{(n)}(t) = |\mathcal{S}_k(n)| p_k^{(n)}(t)$ and since $\mathbb{E}[I_\alpha Z_\alpha]$ is independent of α. As before, by the choice of $z_n(t)$,

$$n^{k-1} p_k^{(n)}(t) \to a_k \mathrm{e}^t/(k-1)!$$

Thus, in particular, $p_k^{(n)}(t) \to 0$, as $n \to \infty$. Further, there exists a constant C_k such that, as $n \to \infty$,

$$\mathbb{E}[Z(\alpha)] = |\mathcal{I}^*(\alpha)| p_k^{(n)}(t) \leq C_k n^{k-2} p_k^{(n)}(t) \to 0.$$

Thus, the first two terms in (**??**) are negligible as $n \to \infty$. The last term requires some more analysis.

We note that

$$\mathbb{E}[I_\alpha Z_\alpha] = \sum_{j=1}^{k-2} |\mathcal{I}^*_{k,j}(\alpha)| p_{k,j}^{(n)}(t). \tag{4.28}$$

Here $\mathcal{I}^*_{k,j}(\alpha) \subset \mathcal{S}_k(n)$ consists of the set of paths of length k which overlap with α in exactly j edges, while

$$p_{k,j}^{(n)}(t) = \mathbb{P}(X_{k,k} < z_n(t), X_{k,j} < z_n(t)),$$

where $X_{k,k} = \sum_{r=1}^k E_r^{-s}$, while $X_{k,j} = \sum_{r=1}^j E_r^{-s} + \sum_{r=j+1}^k \tilde{E}_r^{-s}$, $1 \leq j \leq k-2$, and $(E_i)_{i=1}^k$ and $(\tilde{E}_r)_{r=1}^k$ are two independent vectors of i.i.d. Exp(1) random variables. We bound the probability $p_{k,j}^{(n)}(t)$ in the same way as before, using the saddle point approximation:

Lemma 4.5 (Correlated sums of inverse powers of exponentials) *Fix $k \geq 3$ and $1 \leq j \leq k-2$. Then, for $z \downarrow 0$,*

$$\mathbb{P}(X_{k,k} < z, X_{k,j} < z) \sim C_{k,j} \frac{1}{z^{(k-j-1)p + jp/2}} \exp\left(-z^{-p}[(k-j)\nu + j]^{p+1}\right).$$

Here $\nu = 2^{1/(p+1)}$ and $C_{k,j} > 0$ is a constant.

Proof. The proof is given by straightforward computation using the saddle point approximation:

$$\begin{aligned}
\mathbb{P}(X_{k,k} < z, X_{k,i} < z) &= \mathbb{P}\Big(\sum_{r=1}^k E_r^{-s} < z, \sum_{r=1}^i E_r^{-s} + \sum_{r=i+1}^k (\tilde{E}_r)^{-s} < z\Big) = \int_0^z F_{k-i}^2(z-y)\, dF_i(y) \\
&\sim \int_0^z \Big(a_{k-i}(z-y)^{-(k-i-1)p/2} \mathrm{e}^{-(k-i)^{p+1}(z-y)^{-p}}\Big)^2 \, da_i z^{-(i-1)p/2} \mathrm{e}^{-i^{p+1}y^{-p}} \\
&= a_i a_{k-i}^2 \int_0^z (z-y)^{-(k-i-1)p} \mathrm{e}^{-2(k-i)^{p+1}(z-y)^{-p}} \, dy^{-(i-1)p/2} \mathrm{e}^{-i^{p+1}y^{-p}} \\
&= a_i a_{k-i}^2 \int_0^z y^{-(i-1)p/2-1} \mathrm{e}^{-i^{p+1}y^{-p}} \big(p i^{p+1} y^{-p} - (i-1)p/2\big)(z-y)^{-(k-i-1)p} \mathrm{e}^{-2(k-i)^{p+1}(z-y)^{-p}} \, dy \\
&\qquad \times (1-x)^{-(k-i-1)p} \exp\{-z^{-p} h_{k,i}(x)\}\, dx,
\end{aligned}$$

where we abbreviate

$$h_{k,i}(x) = i^{p+1} x^{-p} + 2(k-i)^{p+1}(1-x)^{-p}.$$

Put $\nu = 2^{\frac{1}{p+1}}$. Then, the saddle point arises in the point $x_{k,i}$, satisfying $h'_{k,i}(x_{k,i}) = 0$, which yields

$$x_{k,i} = \frac{i}{(k-i)\nu + i}.$$

Furthermore, $h_{k,i}(x_{k,i})) = \big((k-i)\nu + i\big)^{p+1}$, while

$$h''_{k,i}(x_{k,i}) = \frac{p(p+1)}{i(k-i)\nu}\big((k-i)\nu + i\big)^{p+3}.$$

Applying the saddle point approximation then yields:

$$\mathbb{P}(X_{k,k} < z, X_{k,i} < z) \sim a_i a_{k-i}^2 z^{-(kp-(i+3)p/2)}(x_{k,i})^{-(i-1)p/2-1}\big(pi^{p+1}(zx_{k,i})^{-p} - (i-1)p/2\big)$$
$$\times(1-x_{k,i})^{-(k-i-1)p}\exp\{-z^{-p}h_{k,i}(x_{k,i})\}\sqrt{\frac{2\pi}{z^{-p}h''_{k,i}(x_{k,i})}}. \tag{4.29}$$

■

Recall (4.27). The first two terms on the right-hand side are negligible, hence it suffices to show that

$$\frac{\mathbb{E}[I_\alpha Z_\alpha]}{p_k^{(n)}(t)} \to 0$$

Using that $p_k^{(n)}(t) = O(n^{-(k-1)})$ and that $|\mathcal{I}^*_{k,j}(\alpha)| \sim n^{k-j-1}$ it follows from (4.28) that we now need to show for $1 \le j \le k-2$,

$$n^{2k-j-2}p_{k,j}^{(n)}(t) \to 0,$$

as $n \to \infty$. Now the polynomial terms (z^k type terms) in the saddle point approximation of $p_{k,j}^{(n)}(t)$ should not play a role. Thus, using the fact that up to the first order

$$(z_n(t))^{-p} \sim \Big(\frac{g_s(k)}{\log^s n}\Big)^{-p} = \frac{(k-1)\log n}{k^{1+p}}, \tag{4.30}$$

we need to show that for $1 \le j \le k-2$ and with $\nu = 2^{1/(p+1)}$,

$$\left[\left(\Big(1-\frac{j}{k}\Big)\nu + \frac{j}{k}\right)^{p+1} - \Big(2 - \frac{j}{k-1}\Big)\right] > 0. \tag{4.31}$$

The above inequality is not true for s close to 0 and larger values of k. However, it is true for $s > 1$ and $k \in \{\lfloor s+1\rfloor, \lceil s+1\rceil\}$ as we will now show. Indeed, define, for $x \in [0,1]$,

$$u_k(x) = \Big[(1-x)2^{s/(s+1)} + x\Big]^{(1+1/s)} - \left(2 - \frac{k}{k-1}x\right),$$

and note that $u_k(j/k)$ is equal to the left side of (4.31). Hence, if we show that for both $k = \lfloor s+1\rfloor$ and $k = \lceil s+1\rceil$, the function $u_k(x) > 0$ for all $x \in (0,1)$, then we are done. Differentiating $x \mapsto u_k(x)$ with respect to x yields

$$u'_k(x) = -a\left[(1-x)2^{1/a} + x\right]^{a-1}(2^{1/a}-1) + \frac{k}{k-1},$$

where $a = (s+1)/s > 1$. The function u' is increasing as can easily be seen from the second derivative

$$u''_k(x) = a(a-1)(1-2^{1/a})^2\left[(1-x)2^{1/a} + x\right]^{a-2} > 0.$$

Hence, we have to show that $u'_k(0) > 0$, for the two indicated values of k.

Claim: *Fix $s > 1$, then the condition $u'_k(0) > 0$ is true for both $k = \lfloor s+1\rfloor$ and $k = \lceil s+1\rceil$.*

Proof of claim. Since $a = (s+1)/s$, the condition $u_k'(0) > 0$ is equivalent to

$$\frac{k}{k-1} > \frac{s+1}{s}(2 - 2^{s/(s+1)}).$$

Now substitute either $k = \lfloor s+1 \rfloor$ or $k = \lceil s+1 \rceil$ and simplifying the claim $u'(0) > 0$ shows that it is equivalent to

$$sk(2^{s/(s+1)} - 1) > (k - (s+1))(2 - 2^{s/(s+1)}).$$

This inequality is trivially true for $k = \lfloor s+1 \rfloor$, since then the right side is negative, whereas the left side is positive. For $k = \lceil s+1 \rceil$, the left side is at least $3(2^{s/(s+1)} - 1)$, whereas the right side is at most $2 - 2^{s/(s+1)}$, while $s > 1$. Furthermore, again using that $s > 1$,

$$3(2^{s/(s+1)} - 1) \geq 3(\sqrt{2} - 1) > 2 - \sqrt{2} \geq 2 - 2^{s/(s+1)}.$$

This shows that the above claim holds and hence that the Poisson approximation holds both for $k = \lfloor s+1 \rfloor$ and $k = \lceil s+1 \rceil$. ■

4.5 The case $s \in \mathcal{S}$, the special set

In this section, we will prove Theorem 2.2. To this end, we fix $s_j \in \mathcal{S}$ and denote by $k = \lfloor s_j + 1 \rfloor$, so that $k+1 = \lceil s_j + 1 \rceil$. Let $N_k^{(n)} = N_k^{(n)}(z_n(x))$ denote the number of paths from 1 to n of length k and with weight at most $z_n(x)$, with $z_n(x)$ given by (4.19), and similarly we denote by $M_k^{(n)} = M_k^{(n)}(z_n(y))$ the number of paths from 1 to n of length $k+1$ and with weight at most $z_n(y)$, where $z_n(y)$ is given by the right-hand side of (4.19), with t replaced by y and k by $k+1$. Note that the change from k to $k+1$ is for many aspects irrelevant, because for $s = s_j$, we have $g_s(k) = g_s(k+1)$. We are therefore in particular allowed to use the same quantity $z_n(y)$ in the definition of $M_k^{(n)}$. We will show below that the total variation distance between $N_k^{(n)} + M_k^{(n)}$ and a Poisson variable with mean $\mu_k^{(n)}(x,y) = \mathbb{E}[N_k^{(n)} + M_k^{(n)}]$ converges to 0 as $n \to \infty$, i.e., we will show that

$$d_{\mathrm{TV}}(N_k^{(n)} + M_k^{(n)}, \mathrm{Poi}(\mu_k^{(n)}(x,y))) \to 0. \tag{4.32}$$

Let us first prove that (4.32) implies Theorem 2.2. Indeed, (4.32) implies that

$$\begin{aligned}\mathbb{P}(W_k(n) > z_n(x), W_{k+1}(n) > z_n(y)) &= \mathbb{P}(N_k^{(n)} = 0, M_k^{(n)} = 0) = \mathbb{P}(N_k^{(n)} + M_k^{(n)} = 0) \\ &\to \mathbb{P}(\mathrm{Poi}(\mu_k(x,y)) = 0),\end{aligned}$$

where $\mu_k(x,y) = \lim_{n\to\infty} \mu_k^{(n)}(x,y) = \lambda_k(x) + \lambda_{k+1}(y)$, by (4.22), and where we define

$$\lambda_l(z) = a_l \mathrm{e}^z/(l-1)!, \qquad l \geq 1, z > 0. \tag{4.33}$$

Thus,

$$\lim_{n\to\infty} \mathbb{P}(W_k(n) > z_n(x), W_{k+1}(n) > z_n(y)) = \lim_{n\to\infty} \mathbb{P}(W_k(n) > z_n(x)) \lim_{n\to\infty} \mathbb{P}(W_{k+1}(n) > z_n(y)). \tag{4.34}$$

Consequently, comparing with (4.23), we see that the events $\{W_k(n) > z_n(x)\}$ and $\{W_{k+1}(n) > z_n(y)\}$ are asymptotically independent. It is then straightforward to conclude that the normalized optimal weight W_n in Theorem 2.2 converges in distribution to the minimum of the *independent* pair $(\Xi_k/(k-1), \Xi_{k+1}/k)$. The lower bound for $(\log n)^s W_k(n)$ of Theorem 4.2 again completes the proof of part (a), and subsequently also part (b), of Theorem 2.2.

In order to prove (4.32), we rely again on the Poisson approximation in [5]. Set $\mathcal{T}_k(n) = \mathcal{S}_k(n) \cup \mathcal{S}_{k+1}(n)$ the index set of all paths from 1 to n having either k or $k+1$ edges, where as before

$k = \lfloor s_j + 1 \rfloor$. To denote that the length of a path is equal to k, we give it a subscript k and write α_k for an element of $\mathcal{S}_k(n)$. For $\alpha_k \in \mathcal{S}_k(n)$, we denote by

$$I_{\alpha_k} = I_{\alpha_k}(z_n(x)) = \mathbb{1}\{w_s(\alpha_k) < z_n(x)\},$$

whereas for a path $\alpha_{k+1} \in \mathcal{S}_{k+1}(n)$, we define

$$I_{\alpha_{k+1}} = I_{\alpha_{k+1}}(z_n(y)) = \mathbb{1}\{w_s(\alpha_{k+1}) < z_n(y)\},$$

so that

$$p_k^{(n)}(x) = \mathbb{P}(w_s(\alpha_k) < z_n(x)) = F_k(z_n(x)), \quad p_{k+1}^{(n)}(y) = \mathbb{P}(w_s(\alpha_{k+1}) < z_n(y)) = F_{k+1}(z_n(y)).$$

Writing α for α_k or α_{k+1}, we denote by $\mathcal{I}^*(\alpha) \subseteq \mathcal{T}_k(n)$ the set of paths (not including α) which have at least one edge in common to α , and by $\mathcal{S}^*(\alpha) \subseteq \mathcal{T}_k(n)$ the set of paths that do not overlap on any edge with α. Finally, let

$$Z_\alpha = \sum_{\beta_k \in \mathcal{I}^*(\alpha)} \mathbb{1}\{w_s(\beta_k) < z_n(x)\} + \sum_{\beta_{k+1} \in \mathcal{I}^*(\alpha)} \mathbb{1}\{w_s(\beta_{k+1}) < z_n(y)\}.$$

The total variation distance in (4.32) is bounded by

$$\frac{\sum_{\alpha \in \mathcal{S}_k(n)} \left[(p_k^{(n)}(x))^2 + p_k^{(n)}(x)\mathbb{E}[Z_\alpha] + \mathbb{E}[I_\alpha Z_\alpha]\right]}{\mu_k^{(n)}(x,y)} + \frac{\sum_{\alpha \in \mathcal{S}_{k+1}(n)} \left[(p_{k+1}^{(n)}(y))^2 + p_{k+1}^{(n)}(y)\mathbb{E}[Z_\alpha] + \mathbb{E}[I_\alpha Z_\alpha]\right]}{\mu_k^{(n)}(x,y)}.$$

Since

$$\mu_k^{(n)}(x,y) = p_k^{(n)}(x)|\mathcal{S}_k(n)| + p_{k+1}^{(n)}(y)|\mathcal{S}_{k+1}(n)| \geq \max\{p_k^{(n)}(x)|\mathcal{S}_k(n)|, p_{k+1}^{(n)}(y)|\mathcal{S}_{k+1}(n)|\},$$

we conclude from the proof of Proposition 4.4 that

$$\frac{\sum_{\alpha \in \mathcal{S}_k(n)} \left[(p_k^{(n)}(x))^2 + p_k^{(n)}(x)\mathbb{E}[Z_\alpha]\right]}{\mu_k^{(n)}(x,y)} + \frac{\sum_{\alpha \in \mathcal{S}_{k+1}(n)} \left[(p_{k+1}^{(n)}(y))^2 + p_{k+1}^{(n)}(y)\mathbb{E}[Z_\alpha]\right]}{\mu_k^{(n)}(x,y)} \to 0.$$

Hence, it remains to prove that

$$\frac{\sum_{\alpha \in \mathcal{S}_k(n)} \mathbb{E}[I_\alpha Z_\alpha] + \sum_{\alpha \in \mathcal{S}_{k+1}(n)} \mathbb{E}[I_\alpha Z_\alpha]}{\mu_k^{(n)}(x,y)} \to 0. \tag{4.35}$$

We next decompose $\mathbb{E}[I_\alpha Z_\alpha]$ into the part where β has k or $k+1$ edges, i.e.,

$$\mathbb{E}[I_\alpha Z_\alpha] = \sum_{\beta_k \in \mathcal{I}^*(\alpha)} \mathbb{P}(I_\alpha = 1, I_{\beta_k} = 1) + \sum_{\beta_{k+1} \in \mathcal{I}^*(\alpha)} \mathbb{P}(I_\alpha = 1, I_{\beta_{k+1}} = 1).$$

By making this decomposition, as well as differentiating between the number of edges of α, the numerator in (4.35) splits into 4 different double sums. The two double sums running over the index sets $\alpha \in \mathcal{S}_k(n), \beta_k \in \mathcal{I}^*(\alpha)$ and $\alpha \in \mathcal{S}_{k+1}(n), \beta_{k+1} \in \mathcal{I}^*(\alpha)$ are treated in the proof of Proposition 4.4, apart from the small change that $z_n(x)$ and $z_n(y)$ are now possibly *different*. Since $z_n(x)/z_n(y) \to 1$, it is straightforward to adapt the argument. Below, we will show that

$$\frac{\sum_{\alpha \in \mathcal{S}_k(n)} \sum_{\beta_{k+1} \in \mathcal{I}^*(\alpha)} \mathbb{P}(I_\alpha = 1, I_{\beta_{k+1}} = 1)}{\mu_k^{(n)}(x,y)} \to 0. \tag{4.36}$$

The term with $\alpha \in \mathcal{S}_{k+1}(n)$ and $\beta_k \in \mathcal{I}^*(\alpha)$ is identical, apart from the fact that x and y are interchanged. Thus, (4.36) completes the proof of (4.32).

To prove (4.36), we write, as in (4.28),

$$\sum_{\beta_{k+1}\in\mathcal{I}^*(\alpha)} \mathbb{P}(I_\alpha = 1, I_{\beta_{k+1}} = 1) = \sum_{j=1}^{k-1} |\mathcal{I}^*_{k+1,j}(\alpha)| p^{(n)}_{k+1,j}(x, y),$$

where $\mathcal{I}^*_{k+1,j}(\alpha) \subset \mathcal{T}_k(n)$ consists of the set of paths of length $k+1$ which overlap with α, which has length k, in exactly j edges, while

$$p^{(n)}_{k+1,j}(x, y) = \mathbb{P}(X_{k,k} < z_n(x), X_{k+1,j} < z_n(y)),$$

where, similarly as in the proof of Proposition 4.4, we now write $X_{k,k} = \sum_{r=1}^{k} E_r^{-s}$, while $X_{k+1,j} = \sum_{r=1}^{j} E_r^{-s} + \sum_{r=j+1}^{k+1} \tilde{E}_r^{-s}$, $1 \le j \le k-1$.

By adapting the saddle point approximation used in the proof of Lemma 4.5, it is readily verified that for $z_1, z_2 \downarrow 0$ such that $\lim_{z_1\to 0} z_2/z_1 = 1$, and $k \ge 3$ and $1 \le j \le k-1$, we have

$$\mathbb{P}(X_{k,k} < z_1, X_{k+1,j} < z_2) = \exp\left(-z_1^{-p}[(k-j)\nu + j + 1]^{p+1}(1 + o(1))\right). \tag{4.37}$$

where, as before, $\nu = 2^{1/(p+1)}$. By (4.30) $(z_n(x))^{-p} \sim \frac{(k-1)\log n}{k^{1+p}}$. Further, since $s \in \mathcal{S}$, we have that $(z_n(y))^{-p} \sim \frac{\log n}{(k+1)^{1+p}} = \frac{(k-1)\log n}{k^{1+p}}$. Thus, $\lim_{n\to\infty} z_n(y)/z_n(x) = 1$, as required.

We conclude that in order that (4.36) holds we need to show that $n^{2k-j-1} p^{(n)}_{k+1,j}(t) \to 0$, or equivalently that

$$\exp\left(-\log n \left[(k-1)\left((1-\frac{j}{k})\nu + \frac{j+1}{k}\right)^{p+1} - (2k-j-1)\right]\right) \to 0, \quad 1 \le j \le k-1.$$

This follows from the facts that $\nu = 2^{1/(p+1)} < 2$ and $p > 0$, since, for $1 \le j \le k-1$,

$$\begin{aligned} \frac{(2k-j-1)}{k-1} &= \left(2 - \frac{j-1}{k-1}\right) \le \left((1-\frac{j-1}{k})\nu + \frac{j-1}{k}\right)^{p+1} \\ &= \left((1-\frac{j}{k})\nu + \frac{j+1}{k} + \frac{\nu-2}{k}\right)^{p+1} < \left((1-\frac{j}{k})\nu + \frac{j+1}{k}\right)^{p+1}. \end{aligned}$$

This proves (4.36), and, thus, completes the proof of (4.32). ∎

4.6 Multipoint distance limits

In this section, we discuss how to prove Corollary 2.3. We shall indicate how to prove this corollary for 2 multipoint distances. The case for general m follows similarly.

More precisely, let $\tilde{W}_n^2, \tilde{W}_n^3$ denote the recentered and rescaled optimal weights between 1 and 2 and 1 and 3. Recall, for any fixed $t \in \mathbb{R}$, the function $z_n(t)$ from (4.19), where we take $k = k^*(s)$. For $j = 2, 3$ and any $t \in \mathbb{R}$, let $N_k^{j,(n)}(z_n(t))$ denote the number of paths between 1 and j having k edges whose weight is less than $z_n(t)$.

The proof of Corollary 2.3 will be an adaptation of the proof of Theorem 2.2 in Section 4.5, and we start by recalling some results we have proved and shall rely on. Recall that we have already proved that, as $n \to \infty$,

$$\lambda_k^{(n)}(t) = \mathbb{E}(N_k^{j,(n)}(z_n(t))) \sim \lambda_k(t),$$

where $\lambda_k(x) = a_k e^x/(k-1)!$ is defined in (4.33) and

$$d_{\mathrm{TV}}(N_k^{j,(n)}(z_n(t)), \mathrm{Poi}(\lambda_k(t))) \to 0.$$

For any fixed $x, y \in \mathbb{R}$, define

$$N_n^* = N_k^{1,(n)}(z_n(x)) + N_k^{2,(n)}(z_n(y)). \tag{4.38}$$

Below, we shall show that

$$N_n^* \xrightarrow{d} \mathrm{Poi}(\lambda_k(x) + \lambda_k(y)). \tag{4.39}$$

Then, the argument leading to (4.34) implies that $\tilde{W}_1^n$ and $\tilde{W}_2^n$ are asymptotically independent, so that

$$\lim_{n\to\infty} \mathbb{P}(\tilde{W}_1^n > x, \tilde{W}_2^n > y) \to \exp(-\lambda_k(x) - \lambda_k(y)),$$

establishing the result we want. We next sketch how one can prove (4.39).

Idea of the proof of (4.39). Fix any path α with k edges between 1 and 2 and path β between 1 and 3. Since the argument is quite close to the proof of Theorem 2.2, we shall keep the discussion brief and focus on the differences. We again rely on the total variation bound [5, Theorem 1.A] that implies

$$d_{\mathrm{TV}}(N_n^*, \mathrm{Poi}(\lambda_k^{(n)}(x) + \lambda_k^{(n)}(y))) \le \frac{(I) + (II) + (III)}{\lambda_k^{(n)}(x) + \lambda_k^{(n)}(y)}.$$

Here,

$$\begin{aligned}
(I) &= p_k^{(n)}(x)\lambda_k^{(n)}(x) + p_k^{(n)}(y)\lambda_k^{(n)}(y),\\
(II) &= (\mathbb{E}[Z_\alpha^1] + \mathbb{E}[Z_\alpha^2])\lambda_k^{(n)}(x) + (\mathbb{E}[Z_\beta^1] + \mathbb{E}[Z_\beta^2])\lambda_k^{(n)}(y),\\
(III) &= \mathbb{E}[I_\alpha Z_\alpha^1 + I_\alpha Z_\alpha^2 + I_\beta Z_\beta^1 + I_\beta Z_\beta^2],
\end{aligned}$$

and, as in (4.25),

$$I_\alpha = \mathbb{1}\{|w_s(\alpha)| < z_n(x)\}, \quad I_\beta = \mathbb{1}\{|w_s(\beta)| < z_n(y)\},$$

while, writing $\mathcal{I}^*(\alpha)$ for the set of paths from 1 to 2 which overlap with α and $\mathcal{I}_2^*(\alpha)$ for the set of paths from 1 to 3 which overlap with α (and similarly for β),

$$Z_\alpha^1 = \sum_{\gamma\in\mathcal{I}^*(\alpha)} \mathbb{1}\{|w_s(\gamma)| < z_n(x)\}, \qquad \text{and} \qquad Z_\alpha^2 = \sum_{\gamma\in\mathcal{I}_2^*(\alpha)} \mathbb{1}\{|w_s(\gamma)| < z_n(y)\},$$

and similarly for Z_β^1 and Z_β^2. Now, we have already shown that the terms (I) and (II) rescaled by the means are negligible as $n \to \infty$. Thus, to complete the proof we just need to show that, as $n \to \infty$,

$$\frac{\mathbb{E}[I_\alpha Z_\alpha^2]}{\lambda_k^{(n)}(x) + \lambda_k^{(n)}(y)} \to 0$$

(and the corresponding term for the path β), namely the contribution of terms consisting of the overlap of the path α (which goes from 1 to 2) with paths which go from 1 to 3. This is a minor adaptation of the proof of (4.36), and we omit the details. ∎

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