LECTURE 2

WORK SHEET

SOLUTIONS

2 Since each new vertex attaches to one of the pre-existing vertices u.a.r, we have

$$E(N_0(n+1) \mid t_n) = \underset{\substack{\uparrow \\ \text{new vertex}}}{N_0(n) + 1} - \underset{\substack{\uparrow \\ \text{Prob this vertex attaches to a leaf.}}}{\frac{N_0(n)}{n}}$$

Thus the unconditional expectation satisfies the recursion

$$p_0(n+1) = p_0(n)\left(1 - \frac{2}{n+1}\right) + \frac{1}{n+1}$$

Iterating and noting that $p_0(1) = 1$, we get

$$p_0(n+1) = \frac{n + n-1 + \cdots + 1}{n(n+1)} = \frac{1}{2}.$$

Similarly for $k=1$ we have

$$E(N_1(n+1) \mid t_n) = N_1(n) + \frac{N_0(n)}{n} - \frac{N_1(n)}{n}$$

$$\Rightarrow p_1(n+1) = \left(1 - \frac{2}{n+1}\right) p_1(n) + \frac{1}{2(n+1)}$$

$$\Rightarrow \lim_{n\to\infty} p_1(n) = \left(\frac{1}{2}\right)^2$$

In general

$$\lim_{n\to\infty} p_k(n) = \left(\frac{1}{2}\right)^{k+1}$$

3. Let $Z(t) = e^{-t} |\mathcal{F}(t)|$

Note that $|\mathcal{F}(t)| \longrightarrow |\mathcal{F}(t)|+1$ at rate $|\mathcal{F}(t)|$

Thus

$$E(dZ(t)|\mathcal{F}_t) = -e^{-t}|\mathcal{F}(t)|\,dt + e^{-t}E(\Delta|\mathcal{F}(t)|\,|\mathcal{F}_t)$$

$$= -e^{-t}|\mathcal{F}(t)|\,dt + e^{-t}|\mathcal{F}(t)|\,dt$$

$$= 0$$

"Thus" $Z(t)$ is a martingale.

In $\mathcal{F}(t)$, if you pick a vertex V_t at random, let T_{V_t} be the time of birth of this vertex. Then the age of V_t i.e $t - T_{V_t} \approx \exp(1) = Y$.

Thus the out-degree of ~~a~~ a vertex V_t

has approx distrⁿ $\mathcal{P}([0, Y])$
where $\mathcal{P}$ = Poisson process, rate 1
independent of Y.

Check that
$$P(\mathcal{P}([0,Y]) = k) = \left(\frac{1}{2}\right)^k.$$

1, ~~Attach to each existing vertex~~

1, Every new vertex picks one of the pre-existing vertices uniformly at random and attaches itself to this vertex.

5. $E(\mathcal{P}[0,T_\Lambda]) = \sum_{k=1}^{\infty} P(\mathcal{P}[0,T_\Lambda] \geq k)$

$$= \sum_{k=1}^{\infty} P(L_1 + L_2 + \cdots + L_k < T_\Lambda)$$

$$= \sum_{k=1}^{\infty} E\left(e^{-\Lambda(L_1+L_2+\cdots+L_k)}\right)$$

$$= \sum_{k=1}^{\infty} \cancel{\frac{f(k)}{\Lambda+f(k)}} \prod_{i=1}^{k} \frac{f(i)}{\Lambda+f(i)}$$

So want

$$\sum_{k=1}^{\infty} \prod_{i=1}^{k} \frac{f(i)}{\Lambda+f(i)} = 1$$

6. Prove this by induction. Assume true for till ~~time~~ $\mathcal{F}(T_{n-1})$, show true for $\mathcal{F}(T_n)$.

7 ~~Since~~ Every vertex reproduces at rate $= 1 +$ # of it's blue children at that time

Let $|\mathcal{F}(t)| =$ size of the process $= Z(t)$

$B(t) =$ # of blue vertices

Then

$$\cancel{\mathcal{F}(t)} \quad Z(t) \to Z(t)+1 \quad \text{at rate } Z(t)+B(t)$$

Since every new vertex is colored blue with probability $q = 1-p$

$$B(t) \to B(t)+1 \quad \text{at rate}$$
$$q\,(Z(t)+B(t))$$

Thus

$$E\left(\Delta\,(Z(t)+B(t)) \mid \mathcal{F}_t\right)$$

$$= (1+q)\,(Z(t)+B(t))\,dt.$$

$$\Rightarrow E\left(d\, e^{-(1+q)t}(Z(t)+B(t)) \mid \mathcal{F}_t\right)$$
$$= 0$$

$$\Rightarrow e^{-(1+q)t}\,(Z(t)+B(t)) = \text{martingale}.$$

4. One can show that

$$e^{-t}|\mathcal{F}(t)| = L^2 \text{ bounded martingale}$$

Thus

$$e^{-t}|\mathcal{F}(t)| \xrightarrow{a.s., L^2} W > 0 \quad \ldots (*)$$

Let

$$T_n = \inf\{t : |\mathcal{F}(t)| = n\}.$$

Then by $(*)$ $T_n = \log n + O(1)$.

~~Conv~~ Also note that

$$t_n \stackrel{d}{=} \mathcal{F}(T_n).$$

Now consider the degree of the root ρ in $\mathcal{F}(t)$.

For any time t

$$\deg_\rho(t) = \text{Poisson}[0, t].$$

Thus by ~~the~~ time T_n

$$\deg_\rho(T_n) \approx \log n + O(1).$$

Thus max degree $= M_n$ at least of size $\log n$.

Claim: There exists $1 < C < \infty$ such that given any $\varepsilon > 0$

$$\limsup_{n \to \infty} P(M_n > c \log n) \le \varepsilon.$$

Proof: Fix B such that

$$\limsup_{n \to \infty} P(T_n > \log n + B) \le \varepsilon/2$$

Enough to show: Can choose c such that

$$\limsup_{n \to \infty} P(\text{max degree of } \mathcal{F}(\log n + B) > c \log n) \le \varepsilon/2.$$

~~Equivalently~~ First moment method:

Enough to show

$$\limsup_{n \to \infty} E(N_B(c \log n)) \le \varepsilon/2 \quad \text{where}$$

$N_B(c\log n) = \#$ of vertices in $\mathcal{F}(\log n + B)$ with more than $c \log n$ vertices.

If a vertex is born at time

$$0 \le S \le \log n + B$$

then it's degree by time $\log n + B$ has distribution

$$\text{Poisson}(\log n + B - S).$$

<u>Thus check</u>

$$E(N_B(c\log n)) = \int_0^{\log n + B} P(\text{Poisson}(\log n + B - S) \ge c\log n) \, e^S \, dS.$$

Show that you can choose c large indep of B, so the above $< \varepsilon/2$