Scaling limits for critical inhomogeneous random graphs with finite third moments

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Abstract

We identify the scaling limits for the sizes of the largest components at criticality for inhomogeneous random graphs when the degree exponent τ satisfies $\tau > 4$. We see that the sizes of the (rescaled) components converge to the excursion lengths of an inhomogeneous Brownian motion, extending results of [1]. We rely heavily on martingale convergence techniques, and concentration properties of (super)martingales. This paper is part of a programme to study the critical behavior in inhomogeneous random graphs of so-called rank-1 initiated in [12].

Key words: critical random graphs, phase transitions, inhomogeneous networks, Brownian excursions, size-biased ordering.
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1 Introduction

1.1 Model

We start by describing the model considered in this paper. While there are many variants available in the literature, the most convenient for our purposes model is the model often referred to as *Poissonian graph process* or *Norros-Reittu model* [17]. See Section 1.2 below for consequences for other models. To define the model, we consider the vertex set $[n] := \{1, 2, \ldots, n\}$, and attach an edge with probability p_{ij} between vertices i and j, where

$$p_{ij} = 1 - \exp\left(-\frac{w_i w_j}{l_n}\right), \tag{1.1}$$

and

$$l_n = \sum_{i=1}^{n} w_i, \tag{1.2}$$

where we assume throughout this paper that $w_i^2/l_n \leq 1$ for all $i \in [n]$. Different edges are independent.

Below, we shall formulate general conditions on the weight sequence $\boldsymbol{w} = (w_1, \ldots, w_n)$, and for now formulate two main examples. The first key example arises when we let the weight sequence $\boldsymbol{w} = (w_1, \ldots, w_n)$ be defined by

$$w_i = [1 - F]^{-1}(i/n), \tag{1.3}$$

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where $F(x)$ is a distribution function satisfying that, when W is a random variable with distribution function F, we have

$$\sigma_3 = \mathbb{E}[W^3] < \infty, \tag{1.4}$$

and where $[1-F]^{-1}$ is the generalized inverse function of $1-F$ defined, for $u \in (0,1)$, by

$$[1-F]^{-1}(u) = \inf\{s : [1-F](s) \leq u\}. \tag{1.5}$$

By convention, we set $[1-F]^{-1}(1) = 0$.

The second key example arises when we take the sequence $\boldsymbol{w} = (w_1, \ldots, w_n)$ to be an i.i.d. sequence of random variables with distribution function F satisfying (1.4).

Suppose $W \sim F$ and write

$$\nu = \frac{\mathbb{E}[W^2]}{\mathbb{E}[W]}. \tag{1.6}$$

Then, by [3], the random graphs we consider are subcritical when $\nu < 1$ and supercritical when $\nu > 1$. Indeed, when $\nu > 1$, then there is one giant component of size $\Theta(n)$ while all other components are of smaller size $o(n)$, while when $\nu \leq 1$ the largest connected component has size $o_{\mathbb{P}}(n)$. Thus, the critical value of the model is $\nu = 1$.

We shall write $\mathcal{G}_n^0(\boldsymbol{w})$ to be the graph constructed via the above procedure, while, for any fixed $t \in \mathbb{R}$, we shall write $\mathcal{G}_n^t(\boldsymbol{w})$ when we use the weight sequence $(1 + tn^{-1/3})\boldsymbol{w}$, for which the probability that i and j are neighbors equals $1 - \exp\left(-(1 + tn^{-1/3})w_i w_j / l_n\right)$.

We now formulate the general conditions on the weight sequence $\boldsymbol{w} = (w_1, \ldots, w_n)$. In Section 3, we shall verify that these conditions are satisfied for i.i.d. weights with finite third moment, as well as for the choice in (1.3). We assume the following three conditions on the weight sequence $\boldsymbol{w} = (w_1, \ldots, w_n)$:

(a) Maximal weight bound. We assume that the maximal weight is $o(n^{1/3})$, i.e.,

$$\max_{i \in [n]} w_i = o(n^{1/3}). \tag{1.7}$$

(b) Weak convergence of weight distribution. We assume that the weight of a uniform vertex converges in distribution to some distribution function F, i.e., let $V_n \in [n]$ be uniform, then we assume that

$$w_{V_n} \xrightarrow{d} W, \tag{1.8}$$

for some limiting random variable W with distribution function F. Condition (1.8) is equivalent to the statement that, for every x which is a continuity point of $x \mapsto F(x)$, we have

$$\frac{1}{n} \#\{i : w_i \leq x\} \to F(x). \tag{1.9}$$

(c) Convergence of first three moments. We assume that

$$\frac{1}{n} \sum_{i=1}^{n} w_i = \mathbb{E}[W] + o(n^{-1/3}), \tag{1.10}$$

$$\frac{1}{n} \sum_{i=1}^{n} w_i^2 = \mathbb{E}[W^2] + o(n^{-1/3}), \tag{1.11}$$

$$\frac{1}{n} \sum_{i=1}^{n} w_i^3 = \mathbb{E}[W^3] + o(1). \tag{1.12}$$

When $\boldsymbol{w}$ is *random*, for example in the case where $(w_i)_{i=1}^n$ are i.i.d. random variables with finite third moment, then we need the estimates in conditions (a), (b) and (c) to hold *in probability*.

We shall simply refer to the above three conditions as conditions (a), (b) and (c). Note that (1.10) and (1.11) in condition (c) also imply that

$$\nu_n = \frac{\sum_{i=1}^n w_i^2}{\sum_{i=1}^n w_i} = \frac{\mathbb{E}[W^2]}{\mathbb{E}[W]} + o(n^{-1/3}) = \nu + o(n^{-1/3}). \tag{1.13}$$

Before we write our main result we shall need one more construct. Now for fixed $t \in \mathbb{R}$ consider the inhomogeneous Brownian motion

$$W^t(s) = \sqrt{\frac{\sigma_3}{\mu}} B(s) + st - \frac{s^2\sigma_3}{2\mu^2}, \tag{1.14}$$

where B is standard Brownian motion. We want to consider this process restricted to be non-negative, i.e., let

$$\bar{W}^t(s) = W^t(s) - \min_{0\leq s'\leq s} W^t(s'). \tag{1.15}$$

It is easy to see (e.g. [1]) that the excursions of $\bar{W}^t$ from 0 can be ranked in increasing order as, say $\gamma_1(t) > \gamma_2(t) > \ldots$.

Now let $\mathcal{C}_n^1(t) \geq \mathcal{C}_n^2(t) \geq \mathcal{C}_n^3(t) \ldots$ denote the sizes of the components in $\mathcal{G}_n^t(\boldsymbol{w})$ arranged in increasing order. To formulate the convergence result define l^2 to be the set of infinite sequences $x = (x_i)_{i=1}^\infty$ with $x_1 \geq x_2 \geq \ldots \geq 0$ and $\sum_{i=1}^\infty x_i^2 < \infty$, and define the l^2 metric by

$$d(x, y) = \sqrt{\sum_{i=1}^\infty (x_i - y_i)^2}. \tag{1.16}$$

Then, our main result is as follows:

Theorem 1.1 (The critical behavior)**.** *Assume that the weight sequence $\boldsymbol{w} = (w_1, \ldots, w_n)$ satisfies conditions* (a)*,* (b) *and* (c)*. Then, as $n \to \infty$,*

$$\left(n^{-2/3}\mathcal{C}_n^i(t)\right)_{i\geq 1} \xrightarrow{d} \left(\gamma_i(t)\right)_{i\geq 1}, \tag{1.17}$$

in distribution and with respect to the l^2 topology.

We next investigate the two key examples, and show that conditions (a), (b) and (c) indeed hold in this case:

Corollary 1.2 (Theorem 1.1 holds for key examples)**.** *Conditions* (a)*,* (b) *and* (c) *are satisfied in the case where $\boldsymbol{w} = (w_1, \ldots, w_n)$ is either as in* (1.3) *where F is a distribution function of a random variable W with $\mathbb{E}[W^3] < \infty$, or when $\boldsymbol{w} = (w_1, \ldots, w_n)$ consists of i.i.d. copies of a random variable W with $\mathbb{E}[W^3] < \infty$.*

Theorem 1.1, in conjunction with Corollary 1.2, proves [12, Conjecture 1.6], where the result in Theorem 1.1 is conjectured in the case where $\boldsymbol{w} = (w_1, \ldots, w_n)$ is as in (1.3) where F is a distribution function of a random variable W with $\mathbb{E}[W^{3+\varepsilon}] < \infty$ for some ε. The current result implies that $\mathbb{E}[W^3]$ is a sufficient condition for this result to hold, and we believe this condition also to be necessary (as the constant $\mathbb{E}[W^3]$ also appears in our results, see (1.14)). Note, however, that in [12, Conjecture 1.6], the constant in front of $-s^2/2$ in (1.14) is erroneously taken as 1, while it should be $\mathbb{E}[W^3]/\mathbb{E}[W]^2$.

We next provide a heuristic that explains the limiting process in (1.14). Note that by our assumptions on the weight sequence, for the graph $\mathcal{G}_n^t(\boldsymbol{w})$

$$p_{ij} = \left(1 + o\left(\frac{1}{n^{1/3}}\right)\right) p_{ij}^* \tag{1.18}$$

where

$$p^*_{ij} = \left(1 + \frac{t}{n^{1/3}}\right) \frac{w_i w_j}{l_n}$$

For the rest of the proof wherever we need p_{ij} we shall use p^*_{ij} and this shall not effect the calculations and simplify the exposition.

We explore the cluster one vertex at a time, in breadth-first search. We choose $v(1)$ according to $\boldsymbol{w}$, i.e., $\mathbb{P}(v(1) = j) = w_j/l_n$. We say that a vertex is *explored* when its neighbors have been investigated, and *unexplored* when it has been found to be part of the cluster found so far, but its neighbors have not been investigated yet. Finally, we say that a vertex is *neutral*, when it has not been considered at all. Then, as long as there are unexplored vertices, we explore the vertices $(v(i))_{i\geq 1}$ in the order of appearance. When there are no unexplored vertices left, then we let $v(i)$ be equal to j, where j is a neutral vertex, with probability proportional to w_j. The size-biased order $v^*(1), v^*(2), \ldots, v^*(n)$ is a random reordering of the above vertex set where $v(1) = i$ with probability equal to w_i/l_n. Then, given $v^*(1)$, we have that $v^*(2) = j \in [n] \setminus \{v^*(1)\}$ with probability proportional to w_j and so on. Then, an important ingredient in our proof is that $(v(i))_{i=1}^n$ is a *size-biased reordering of* $[n]$ (see Lemma 2.1 below).

Let $c(i)$ denote the number of neutral neighbors of $v(i)$, and denote the process $(Z_n(l))_{l\in[n]}$ by $Z_n(l) = 0$ and $Z_n(l) = Z_n(l-1) + c(l) - 1$. The clusters of our random graph are found in between successive times in which $(Z_n(l))_{l\in[n]}$ reaches a new minimum (see Section 2 for more details). Now, Theorem 1.1 follows from the fact that $\bar{Z}_n(s) = \frac{1}{n^{1/3}} Z_n(\lfloor n^{2/3}s \rfloor)$ weakly converges to $(W^t(s))_{s\geq 0}$ defined in (1.14). General techniques from [1] show that this also implies that the ordered excursions between successive minima of $(\bar{Z}_n(s))_{s\geq 0}$ also converge to the ones of $(W^t(s))_{s\geq 0}$, which then proves Theorem 1.1. We complete the sketch of proof by giving a heuristic argument that indeed $\bar{Z}_n(s) = \frac{1}{n^{1/3}} Z_n(\lfloor n^{2/3}s \rfloor)$ weakly converges to $(W^t(s))_{s\geq 0}$. For this, we investigate $c(i)$, the number of neutral neighbors of $v(i)$. Throughout this paper, we shall denote $\tilde{w}_j = w_j(1 + tn^{-1/3})$, so that the $\mathcal{G}^t_n(\boldsymbol{w})$ has weights $\tilde{\boldsymbol{w}} = (\tilde{w}_j)_{j\in[n]}$.

We note that since p_{ij} in (1.1) is quite small, the number of neighbors of a vertex j is close to $\mathrm{Poi}(\tilde{w}_j)$, where $\mathrm{Poi}(\lambda)$ denotes a Poisson random variable with mean λ. Thus, the number of neutral neighbors is close to the total number of neighbors minus the active neighbors, i.e.,

$$c(i) \approx \mathrm{Poi}(\tilde{w}_{v(i)}) - \mathrm{Poi}\Big(\sum_{j=1}^{i} \frac{\tilde{w}_{v(i)}\tilde{w}_{v(j)}}{l_n}\Big), \tag{1.19}$$

since $\sum_{j=1}^{i} \frac{\tilde{w}_{v(i)}\tilde{w}_{v(j)}}{l_n}$ is, conditionally on $(v(j))_{j=1}^{i}$, the expected number of edges between $v(i)$ and $(v(j)_{j=1}^{i-1}$.

We conclude that the increase of the process $Z_n(l)$ equals

$$c(i) - 1 \approx \mathrm{Poi}(\tilde{w}_{v(i)}) - 1 - \mathrm{Poi}\Big(\sum_{j=1}^{i} \frac{\tilde{w}_{v(i)}\tilde{w}_{v(j)}}{l_n}\Big), \tag{1.20}$$

so that

$$Z_n(l) \approx \sum_{i=1}^{l} (\mathrm{Poi}(\tilde{w}_{v(i)}) - 1) - \mathrm{Poi}\Big(\sum_{j=1}^{i} \frac{\tilde{w}_{v(i)}\tilde{w}_{v(j)}}{l_n}\Big). \tag{1.21}$$

The change in $Z_n(l)$ is not stationary, and decreases on the average as l increases, due to two reasons. First of all, the number of neutral vertices decreases (as is apparent from the sum which is subtracted in (1.21)), and the law of $\tilde{w}_{v(l)}$ becomes stochastically smaller as l increases. the latter can be understood by noting that $\mathbb{E}[\tilde{w}_{v(1)}] = (1 + tn^{-1/3})\nu_n = 1 + tn^{-1/3} + o(n^{-1/3})$, while $\frac{1}{n}\sum_{j\in[n]} \tilde{w}_{v(j)} = (1 + tn^{-1/3}) l_n/n$, and, by Cauchy-Schwarz,

$$l_n/n \approx \mathbb{E}[W] \leq \mathbb{E}[W^2]^{1/2} = \mathbb{E}[W]^{1/2}\nu^{1/2} = \mathbb{E}[W]^{1/2}, \tag{1.22}$$

so that $l_n/n \leq 1+o(1)$, and the inequality becomes strict when $\mathrm{Var}(W) > 0$. We now study these two effects in more detail.

The random variable $\mathrm{Poi}(\tilde{w}_{v(i)}) - 1$ has asymptotic mean

$$\mathbb{E}[\mathrm{Poi}(\tilde{w}_{v(i)}) - 1] \sim \sum_{j\in[n]} \tilde{w}_j \mathbb{P}(v(i)=j) - 1 = \sum_{j\in[n]} \tilde{w}_j \frac{w_j}{l_n} - 1 = \nu_n(1+tn^{-1/3}) - 1 \approx 0. \tag{1.23}$$

However, since we sum $\Theta(n^{2/3})$ contributions, and we multiply by $n^{-1/3}$, we need to be rather precise and compute error terms up to order $n^{-1/3}$ in the above computation. We shall do this rather precisely now, by conditioning on $(v(j))_{j=1}^{i-1}$. Indeed,

$$\begin{aligned}
\mathbb{E}[\tilde{w}_{v(i)} - 1] &\sim \nu_n t n^{-1/3} + \mathbb{E}\big[\mathbb{E}[w_{v(i)} - 1 \mid (v(j))_{j=1}^{i-1}]\big] \\
&\sim tn^{-1/3} + \mathbb{E}\big[\sum_{l=1}^{n} w_l \mathbb{1}_{\{l\notin\{v(j)\}_{j=1}^{i-1}\}} \frac{w_l}{l_n - \sum_{j=1}^{i-1} w_{v(j)}}\big] - 1 \\
&\sim tn^{-1/3} + \sum_{j\in[n]} \frac{w_j^2}{l_n} + \mathbb{E}\big[\frac{1}{l_n^2}\sum_{j=1}^{i-1} w_{v(j)} \sum_{l=1}^{n} w_l\big] - \mathbb{E}\big[\frac{1}{l_n}\sum_{j=1}^{i-1} w_{v(j)}^2\big] - 1 \\
&\sim tn^{-1/3} + i\big(\frac{\nu_n}{l_n} - \frac{1}{l_n}\sum_{j=1}^{n} w_j^3\big) \sim tn^{-1/3} + \frac{i}{l_n}\big(1 - \frac{1}{l_n}\sum_{j=1}^{n} w_j^3\big).
\end{aligned} \tag{1.24}$$

When $i = \Theta(n^{2/3})$, these terms are indeed both of order $n^{-1/3}$, and shall thus contribute to the scaling limit of $(Z_n(l))_{l\geq 0}$.

The variance of $\mathrm{Poi}(\tilde{w}_{v(i)})$ is approximately equal to

$$\begin{aligned}
\mathrm{Var}(\mathrm{Poi}(\tilde{w}_{v(i)})) &= \mathbb{E}[\mathrm{Var}(\mathrm{Poi}(\tilde{w}_{v(i)})) \mid v(i)] + \mathrm{Var}(\mathbb{E}[\mathrm{Poi}(\tilde{w}_{v(i)}) \mid v(i)]) \\
&= \mathbb{E}[\tilde{w}_{v(i)}] + \mathrm{Var}(w_{v(i)}) \sim \mathbb{E}[\tilde{w}_{v(i)}^2] \sim \mathbb{E}[w_{v(i)}^2],
\end{aligned} \tag{1.25}$$

since $\mathbb{E}[w_{v(i)}] = 1+\Theta(n^{-1/3})$. Summing the above over $i = 1, \ldots, sn^{2/3}$ and multiplying through by $n^{-1/3}$ intuitively explains that

$$n^{-1/3}\sum_{i=1}^{sn^{2/3}} (\mathrm{Poi}(\tilde{w}_{v(i)}) - 1) \xrightarrow{d} \sigma B(s) + st + \frac{s^2}{2\mathbb{E}[W]}(1-\sigma), \tag{1.26}$$

where we write $\sigma = \mathbb{E}[W^3]/\mathbb{E}[W]$ and we let $(B(s))_{s\geq 0}$ denote a standard Brownian motion. Note that when $\mathrm{Var}(W) > 0$, then $\mathbb{E}[W] < 1, \mathbb{E}[W^2] > 1$, so that also $\mathbb{E}[W^3]/\mathbb{E}[W] > 1$ and the constant in front of s^2 is *negative*. We shall make the limit in (1.26) precise by using a martingale functional central limit theorem.

The second term in (1.21) turns out to be well-concentrated around its mean, so that, in this heuristic, we shall replace it by its mean. The concentration shall be proved using concentration techniques on appropriate supermartingales. This leads us to compute

$$\begin{aligned}
\mathbb{E}\Big[\sum_{i=1}^{l} \mathrm{Poi}\Big(\sum_{j=1}^{i} \frac{\tilde{w}_{v(i)}\tilde{w}_{v(j)}}{l_n}\Big)\Big] &\sim \mathbb{E}\Big[\sum_{i=1}^{l}\sum_{j=1}^{i} \frac{\tilde{w}_{v(i)}\tilde{w}_{v(j)}}{l_n}\Big] \sim \mathbb{E}\Big[\sum_{i=1}^{l}\sum_{j=1}^{i} \frac{w_{v(i)}w_{v(j)}}{l_n}\Big] \\
&\sim \frac{1}{2}\mathbb{E}\Big[\frac{1}{l_n}\Big(\sum_{j=1}^{i} w_{v(j)}\Big)^2\Big] \sim \frac{1}{2l_n}\mathbb{E}\Big[\sum_{j=1}^{i} w_{v(j)}\Big]^2,
\end{aligned} \tag{1.27}$$

the last asymptotic equality again following from the fact that the random variable involved is concentrated.

We conclude that

$$n^{-1/3}\mathbb{E}\Big[\sum_{i=1}^{sn^{2/3}} \mathrm{Poi}\Big(\sum_{j=1}^{i} \frac{\tilde{w}_{v(i)}\tilde{w}_{v(j)}}{l_n}\Big)\Big] \sim \frac{s^2}{2\mathbb{E}[W]}. \tag{1.28}$$

Subtracting (1.28) from (1.26), these computations suggest, informally, that

$$\bar{Z}_n(s) = \frac{1}{n^{1/3}} Z_n(\lfloor n^{2/3}s\rfloor) \xrightarrow{d} \sigma B(s) + st - \frac{s^2\mathbb{E}[W^3]}{2\mathbb{E}[W]^2} = \sqrt{\mathbb{E}[W^3]^{1/2}}\mathbb{E}[W]^{1/2}B(s) + st - \frac{s^2\mathbb{E}[W^3]}{2\mathbb{E}[W]^2}, \tag{1.29}$$

as required. Note the cancelation of the terms $\frac{s^2}{2\mathbb{E}[W]}$ in (1.26) and (1.28), where they appear with an opposite sign. Our proof will make this analysis precise.

1.2 Discussion

Our results are generalizations of the critical behavior of Erdős-Rényi random graphs, which have received tremendous attention over the past decades. We refer to [1], [2], [15] and the references therein. Properties of the limiting distribution of the largest component $\gamma_1(t)$ can be found in [18], which, together with the recent local limit theorems in [13], give excellent control over the joint tail behavior of several of the largest connected components.

Comparison to results of Aldous. Aldous [1] identifies the scaling limit of the largest connected components in the Erdős-Rényi random graph. Let us denote the limiting variables in the Erdős-Rényi random graph with parameter $p = (1 + tn^{-1/3})/n$ to be $\big(\gamma_i'(t)\big)_{i\geq 1}$ that correspond the the excursions of the reflected process in (1.15) with $\mu = \sigma_3 = 1$.

We claim that, by Brownian scaling, the limit $\big(\gamma_i(t)\big)_{i\geq 1}$ in Theorem 1.1 is equal to

$$\big((\mathbb{E}[W]/\mathbb{E}[W^3]^{1/3}\gamma_i'(t\mathbb{E}[W]/\mathbb{E}[W^3]^{2/3})\big)_{i\geq 1}. \tag{1.30}$$

Indeed, fix $a > 0$ and note that $\{B(a^2s)\}_{s=0}^{\infty}$ has the same distribution as $\{aB(s)\}_{s=0}^{\infty}$. Thus, we obtain, with $\{W_{\sigma,\kappa}^t(s)\}_{s\geq 0}$ defined by

$$W_{\sigma,\kappa}^t(s) = \sigma B(s) + st - \kappa s^2/2, \tag{1.31}$$

the scaling relation

$$\{W_{\sigma,\kappa}^t(s)\}_{s\geq 0} \stackrel{d}{=} \{\sigma/aW_{1,a^{-3}\kappa/\sigma}^{t/(a\sigma)}(a^2 s)\}_{s\geq 0}. \tag{1.32}$$

Using this for $\kappa = \sigma^2/\mu$ and $a = (\kappa/\sigma)^{1/3} = (\sigma/\mu)^{1/3}$, we note that

$$\{W_{\sigma,\sigma^2/\mu}^t(s)\}_{s\geq 0} \stackrel{d}{=} \{\sigma^{2/3}\mu^{1/3}W_{1,1}^{t\sigma^{-4/3}\mu^{1/3}}(\sigma^{2/3}\mu^{-2/3}s)\}_{s\geq 0}, \tag{1.33}$$

which implies that

$$\big(\gamma_i(t)\big)_{i\geq 1} \stackrel{d}{=} \big(\sigma^{-2/3}\mu^{2/3}\gamma_i'(t\sigma^{-4/3}\mu^{1/3})\big)_{i\geq 1} = \big((\mathbb{E}[W]/\mathbb{E}[W^3]^{1/3}\gamma_i'(t\mathbb{E}[W]/\mathbb{E}[W^3]^{2/3})\big)_{i\geq 1}. \tag{1.34}$$

Theorem 1.1 is related to another results of Aldous [1, Proposition 4], which is less well known, and which investigates a kind of Norros-Reittu model (see [17]) for which the ordered *weights* of the clusters are determined. Here, the weight of a set of vertices $A \subseteq [n]$ is defined by $\bar{w}_A = \sum_{a\in A} w_a$. Indeed, Aldous defines an inhomogeneous random graph where the edge probability is equal to

$$p_{ij} = 1 - \mathrm{e}^{-qx_ix_j}, \tag{1.35}$$

and assumes that the pair $(q, (x_i)_{i=1}^n)$ satisfies the following scaling relation:

$$\frac{\sum_{i=1}^n x_i^3}{\Big(\sum_{i=1}^n x_i^2\Big)^3} \to 1, \qquad q - \Big(\sum_{i=1}^n x_i^2\Big)^{-1} \to t, \qquad \max_{j\in[n]} x_j = o(\sum_{i=1}^n x_i^2). \tag{1.36}$$

When we pick

$$x_j = w_j \frac{\Big(\sum_{i=1}^n w_i^3\Big)^{1/3}}{\sum_{i=1}^n w_i^2}, \qquad q = \frac{\Big(\sum_{i=1}^n w_i^2\Big)^2}{\Big(\sum_{i=1}^n w_i^3\Big)^{2/3} l_n}(1+tn^{-1/3}), \tag{1.37}$$

then these assumptions are close to equivalent to conditions (a)-(c). However, the asymptotics of q in (1.36) is replaced with

$$q - \Big(\sum_{i=1}^n x_i^2\Big)^{-1} = \frac{\frac{1}{n}\sum_{i=1}^n w_i^2}{\Big(\frac{1}{n}\sum_{i=1}^n w_i^3\Big)^{2/3}}(n^{1/3}\nu_n(1+tn^{-1/3}) - n^{1/3}) \to \frac{\mathbb{E}[W^2]}{\mathbb{E}[W^3]^{2/3}}t = \frac{\mathbb{E}[W]}{\mathbb{E}[W^3]^{2/3}}t, \tag{1.38}$$

where the last equality follows from the fact that $\nu = \mathbb{E}[W^2]/\mathbb{E}[W] = 1$. This scaling in t simply means that the parameter t in the process W^t in (1.14) is rescaled, which is explained in more detail in the scaling relations in (1.34). Writing $\mathcal{C}_n^i(t)$ to be the component with the i^{th} largest **weight**, Aldous [1] proves that

$$\Big(\frac{\Big(\sum_{i=1}^n w_i^3\Big)^{1/3}}{\sum_{i=1}^n w_i^2}\bar{w}_{\mathcal{C}_n^i(t)}\Big)_{i\geq 1} \xrightarrow{d} \big(\gamma_i'(t\mathbb{E}[W]/\mathbb{E}[W^3]^{2/3})\big)_{i\geq 1}, \tag{1.39}$$

where we recall that $\big(\gamma_i'(t)\big)_{1\geq 1}$ is the scaling limit of the ordered component sizes in the Erdős-Rényi random graph with parameter $p = (1+tn^{-1/3})/n$. Now,

$$\frac{\Big(\sum_{i=1}^n w_i^3\Big)^{1/3}}{\sum_{i=1}^n w_i^2} \sim n^{-2/3}\mathbb{E}[W^3]^{1/3}/\mathbb{E}[W^2] = n^{2/3}\mathbb{E}[W^3]^{1/3}/\mathbb{E}[W], \tag{1.40}$$

and one would expect that $\bar{w}_{\mathcal{C}_n^i(t)} \sim \mathcal{C}_n^i(t)$, which is consistent with (1.29) and (1.34).

Related models. The model studied here is asymptotically equivalent with many related models appearing in the literature,for example to the *random graph with prescribed expected degrees* that has been studied intensively by Chung and Lu (see [5, 6, 7, 8, 9]). This model corresponds to the rank-1 case of the general inhomogeneous random graphs studied in [3]. Here

$$p_{ij} = \frac{w_i w_j}{l_n}, \tag{1.41}$$

and the *generalized random graph* [4], for which

$$p_{ij} = \frac{w_i w_j}{l_n + w_i w_j}, \tag{1.42}$$

See [12, Section 2], which in turn is based on [14], for more details on the asymptotic equivalence of such inhomogeneous random graphs. Further, Nachmias and Peres [16] recently proved similar scaling limits for critical percolation on random regular graphs.

Recent result by Turova. Turova [19] studies a problem that is intimately related with this paper. In Turova's setting, the edge probabilities are given by $p_{ij} = \min\{x_i x_j/n, 1\}$, and it is assumed that $(x_i)_{i=1}^n$ are i.i.d. random variables. This setting follows from our setting when we take

$$w_i = x_i\Big(\frac{1}{n}\sum_{j=1}^n x_j\Big)^{1/2}. \tag{1.43}$$

Naturally, the critical point changes in this setting, and is now equal to $\mathbb{E}[X^2] = 1$. Turova proves a similar result as ours, apart from the fact that the drift term does not contain a factor $\mathbb{E}[X^3]$. A possible explanation could be that the term $\frac{s^2}{2\mathbb{E}[W]}(1 - \frac{\mathbb{E}[W^3]}{\mathbb{E}[W]})$ appearing in (1.26) has been left out of the analysis.

Comparing the two results, we see that they are quite similar, while the proofs are rather different. We lean more heavily on martingale techniques, while the proof in [19] is more analytical. Further, our result is slightly more general that the one in [19]. We have been inspired to add the i.i.d. weights to our analysis by [19]. We aim to combine the two approaches into a single four-authored paper.

2 Cluster exploration

Let us now delve into the proof of Theorem 1.1. The proof involves two key ingredients:

- The exploration of components via breath-first search; and
- The labeling of vertices in a size-biased order of their weights $\boldsymbol{w}$.

More precisely, we shall explore components and simultaneously construct the graph $\mathcal{G}_n^t(\boldsymbol{w})$ in the following manner: First, for all ordered pairs of vertices (i,j), let $V(i,j)$ be exponential random variables with rate $\left(1+tn^{-1/3}\right) w_j/l_n$ random variables. Choose vertex $v(1)$ with probability proportional to $\boldsymbol{w}$, so that

$$\mathbb{P}(v(1)=i) = w_i/l_n. \tag{2.1}$$

The children of $v(1)$ are all the vertices j such that

$$V(v(1),j) \le w_{v(1)}. \tag{2.2}$$

Suppose $v(1)$ has $c(1)$ children. Label these as $v(2), v(3), \dots v(c(1)+1)$ in increasing order of their $V(v(1),\cdot)$ values. Now move on to $v(2)$ and explore all of its children (say $c(2)$ of them) and label them as before. Note that when we explore the children of $v(2)$, its potential children cannot include the vertices that we have already identified. More precisely, the children of $v(2)$ consists of the set

$$\{v \notin \{v(1), \dots v(c(1)+1)\} : V(v(2),v) \le w_{v(2)}\}$$

and so on. Once we finish exploring one component, we move onto the next component by choosing the starting vertex in a size-biased manner amongst remaining vertices and start exploring its component. It is obvious that this constructs all the components of our graph $\mathcal{G}_n^t(\boldsymbol{w})$.

Write the breadth-first walk associated to this exploration process as

$$Z_n(0) = 0, \quad Z_n(i) = Z_n(i-1) + c(i) - 1, \tag{2.3}$$

for $i = 1, \dots, n$. Suppose $\mathcal{C}^*(i)$ is the size of the i^{th} component explored in this manner (here we write $\mathcal{C}^*(i)$ to distinguish this from $\mathcal{C}_n^i(t)$, the i^{th} *largest* component). Then these can be easily recovered from the above walk by the following prescription: For $j \ge 0$, write $\eta(j)$ as the stopping time

$$\eta(j) = \min\{i : Z_n(i) = -j\}. \tag{2.4}$$

Then

$$\mathcal{C}^*(j) = \eta(j) - \eta(j-1). \tag{2.5}$$

Further,

$$Z_n(\eta(j)) = -j, \quad Z_n(i) \ge -j \text{ for all } \eta(j) < i < \eta(j+1). \tag{2.6}$$

Recall that we started with vertices labeled $1, 2, \dots, n$ with corresponding weights $\boldsymbol{w} = (w_1, w_2, \dots w_n)$. The size-biased order $v^*(1), v^*(2), \dots, v^*(n)$ is a random reordering of the above vertex set where $v(1) = i$ with probability equal to w_i/l_n. Then, given $v^*(1)$, we have that $v^*(2) = j \in [n] \setminus \{v^*(1)\}$ with probability proportional to w_j and so on. By construction and the properties of the exponential random variables, we have the following representation:

Lemma 2.1 (Size-biased reordering of vertices)**.** *The order* $v(1), v(2), \ldots, v(n)$ *in the above construction of the breadth-first exploration process is the size-biased ordering* $v^*(1), v^*(2), \ldots, v^*(n)$ *of the vertex set* $[n]$ *with weights proportional to* $\boldsymbol{w}$*.*

Proof. The first vertex $v(1)$ is chosen from $[n]$ via the size-biased distribution. Suppose it has no neighbors. Then, by construction, the next vertex is chosen via the size-biased distribution amongst all remaining vertices. If vertex $v(1)$ does have neighbors, we shall use the following construction.

For $j \in [n] \setminus \{v(1)\}$, let $V(v(1), j)$ be exponentially distributed with rate $(1+tn^{-1/3})w_j/l_n$. Rearrange the vertices in increasing order of their $V(v(1), j)$ values (so that $v'(2)$ is the vertex with the smallest $V(v(1), j)$ value, $v'(3)$ is the vertex with the second smallest value and so on). Note that by the properties of the exponential distribution

$$\mathbb{P}(v(2) = i \mid v(1)) = \frac{w_i}{\sum_{j \neq v(1)} w_j} \quad \text{for } j \in [n] \setminus v(1). \tag{2.7}$$

Similarly, given the values of $v(1), v(2)$ we have,

$$\mathbb{P}(v(3) = i \mid v(1), v(2)) = \frac{w_i}{\sum_{j \neq v(1), v(2)} w_j}, \tag{2.8}$$

and so on. Thus the above gives us a size-biased ordering of the vertex sex $[n] \setminus \{v(1)\}$. Suppose $c(1)$ of the exponential random variables are less than w_1. Then set $v(j) = v'(j)$ for $2 \leq j \leq c(1)+1$ and discard all the other labels. This gives us the first $c(1)+1$ values of our size-biased ordering.

Once we are done with $v(1)$, let the potentially unexplored neighbors of $v(2)$ be

$$\mathcal{U}_2 = [n] \setminus \{v(1), \ldots v(c(1)+1)\}, \tag{2.9}$$

and, again, for j in $\mathcal{U}_2$, we let $V(v(2), j)$ be exponential with rate $(1+tn^{-1/3})w_j/l_n$ and proceed as above to obtain the actual neighbors of $v(2)$ and their ordering.

Proceeding this way, it is clear that at the end, the random ordering $v(1), v(2), \ldots, v(n)$ that we obtain is a size-biased random ordering of the vertex set $[n]$. This proves the Lemma. □

2.1 Weak convergence of cluster exploration

In this section, we shall study the scaling limit of the cluster exploration studied in Section 2 above. The main result in this paper is the following theorem:

Theorem 2.2 (Weak convergence of cluster exploration)**.** *Assume that the weight sequence* $\boldsymbol{w} = (w_1, \ldots, w_n)$ *satisfies conditions* (a), (b) *and* (c)*. Consider the breadth-first walk* $Z_n(\cdot)$ *of* (2.6) *exploring the components of the random graph* $\mathcal{G}_n^t(\boldsymbol{w})$*. Define*

$$\bar{Z}_n(s) = \frac{1}{n^{1/3}} Z_n(\lfloor n^{2/3} s \rfloor). \tag{2.10}$$

Then, as $n \to \infty$*,*

$$\bar{Z}_n \xrightarrow{d} W^t, \tag{2.11}$$

where W^t *is the process defined in* (1.14)*, in the sense of convergence in the* J_1 *Skorohod topology on the space of right-continuous left-limited functions on* $\mathbb{R}^+$*.*

Assume this theorem for the time being and let us show how this immediately proves Theorem 1.1. Comparing (1.15) and (2.6), Theorem 2.2 suggests that also the excursions of $\bar{Z}_n$ beyond past minima arranged in increasing order also converge to the corresponding excursions of W^t beyond past minima arranged in increasing order. See Aldous [1, Section 3.3] for a proof of this fact. Therefore, Theorem 2.2 implies Theorem 1.1. The remainder of this paper is devoted to the proof of Theorem 2.2.

Proof of Theorem 2.2. We shall make use of a martingale central limit theorem. Recall that from Equation (1.18) we had

$$p_{ij} \approx \left(1 + \frac{t}{n^{1/3}}\right) \frac{w_i w_j}{l_n}$$

We shall use the above as an equality for the rest of the proof as this shall simplify exposition. It is quite easy to show that the error made is negligible in the limit.

Recall from (2.3) that

$$Z_n(k) = \sum_{i=0}^{k} (c(i) - 1). \tag{2.12}$$

Then, we decompose

$$Z_n(k) = M_n(k) + A_n(k), \tag{2.13}$$

where

$$M_n(k) = \sum_{i=0}^{k} (c(i) - \mathbb{E}[c(i) \mid \mathcal{F}_{i-1}]), \qquad A_n(k) = \sum_{i=0}^{k} \mathbb{E}[c(i) - 1 \mid \mathcal{F}_{i-1}], \tag{2.14}$$

with $\mathcal{F}_i$ the natural filtration of Z_n. Then, clearly, $\{M_n(k)\}_{k=0}^n$ is a martingale. For a process $\{S_k\}_{k=0}^n$, we further write

$$\bar{S}_n(u) = n^{-1/3} S_n(\lfloor un^{2/3} \rfloor). \tag{2.15}$$

Furthermore, let

$$B_n(k) = \sum_{i=0}^{k} \mathbb{E}[c(i)^2 \mid \mathcal{F}_{i-1}] - \mathbb{E}[c(i) \mid \mathcal{F}_{i-1}]^2. \tag{2.16}$$

Then, by the Martingale Central limit theorem ([11, Theorem 7.1.4]), Theorem 2.2 follows when the following three conditions hold:

$$\sup_{s \le u} \left| \bar{A}_n(s) + \frac{s^2 \sigma_3}{2\mu^2} - st \right| \xrightarrow{\mathbb{P}} 0, \tag{2.17}$$

$$n^{-2/3} B_n(n^{2/3} u) \xrightarrow{\mathbb{P}} \frac{\sigma_3 u}{\mu}, \tag{2.18}$$

$$\mathbb{E}(\sup_{s \le u} |\bar{M}_n(s) - \bar{M}_n(s-)|^2) \to 0. \tag{2.19}$$

Indeed, the last two equations, by [11, Theorem 7.1.4] imply that the process $\bar{M}_n(s) = n^{-1/3} M_n(n^{2/3} s)$ satisfies the asymptotics

$$\bar{M}_n \xrightarrow{d} \sqrt{\frac{\sigma_3}{\mu}} B, \tag{2.20}$$

where as before B is standard Brownian motion, while (2.17) gives the drift term in (1.14) and this completes the proof.

We shall now start to verify the conditions (2.17), (2.18) and (2.19). Throughout the proof, we shall assume, without loss of generality, that $w_1 \ge w_2 \ge w_3 \ge \ldots \ge w_n$. Recall that we shall work with weight sequence $\tilde{\boldsymbol{w}} = (1 + tn^{-1/3})\boldsymbol{w}$, for which the edge probabilities are equal to $w_i w_j (1 + tn^{-1/3})/l_n$.

We note that, since $M_n(k)$ is a discrete martingale,

$$\begin{aligned} \sup_{s \le u} |\bar{M}_n(s) - \bar{M}_n(s-)|^2 &= n^{-2/3} \sup_{k \le un^{2/3}} (M_n(k) - M_n(k-1))^2 \le n^{-2/3} (1 + \sup_{k \le un^{2/3}} c(k)^2) \\ &\le n^{-2/3} (1 + \Delta_n^2), \end{aligned} \tag{2.21}$$

where Δ_n is the maximal degree in the graph. It is not hard to see that, by condition (a), $\tilde{w}_i = o(n^{1/3})$, so that

$$\mathbb{E}(\sup_{s \le u} |\bar{M}_n(s) - \bar{M}_n(s-)|^2) \le n^{-2/3} (1 + \mathbb{E}[\Delta_n^2]) = o(n^{-2/3} n^{2/3}) = o(1). \tag{2.22}$$

This proves (2.19).

We continue with (2.17) and (2.18), for which we first analyse $c(i)$. In the course of the proof, we shall make use of the following lemma, which lies at the core of the argument:

Lemma 2.3 (Sums over sized-biased orderings)**.** *As $n \to \infty$, for all $u > 0$,*

$$\sup_{u\leq t} \big|n^{-2/3} \sum_{i=1}^{n^{2/3}u} w_{v(i)}^2 - \frac{\sigma_3 u}{\mu}\big| \xrightarrow{\mathbb{P}} 0, \tag{2.23}$$

$$n^{-2/3} \sum_{i=1}^{n^{2/3}u} \mathbb{E}[w_{v(i)}^2 \mid \mathcal{F}_{i-1}] \xrightarrow{\mathbb{P}} \frac{\sigma_3 u}{\mu}. \tag{2.24}$$

Proof. We start by proving (2.23), for which we write

$$H_n(u) = n^{-2/3} \sum_{i=1}^{\lfloor un^{2/3} \rfloor} w_{v(i)}^2. \tag{2.25}$$

We shall use a randomization trick introduced by Aldous [1]. Indeed, let T_j be a sequence of independent exponential random variables with rate w_j/l_n and define

$$\tilde{H}_n(v) = n^{-2/3} \sum_{i=1}^{n} w_j^2 \mathbb{1}\{T_j \leq n^{2/3} v\}. \tag{2.26}$$

Note that by the properties of the exponential random variables, if we rank the vertices according to the order in which they arrive then they appear in size-biased order. More precisely, for any v,

$$\sum_{j=1}^{n} w_j^2 \mathbb{1}\{T_j \leq n^{2/3} v\} = \sum_{i=1}^{N(vn^{2/3})} w_{v(i)}^2 = H_n(N(vn^{2/3})), \tag{2.27}$$

where

$$N(t) := \#\{j : T_j \leq t\}. \tag{2.28}$$

As a result, when $N(2tn^{2/3}) \geq tn^{2/3}$ **whp**, we have that

$$\begin{aligned} \sup_{u\leq t} \big|n^{-2/3} \sum_{i=1}^{n^{2/3}u} w_{v(i)}^2 - \sigma_3 u\big| &\leq \sup_{u\leq 2t} \big|n^{-2/3} \sum_{i=1}^{N(un^{2/3})} w_{v(i)}^2 - \sigma_3 n^{-2/3} N(un^{2/3})\big| \\ &\leq \sup_{u\leq 2t} \big|n^{-2/3} \tilde{H}_n(u) - \sigma_3 u\big| + \sigma_3 \sup_{u\leq 2t} \big|n^{-2/3} N(un^{2/3}) - u\big|. \end{aligned} \tag{2.29}$$

We shall prove that both terms converge to zero in probability. We start with the second, for which we use that the process

$$Y_0(s) = \frac{1}{n^{1/3}} \left(N(sn^{2/3}) - sn^{2/3} \right), \tag{2.30}$$

is a supermartingale, since

$$\begin{aligned} \mathbb{E}[N(t+s) \mid \mathcal{F}_t] &= N(t) + \mathbb{E}[N(t+s) - N(t) \mid \mathcal{F}_t] \leq \mathbb{E}[\#\{j : T_j \in (t, t+s]\} \mid \mathcal{F}_t] \\ &\leq \sum_{j=1}^{n} (1 - \mathrm{e}^{-w_j s/l_n}) \leq \sum_{j=1}^{n} \frac{w_j s}{l_n} = s, \end{aligned} \tag{2.31}$$

as required. Therefore,

$$|\mathbb{E}[Y_0(t)]| = -\mathbb{E}[Y_0(t)] = \frac{1}{n^{1/3}}\left[tn^{2/3} - \sum_{i=1}^{n}(1-\exp(-tn^{2/3}w_i/l_n))\right]. \tag{2.32}$$

Using the fact that

$$1 - e^{-x} \leq x - x^2/2,$$

we obtain that, also using that $\nu_n = 1 + o(1)$,

$$|\mathbb{E}[Y_0(t)]| \leq \sum_{i=1}^{n}\frac{w_i^2 t^2}{2l_n^2} = \frac{n\nu_n}{l_n}\frac{t^2}{2} = \frac{t^2}{2\mu} + o(1). \tag{2.33}$$

Similarly, by the independence of $\{T_j\}_{j\in[n]}$,

$$\begin{aligned}\mathrm{Var}(Y_0(t)) = n^{-2/3}\mathrm{Var}(N(sn^{2/3})) &= n^{-2/3}\sum_{j=1}^{n}\mathbb{P}(T_j \leq tn^{2/3})(1-\mathbb{P}(T_j \leq tn^{2/3}))\\ &\leq n^{-2/3}\sum_{j=1}^{n}\frac{w_j tn^{2/3}}{l_n} = t.\end{aligned} \tag{2.34}$$

Now we use the supermartingale inequality (Aldous [1, Page 831, proof of Lemma 12]), stating that, for any supermartingale $Y = (Y(s))_{s\geq 0}$,

$$\varepsilon\mathbb{P}(\sup_{s\leq t}|Y(s)| > 3\varepsilon) \leq 3\mathbb{E}(|Y(t)|) \leq 3\left(|\mathbb{E}(Y(t))| + \sqrt{\mathrm{Var}(Y(t))}\right). \tag{2.35}$$

Equation (2.35) shows that, for any large A,

$$\mathbb{P}(\sup_{s\leq t}|N(sn^{2/3}) - sn^{2/3}| > 3An^{1/3}) \leq \frac{3(t^2/2\mu + t)}{A}. \tag{2.36}$$

This clearly proves that, for every $t > 0$,

$$\sup_{u\leq 2t}\left|n^{-2/3}N(un^{2/3}) - u\right| \xrightarrow{\mathbb{P}} 0. \tag{2.37}$$

Observe that (2.37) also immediately proves that, **whp**, $N(2tn^{2/3}) \geq tn^{2/3}$.

To deal with $\tilde{H}_n(v)$, we define

$$Y_1(u) = \tilde{H}_n(u) - \mu_3(n)u, \tag{2.38}$$

where

$$\mu_3(n) = \sum_{j=1}^{n}\frac{w_j^3}{l_n} = \frac{\sigma_3}{\mu} + o(1), \tag{2.39}$$

and note that $Y_1(u)$ is a supermartingale. Indeed, writing $\mathcal{F}_t$ to be the natural filtration of the above process, we have, for $s < t$ and letting $V_s = \left\{v : T_v < sn^{2/3}\right\}$

$$\mathbb{E}(Y_1(t)|\mathcal{F}_s) = Y_1(s) + \frac{1}{n^{2/3}}\sum_{j\notin V_s}w_j^2\left(1-\exp\left(-\frac{n^{2/3}(t-s)w_j}{l_n}\right)\right) - \mu_3(n)(t-s). \tag{2.40}$$

Now using the inequality $1 - \mathrm{e}^{-x} \leq x$ for $x \in [0,1]$ we get that

$$\mathbb{E}(Y_1(t)|\mathcal{F}_s) \leq Y_1(s), \tag{2.41}$$

as required. Again we can easily compute, using condition (a), that

$$\begin{aligned}|\mathbb{E}[Y_1(t)]| &= -\mathbb{E}[Y_1(t)] = \mu_3(n)t - n^{-2/3}\sum_{i=1}^{n} w_i^2(1-\exp(-tn^{2/3}w_i/l_n)) \\ &= n^{-2/3}\sum_{i=1}^{n} w_i^2\big(\exp(-tn^{2/3}w_i/l_n) - 1 + tw_i\big) \\ &\le n^{-2/3}\sum_{i=1}^{n} w_i^2\frac{(tn^{2/3}w_i)^2}{2l_n^2} \le n^{2/3}\sum_{i=1}^{n}\frac{w_i^4}{2l_n^2} = o(n^{2/3}n^{1/3})\sum_{i=1}^{n}\frac{w_i^3}{l_n^2} = o(1). \qquad (2.42)\end{aligned}$$

By independence,

$$\begin{aligned}\mathrm{Var}(Y_1(t)) &= n^{-4/3}\sum_{j=1}^{n} w_i^4(1-\exp(-tn^{2/3}w_i/l_n))\exp(-tn^{2/3}w_i/l_n) \\ &\le n^{-2/3}t\sum_{j=1}^{n}\frac{w_i^5}{l_n} = o(1)\sum_{j=1}^{n}\frac{w_i^3}{l_n} = o(1). \qquad (2.43)\end{aligned}$$

Therefore, (2.35) completes the proof of (2.23).

The proof of (2.24) is a little easier. We denote

$$\mathcal{V}_i = \{v(j)\}_{j=1}^{i}. \qquad (2.44)$$

Then, we compute explicitly

$$\begin{aligned}\mathbb{E}[w_{v(i)}^2 \mid \mathcal{F}_{i-1}] &= \sum_{j\in[n]} w_j^2\mathbb{P}(v(i)=j \mid \mathcal{F}_{i-1}) \\ &= \frac{\sum_{j\notin\mathcal{V}_{i-1}} w_j^3}{\sum_{j\notin\mathcal{V}_{i-1}} w_j}. \qquad (2.45)\end{aligned}$$

Now, uniformly in $i \le sn^{2/3}$, again using condition (a),

$$\sum_{j\notin\mathcal{V}_{i-1}} w_j = \sum_{j\in[n]} w_j + O((\max_{j\in[n]} w_j)i) = l_n + o(n) \qquad (2.46)$$

for every $i \le sn^{2/3}$ and since $1/(\tau-1) < 1/3$ for $\tau > 4$. Similarly, again uniformly in $i \le sn^{2/3}$, and using that $j \mapsto w_j$ is non-increasing,

$$\Big|\sum_{j\notin\mathcal{V}_{i-1}} w_j^3 - l_n\sigma_3(n)\Big| \le \sum_{j=1}^{sn^{2/3}} w_j^3 = o(n), \qquad (2.47)$$

Indeed, we shall show that conditions (b) and (c) imply that

$$\lim_{K\to\infty}\lim_{n\to\infty}\frac{1}{n}\sum_{j\in[n]}\mathbb{1}_{\{w_j>K\}}w_j^3 = 0. \qquad (2.48)$$

By the weak convergence stated in condition (b),

$$\frac{1}{n}\sum_{j\in[n]}\mathbb{1}_{\{w_j\le K\}}w_j^3 = \mathbb{E}[\mathbb{1}_{\{w_{V_n}\le K\}}w_{V_n}^3] \to \mathbb{E}[\mathbb{1}_{\{W\le K\}}W^3]. \qquad (2.49)$$

As a result, we have that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j\in[n]}\mathbb{1}_{\{w_j>K\}}w_j^3=\lim_{n\to\infty}\frac{1}{n}\sum_{j\in[n]}w_j^3-\lim_{n\to\infty}\frac{1}{n}\sum_{j\in[n]}\mathbb{1}_{\{w_j\le K\}}w_j^3=\mathbb{E}[\mathbb{1}_{\{W>K\}}W^3]. \tag{2.50}$$

The latter converges to 0 when $K\to\infty$, since $\mathbb{E}[W^3]<\infty$. We finally show that (2.48) implies that $\sum_{j=1}^{sn^{2/3}}w_j^3=o(n)$. For this, we note that, for each K,

$$\frac{1}{n}\sum_{j=1}^{sn^{2/3}}w_j^3\le+\frac{1}{n}\sum_{j=1}^{sn^{2/3}}\mathbb{1}_{\{w_j\le K\}}w_j^3\le\frac{1}{n}\sum_{j\in[n]}\mathbb{1}_{\{w_j>K\}}w_j^3\frac{1}{n}\sum_{j\in[n]}\mathbb{1}_{\{w_j>K\}}w_j^3+K^3sn^{-1/3}=o(1), \tag{2.51}$$

when we first let $n\to\infty$, followed by $K\to\infty$.

We conclude that, uniformly for $i\le sn^{2/3}$,

$$\mathbb{E}[w_{v(i)}^2\mid\mathcal{F}_{i-1}]=\frac{\sigma_3}{\mu}+o(1). \tag{2.52}$$

This proves (2.24). □

To complete the proof of Theorem 2.2, we proceed to investigate $c(i)$. By construction, we have that, conditionally on $\mathcal{V}_i$,

$$c(i)\stackrel{d}{=}\sum_{j\notin\mathcal{V}_i}I_{ij}, \tag{2.53}$$

where I_{ij} are (conditionally) independent indicators with

$$\mathbb{P}(I_{ij}=1\mid\mathcal{V}_i)=\frac{w_{v(i)}w_j(1+tn^{-1/3})}{l_n}, \tag{2.54}$$

for all $j\notin\mathcal{V}_i$. Furthermore, when we condition on $\mathcal{F}_{i-1}$, we know $\mathcal{V}_{i-1}$, and we have that, for all $j\notin\mathcal{V}_{i-1}$,

$$\mathbb{P}(v(i)=j\mid\mathcal{F}_{i-1})=\frac{w_j}{\sum_{s\in\mathcal{V}_{i-1}^c}w_s}. \tag{2.55}$$

Since $\mathcal{V}_i=\mathcal{V}_{i-1}\cup\{v(i)\}=\mathcal{V}_{i-1}\cup\{j\}$ when $v(i)=j$, this gives us all we need to know to compute conditional expectations involving $c(i)$ given $\mathcal{F}_{i-1}$.

Now we start to prove (2.17), for which we note that

$$\begin{aligned}\mathbb{E}[c(i)\mid\mathcal{F}_{i-1}]&=\sum_{j\in\mathcal{V}_{i-1}^c}\mathbb{P}(v(i)=j)\mathbb{E}[c(i)\mid\mathcal{F}_{i-1},v(i)=j]\\&=\sum_{j\in\mathcal{V}_{i-1}^c}\mathbb{P}(v(i)=j)\sum_{l\notin\mathcal{V}_{i-1}\cup\{j\}}\frac{\tilde{w}_jw_l}{l_n}.\end{aligned} \tag{2.56}$$

Then we split

$$\begin{aligned}\mathbb{E}[c(i)-1\mid\mathcal{F}_{i-1}]&=\sum_{j\in\mathcal{V}_{i-1}^c}\mathbb{P}(v(i)=j)\tilde{w}_j-1-\sum_{j\in\mathcal{V}_{i-1}^c}\mathbb{P}(v(i)=j)\tilde{w}_j\sum_{l\in\mathcal{V}_{i-1}\cup\{j\}}\frac{w_l}{l_n}\\&=\mathbb{E}[\tilde{w}_{v(i)}-1\mid\mathcal{F}_{i-1}]-\mathbb{E}[\tilde{w}_{v(i)}\mid\mathcal{F}_{i-1}]\sum_{s=1}^{i-1}\frac{w_{v(s)}}{l_n}-\mathbb{E}[\frac{w_{v(i)}^2(1+tn^{-1/3})}{l_n}\mid\mathcal{F}_{i-1}].\end{aligned} \tag{2.57}$$

By condition (a), the last term is, a.s., bounded by $O(w_1^2/l_n) = o(n^{-1/3})$ and is therefore an error term. We continue to compute

$$\begin{aligned}
\mathbb{E}[\tilde{w}_{v(i)} - 1 \mid \mathcal{F}_{i-1}] &= \sum_{j \in \mathcal{V}_{i-1}^c} \frac{w_j^2(1+tn^{-1/3})}{\sum_{s \in \mathcal{V}_{i-1}^c} w_s} - 1 \\
&= \sum_{j \in \mathcal{V}_{i-1}^c} \frac{w_j^2(1+tn^{-1/3})}{l_n} - 1 + \sum_{j \in \mathcal{V}_{i-1}^c} \frac{w_j^2(1+tn^{-1/3})}{l_n \sum_{s \in \mathcal{V}_{i-1}^c} w_s} \Big(\sum_{s \in \mathcal{V}_{i-1}} w_s \Big).
\end{aligned} \tag{2.58}$$

The last term equals

$$\mathbb{E}[\frac{\tilde{w}_{v(i)}}{l_n} \mid \mathcal{F}_{i-1}] \sum_{s=1}^{i-1} w_{v(s)}, \tag{2.59}$$

which equals the second term in (2.57). Therefore, writing $\tilde{\nu}_n = \nu_n(1+tn^{-1/3})$,

$$\begin{aligned}
\mathbb{E}[c(i) - 1 \mid \mathcal{F}_{i-1}] &= \sum_{j \in \mathcal{V}_{i-1}^c} \frac{w_j^2(1+tn^{-1/3})}{l_n} - 1 + o(n^{-1/3}) \\
&= \sum_{j=1}^{n} \frac{w_j^2(1+tn^{-1/3})}{l_n} - 1 - \sum_{j \in \mathcal{V}_{i-1}} \frac{w_j^2(1+tn^{-1/3})}{l_n} + o(n^{-1/3}) \\
&= (\tilde{\nu}_n - 1) - \sum_{s=1}^{i-1} \frac{w_{v(s)}^2(1+tn^{-1/3})}{l_n} + o(n^{-1/3}) \\
&= (\tilde{\nu}_n - 1) - \sum_{s=1}^{i-1} \frac{w_{v(s)}^2}{l_n} + o(n^{-1/3}).
\end{aligned} \tag{2.60}$$

As a result, we obtain that

$$A_n(k) = \sum_{i=0}^{k} \mathbb{E}[c(i) - 1 \mid \mathcal{F}_{i-1}] = k(\tilde{\nu}_n - 1) - \sum_{i=0}^{k} \sum_{s=1}^{i-1} \frac{w_{v(s)}^2}{l_n} + o(kn^{-1/3}). \tag{2.61}$$

Thus,

$$\bar{A}_n(s) = ts - n^{-1/3} \sum_{i=0}^{sn^{2/3}} \sum_{l=1}^{i-1} \frac{w_{v(l)}^2}{l_n} + o(1). \tag{2.62}$$

By (2.23) in Lemma 2.3, we have that

$$\sup_{t \le u} |n^{-2/3} \sum_{s=1}^{tn^{2/3}} w_{v(s)}^2 - \frac{\sigma_3}{\mu} t| \xrightarrow{\mathbb{P}} 0, \tag{2.63}$$

so that

$$\sup_{t \le u} \big| \bar{A}_n(s) - ts + \frac{s^2 \sigma_3}{2\mu^2} \big| \xrightarrow{\mathbb{P}} 0. \tag{2.64}$$

This proves (2.17).

The proof for (2.18) is similar, and we start by noting that (2.60) gives that

$$B_n(k) = \sum_{i=0}^{k} \mathbb{E}[c(i)^2 \mid \mathcal{F}_{i-1}] - \mathbb{E}[c(i) \mid \mathcal{F}_{i-1}]^2 = \sum_{i=0}^{k} \mathbb{E}[c(i)^2 - 1 \mid \mathcal{F}_{i-1}] + O(kn^{-1/3}). \tag{2.65}$$

Now, as above, we obtain that

$$\mathbb{E}[c(i)^2 \mid \mathcal{F}_{i-1}] = \sum_{j\in\mathcal{V}_{i-1}^c} \mathbb{P}(v(i)=j) \sum_{\substack{s_1,s_2\notin\mathcal{V}_{i-1}\cup\{j\}\\ s_1\neq s_2}} \frac{\tilde{w}_j w_{s_1}}{l_n}\frac{\tilde{w}_j w_{s_2}}{l_n} + \sum_{j\in\mathcal{V}_{i-1}^c} \mathbb{P}(v(i)=j) \sum_{s\notin\mathcal{V}_{i-1}\cup\{j\}} \frac{\tilde{w}_j w_s}{l_n}. \tag{2.66}$$

We compute the second term as, using condition (a),

$$\sum_{j\in\mathcal{V}_{i-1}^c} \mathbb{P}(v(i)=j) \sum_{s\notin\mathcal{V}_{i-1}\cup\{j\}} \frac{\tilde{w}_j w_s}{l_n} = \sum_{j\in\mathcal{V}_{i-1}^c} \mathbb{P}(v(i)=j) w_j(1+n(in^{-2/3})) = 1 + o_{\mathbb{P}}(1). \tag{2.67}$$

The first sum is similarly computed as

$$\sum_{j\in\mathcal{V}_{i-1}^c} \mathbb{P}(v(i)=j) \sum_{\substack{s_1,s_2\notin\mathcal{V}_{i-1}\cup\{j\}\\ s_1\neq s_2}} \frac{\tilde{w}_j w_{s_1}}{l_n}\frac{\tilde{w}_j w_{s_2}}{l_n} = \sum_{j\in\mathcal{V}_{i-1}^c} \mathbb{P}(v(i)=j)\tilde{w}_j^2(1+o(1)) = \mathbb{E}[\tilde{w}_{v(i)}^2 \mid \mathcal{F}_{i-1}] + o(1), \tag{2.68}$$

so that

$$\begin{aligned} n^{-2/3}B_n(n^{2/3}u) &= n^{-2/3}\sum_{i=1}^{n^{2/3}u} \mathbb{E}[w_{v(i)}^2 \mid \mathcal{F}_{i-1}] + o(1) \\ &= \frac{\sigma_3}{\mu}u + o(1), \end{aligned} \tag{2.69}$$

where the last equality follows from (2.24) in Lemma 2.3. Thus,

$$n^{-2/3}B_n(n^{2/3}u) \xrightarrow{\mathbb{P}} \frac{\sigma_3}{\mu}u. \tag{2.70}$$

This proves (2.18).

The proofs of (2.17), (2.18) and (2.19) complete the proof of Theorem 2.2. □

3 Verification of conditions (a)–(c): Proof of Corollary 1.2

3.1 Verification of conditions for i.i.d. weights

We now check conditions (a), (b) and (c) for the case that $\boldsymbol{w} = (W_1,\ldots,W_n)$ where $\{W_i\}_{i\in[n]}$ are i.i.d. random variables with $\mathbb{E}[W^3] < \infty$. Condition (a) follows from the fact that when $\{X_i\}_{i=1}^n$ are i.i.d. with $\mathbb{E}[X] < \infty$, then $\max_{i\in[n]} X_i = o_{\mathbb{P}}(n)$. Applying this to $X_i = W_i^3$ proves the claim. Condition (b) is equivalent to the a.s. convergence of the empirical distribution function, while (1.12) in condition (c) holds by the strong law of large numbers. Equation (1.10) in condition (c) holds by the central limit theorem (even with $o(n^{-1/3})$ replaced with $O_{\mathbb{P}}(n^{-1/2})$. The bound in (1.11) in condition (c) is a bit more delicate.

To bound (1.11) in condition (c), we will first split

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^n \big(W_i^2 - \mathbb{E}[W^2]\big) &= \frac{1}{n}\sum_{i=1}^n \Big(W_i^2\mathbb{1}_{\{W_i\leq n^{1/3}\}} - \mathbb{E}[W^2\mathbb{1}_{\{W\leq n^{1/3}\}}]\Big) \\ &\quad + \frac{1}{n}\sum_{i=1}^n \Big(W_i^2\mathbb{1}_{\{W_i> n^{1/3}\}} - \mathbb{E}[W^2\mathbb{1}_{\{W> n^{1/3}\}}]\Big). \end{aligned} \tag{3.1}$$

On the second term, we use the Markov inequality to show that, for every $\varepsilon > 0$,

$$\begin{aligned} \mathbb{P}\Big(\frac{1}{n}\sum_{i=1}^{n}\Big|W_i^2\mathbb{1}_{\{W_i>n^{1/3}\}} - \mathbb{E}[W^2\mathbb{1}_{\{W>n^{1/3}\}}]\Big| > \varepsilon n^{-1/3}\Big) \\ \leq n^{1/3}\mathbb{E}[W^2\mathbb{1}_{\{W>n^{1/3}\}}] \leq \mathbb{E}[W^3\mathbb{1}_{\{W>n^{1/3}\}}] = o(1), \end{aligned} \tag{3.2}$$

since $\mathbb{E}[W^3] < \infty$. Thus,

$$\frac{n^{1/3}}{n}\sum_{i=1}^{n}\Big|W_i^2\mathbb{1}_{\{W_i>n^{1/3}\}} - \mathbb{E}[W^2\mathbb{1}_{\{W>n^{1/3}\}}]\Big| \xrightarrow{\mathbb{P}} 0. \tag{3.3}$$

For the first sum, we use the Chebycheff inequality to obtain

$$\begin{aligned} \mathbb{P}\Big(\Big|\frac{1}{n}\sum_{i=1}^{n}W_i^2\mathbb{1}_{\{W_i\leq n^{1/3}\}} - \mathbb{E}[W^2]\Big| > \varepsilon n^{-1/3}\Big) &= \varepsilon^{-2}n^{2/3}\mathrm{Var}\Big(\frac{1}{n}\sum_{i=1}^{n}W_i^2\mathbb{1}_{\{W_i\leq n^{1/3}\}}\Big) \\ &\leq \varepsilon^{-2}n^{-1/3}\mathbb{E}\Big[W^4\mathbb{1}_{\{W\leq n^{1/3}\}}\Big] = o(1), \end{aligned} \tag{3.4}$$

since, when $\mathbb{E}[W^3] < \infty$, we have that $\mathbb{E}[W^4\mathbb{1}_{\{W>x\}}] = o(x)$. Thus, also,

$$\frac{n^{1/3}}{n}\sum_{i=1}^{n}\Big|W_i^2\mathbb{1}_{\{W_i\leq n^{1/3}\}} - \mathbb{E}[W^2\mathbb{1}_{\{W\leq n^{1/3}\}}]\Big| \xrightarrow{\mathbb{P}} 0. \tag{3.5}$$

This proves that conditions (a)-(c) hold in probability.

3.2 Verification of conditions for weights as in (1.3)

Here we check conditions (a), (b) and (c) for the case that $\boldsymbol{w} = (w_1, \ldots, w_n)$ where w_i is chosen as in (1.3). We shall frequently make use of the fact that (1.4) implies that $1 - F(x) = o(x^{-3})$ as $x \to \infty$, which, in turn implies that (see e.g., [10, (B.9)]), as $u \downarrow 0$,

$$[1-F]^{-1}(u) = o(u^{-1/3}). \tag{3.6}$$

Equation (3.6) immediately implies that

$$\max_{i\in[n]} w_i = w_1 = [1-F]^{-1}(1/n) = o(n^{1/3}), \tag{3.7}$$

which proves condition (a). To verify condition (b), we note that by [12, (4.2)], w_{V_n} has distribution function

$$F_n(x) = \frac{1}{n}\big(\lfloor nF(x)\rfloor + 1\big) \wedge 1. \tag{3.8}$$

This converges to $F(x)$ for every $x \geq 0$, which proves that condition (b) holds. To verify condition (c), we note that since $i \mapsto [1-F]^{-1}(i/n)$ is monotonically decreasing, for any $s > 0$, we have

$$\mathbb{E}[W^s] - \int_0^{1/n}(1-F^{-1}(u))^s du \leq \frac{1}{n}\sum_{i=1}^{n} w_i^s \leq \mathbb{E}[W^s]. \tag{3.9}$$

Now, by (3.6), we have that, for $s = 1, 2, 3$,

$$\int_0^{1/n}(1-F^{-1}(u))^s du = o(n^{s/3-1}), \tag{3.10}$$

which proves all necessary bounds for condition (c) at once.

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