

Universality for first passage percolation on sparse random graphs

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Abstract

We consider first passage percolation on sparse random graphs with prescribed degree distributions and general independent and identically distributed edge weights assumed to have a density. Assuming that the degree distribution satisfies a uniform $X^2 \log X$ -condition, we analyze the asymptotic distribution for the minimal weight path between a pair of typical vertices, as well the number of edges on this path or hopcount.

The hopcount satisfies a central limit theorem where the norming constants are expressible in terms of the parameters of an associated continuous-time branching process. Centered by a multiple of $\log n$, where the constant is the inverse of the Malthusian rate of growth of the associated branching process, the minimal weight converges in distribution. The limiting random variable equals the sum of the logarithms of the martingale limits of the branching processes that measure the relative growth of neighborhoods about the two vertices, and a Gumbel random variable, and thus shows a remarkably universal behavior. The proofs rely on a refined coupling between the shortest path problems on these graphs and continuous-time branching processes, and on a Poisson point process limit for the potential closing edges of shortest-weight paths between the source and destination.

The results extend to a host of related random graph models, ranging from random r -regular graphs, inhomogeneous random graphs and uniform random graphs with a prescribed degree sequence.

Key words: central limit theorem, continuous-time branching processes, extreme value theory, first passage percolation, flows, hopcount, Malthusian rate of growth, point process convergence, Poisson point process, stable-age distribution, random graphs.

MSC2000 subject classification: 60C05, 05C80, 90B15.

1 Introduction and results

1.1 Motivation

First passage percolation is an important topic in modern probability, due to the inherent applications in a number of fields such as disordered systems in statistical physics, and since it arises as a building block in the analysis of many more complicated interacting particle systems such as the contact process, other epidemic models and the voter model.

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Let us start by describing the basic model. Let \mathcal{G} be a finite simply connected graph on n vertices (for example the box $[-N, N]^d$ in the \mathbb{Z}^d lattice, so that $n = (2N + 1)^d$). We assign a random edge *weight or length* independently and identically distributed (i.i.d.) to each of the edges. Due to the random weights, this is an example of a disordered system entrusted with carrying flow between vertices in the graph. Fix two vertices in \mathcal{G} . Two functionals of interest are the minimal weight L_n of a path between the two vertices and the number of edges H_n on the minimal path, often referred to as the *hopcount*. We assume that the common distribution of the edge weights is continuous, so that the optimal paths are unique and one can talk about objects such as the number of edges on *the* optimal path.

This model has been intensively studied, largely in the context of the integer lattice $[-N, N]^d$ (see e.g. [42, 28, 32, 54]). For the power of this model to analyze more complicated interacting particle systems, see [45] and [24] and the references therein. Due to the interest in complex networks, such as social networks or the Internet, recently, this model has attracted attention on general random graph models. Indeed, stimulated by the availability of an enormous amount of empirical data on real-world networks, the last decade has witnessed the formulation and development of many new mathematical graph models for real-world networks. These models are used to study various dynamics, such as models of epidemics or random walks to search through the network (see e.g. [1, 51]).

In the modern context, first passage percolation problems take on an added significance. Many real-world networks (such as the Internet at the router level or various road and rail networks) are entrusted with carrying flow between various parts of the network. These networks have both a graph theoretic structure as well as weights on edges, representing for example congestion. In the applied setting understanding properties of both the hopcount and the optimal weight are crucial, since whilst routing is done via least weight paths, the actual time delay experienced by users scales like the hopcount (the number of “hops” a message has to perform in getting from the source to the destination). Simulation-based studies (see e.g., [15]) suggest that random edge weights have a marked effect on the geometry of the network. This has been rigorously established in various works [5, 9, 10, 11], in the specific situation of *exponential* edge weights.

In this paper, we study the behavior of the hopcount and minimal weight in the setting of random graphs with finite variance degrees and *general* continuous edge weights. Since in many applications, the distribution of edge weights is unknown, the assumption of general weights is highly relevant. From a mathematical point of view, working with general instead of exponential edge weights implies that our exploration process is *non-Markovian*. This is the first paper that studies first passage percolation on random graph models in this general setting. Further, due to the choices of degree distribution, our results immediately carry over to various other random graph models, such as rank-1 inhomogeneous random graphs as introduced in [14].

Organization of this section. We start by introducing the configuration model in Section 1.2, where we also state our main result Theorem 1.2. In Section 1.3, we discuss a continuous-time branching process approximation, which is necessary in order to be able to identify the limiting variables in Theorem 1.2. In Section 1.4, we extend our results to related random graph models, and in Section 1.5 we study some examples that allow us to relate our results to results in the literature. We close with Section 1.6 where we present a discussion of our results and some open problems.

Throughout this paper, we make use of the following standard notation. We let $\xrightarrow{a.s.}$ denote convergence almost surely, $\xrightarrow{L^1}$ denote convergence in means, \xrightarrow{d} denote convergence in distribution, and $\xrightarrow{\mathbb{P}}$ convergence in probability. For a sequence of random variables $(X_n)_{n \geq 1}$, we write $X_n = O_{\mathbb{P}}(b_n)$ when $|X_n|/b_n$ is a tight sequence of random variables, and $X_n = o_{\mathbb{P}}(b_n)$ when $|X_n|/b_n \xrightarrow{\mathbb{P}} 0$ as $n \rightarrow \infty$. To denote that the random variable D has distribution function F , we

write $D \sim F$. For non-negative functions $n \mapsto f(n)$, $n \mapsto g(n)$ we write $f(n) = O(g(n))$ when $|f(n)|/g(n)$ is uniformly bounded, and $f(n) = o(g(n))$ when $\lim_{n \rightarrow \infty} f(n)/g(n) = 0$. Furthermore, we write $f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ and $g(n) = O(f(n))$. Finally, we write that a sequence of events $(\mathcal{E}_n)_{n \geq 1}$ occurs *with high probability* (whp) when $\mathbb{P}(\mathcal{E}_n) \rightarrow 1$.

1.2 Configuration model and main result

We work with the configuration model on n vertices $[n] := \{1, 2, \dots, n\}$. Let us first describe the random graph model.

Configuration model. We are interested in constructing a random graph on n vertices with prescribed degrees. Given a *degree sequence*, namely a sequence of n positive integers $\mathbf{d} = (d_1, d_2, \dots, d_n)$ with $\sum_{i \in [n]} d_i$ assumed to be even, the configuration model (CM) on n vertices with degree sequence \mathbf{d} is constructed as follows:

Start with n vertices and d_i half-edges adjacent to vertex i . The graph is constructed by randomly pairing each half-edge to some other half-edge to form edges. Let

$$\ell_n = \sum_{i \in [n]} d_i, \quad (1.1)$$

denote the total degree. Number the half-edges from 1 to ℓ_n in some arbitrary order. Then, at each step, two half-edges which are not already paired are chosen uniformly at random among all the unpaired or *free* half-edges and have been paired to form a single edge in the graph. These half-edges are no longer free and removed from the list of free half-edges. We continue with this procedure of choosing and pairing two half-edges until all the half-edges are paired. Observe that the order in which we choose the half-edges does not matter. Although self-loops may occur, these become rare as $n \rightarrow \infty$ (see e.g. [13] or [37] for more precise results in this direction). We denote the resulting graph by $\text{CM}_n(\mathbf{d})$, its vertex set by $[n]$ and its edge set by \mathcal{E}_n .

Regularity of vertex degrees. Above, we have described the construction of the CM when the degree sequence is given. Here, we shall specify how we construct the actual degree sequence \mathbf{d} . We start by formulating conditions on \mathbf{d} . We denote the degree of a uniformly chosen vertex V in $[n]$ by $D_n = d_V$. The random variable D_n has distribution function F_n given by

$$F_n(x) = \frac{1}{n} \sum_{j \in [n]} \mathbb{1}_{\{d_j \leq x\}}. \quad (1.2)$$

Write $\log(x)_+ = \log(x)$ for $x \geq 1$ and $\log(x)_+ = 0$ for $x \leq 1$. We assume that the vertex degrees satisfy the following *regularity conditions*:

Condition 1.1 (Regularity conditions for vertex degrees)

(a) Weak convergence of vertex degree.

There exists a distribution function F on \mathbb{N} such that

$$D_n \xrightarrow{d} D, \quad (1.3)$$

where D_n and D have distribution functions F_n and F , respectively.

Equivalently, for any continuity point x of F ,

$$\lim_{n \rightarrow \infty} F_n(x) = F(x). \quad (1.4)$$

(b) Convergence of second moment vertex degrees.

$$\lim_{n \rightarrow \infty} \mathbb{E}[D_n^2] = \mathbb{E}[D^2], \quad (1.5)$$

where D_n and D have distribution functions F_n and F , respectively, and we assume that

$$\nu = \mathbb{E}[D(D-1)]/\mathbb{E}[D] > 1. \quad (1.6)$$

(c) Uniform $X^2 \log X$ condition. For every $K_n \rightarrow \infty$,

$$\limsup_{n \rightarrow \infty} \mathbb{E}[D_n^2 \log (D_n/K_n)_+] = 0. \quad (1.7)$$

The degree of a vertex chosen uniformly at random has distribution D_n as given in Condition 1.1(a). By Condition 1.1(c), the degree distribution D_n satisfies a uniform $X^2 \log X$ condition. A vertex incident to a half-edge that is chosen uniformly at random from all half-edges has the same distribution as the random variable D_n^* given in (4.1), which is the size-biased version of D_n . The latter random variable satisfies a uniform $X \log X$ condition if and only if D_n satisfies a uniform $X^2 \log X$ condition. As explained in more detail in Section 1.3 below, D_n^* is closely related to a *branching-process approximation* of neighborhoods of a uniform vertex, and thus Condition 1.1(c) implies that this branching process satisfies a uniform $X \log X$ condition. By uniform integrability, Condition 1.1(c) follows from the assumption that $\lim_{n \rightarrow \infty} \mathbb{E}[D_n^2 \log (D_n)_+] = \mathbb{E}[D^2 \log (D)_+]$.

Note that that Condition 1.1(a) and (c) imply that $\mathbb{E}[D_n^i] \rightarrow \mathbb{E}[D^i]$, $i = 1, 2$. When the degrees are *random* themselves, then we assume that the above convergence conditions hold *in probability*. Condition (1.6) is equivalent to a giant component existing in $\text{CM}_n(\mathbf{d})$, see e.g. [40, 48, 49]. We often abbreviate $\mu = \mathbb{E}[D]$. Let F be a distribution function of a random variable D , satisfying that $\mathbb{E}[D^2 \log (D)_+] < \infty$. We give two canonical examples in which Condition 1.1 hold. The first is when there are precisely $n_k = \lceil nF(k) \rceil - \lceil nF(k-1) \rceil$ vertices having degree k . The second is when $(d_i)_{i \in [n]}$ is an i.i.d. sequence of random variables with distribution function F (in the case where $\sum_{i \in [n]} d_i$ is odd, we increase d_n by 1, this does not affect the results).

As we will describe in more detail in Section 1.4, Condition 1.1 is such that it allows to extend our results to a range of other random graph models.

Edge weights and shortest paths. Once the graph has been constructed, we attach edge weight X_e to every edge e , where $(X_e)_{e \in \mathcal{E}_n}$ are i.i.d. continuous random variables with density $g: [0, \infty) \rightarrow [0, \infty)$ and corresponding distribution function G . Pick two vertices U_1 and U_2 at random and let Γ_{12} denote the set of all paths in $\text{CM}_n(\mathbf{d})$ between these two vertices. For any path $\pi \in \Gamma_{12}$, the weight of the path is defined as

$$c_{\text{wk-dis}}(\pi) = \sum_{e \in \pi} X_e. \quad (1.8)$$

Let

$$L_n = \min_{\pi \in \Gamma_{12}} c_{\text{wk-dis}}(\pi), \quad (1.9)$$

denote the weight of the optimal (i.e., minimal weight) path between the two vertices and let H_n denote number of edges or the *hopcount* of this path. If the two vertices are in different components of the graph, then we let $L_n, H_n = \infty$. Now we are ready to state our main result. Due to the complexity of the various constructs (constants and limiting random variables) arising in the theorem, we defer a complete description of these constructs to the next section.

Theorem 1.2 (Joint convergence hopcount and weight) *Consider the configuration model $\text{CM}_n(\mathbf{d})$ with degrees \mathbf{d} satisfying Condition 1.1, and with i.i.d. edge weights distributed according to the continuous distribution G . Then, there exist constants $\alpha, \gamma, \beta \in (0, \infty)$ and α_n, γ_n with $\alpha_n \rightarrow \alpha, \gamma_n \rightarrow \gamma$, such that the hopcount H_n and weight L_n of the optimal path between two uniformly selected vertices conditioned on being connected, satisfy*

$$\left(\frac{H_n - \gamma_n \log n}{\sqrt{\beta \log n}}, L_n - \frac{1}{\alpha_n} \log n \right) \xrightarrow{d} (Z, Q), \quad (1.10)$$

as $n \rightarrow \infty$, where Z and Q are independent and Z has a standard normal distribution, while Q has a continuous distribution.

In Remark 1.4 below, we will state conditions that imply that we can replace α_n and γ_n by their limits α and γ , respectively. Theorem 1.2 shows a remarkable kind of universality. For the configuration model with finite-variance degrees satisfying Condition 1.1, the hopcount always satisfies a central limit theorem with mean and variance proportional to $\log n$. Also, the weight of the shortest weight path between two uniformly chosen vertices always is of order $\log n$, and the fluctuations converge in distribution. We will see that even the limit distribution Q of L_n has a large degree of universality. In order to do this, as well as to define the parameters $\alpha, \alpha_n, \beta, \gamma, \gamma_n$, we first need to describe a continuous-time branching process approximation for the flow on the configuration model with i.i.d. edge weights.

1.3 Continuous-time branching processes

Before stating our results we recall a standard model of continuous-time branching process (CTBP), the splitting process or the Bellman-Harris process as well as various associated processes.

Define the size-biased distribution F^* of the random variable D with distribution function F by

$$F^*(x) = \mathbb{E}[D \mathbb{1}_{\{D \leq x\}}] / \mathbb{E}[D], \quad x \in \mathbb{R}. \quad (1.11)$$

Now let $(\text{BP}^*(t))_{t \geq 0}$ denote the following CTBP:

- (a) At time $t = 0$, we start with one individual which we shall refer to as the original ancestor or the root of the branching process. Generate D^* having the size-biased distribution F^* given in (1.11). This individual immediately dies giving rise to $D^* - 1$ children.
- (b) Each new individual v in the branching process lives for a random amount of time which has distribution G , i.e., the edge weight distribution, and then dies. At the time of death again the individual gives birth to $D_v^* - 1$ children, where $D_v^* \sim F^*$. Lifetimes and number of offspring across individuals are independent.

Note that in the above construction, by Condition 1.1(b), if we let $X_v = D_v^* - 1$ be the number of children of an individual then the expected number of children satisfies

$$\mathbb{E}[X_v] = \mathbb{E}[D_v^* - 1] = \nu > 1, \quad (1.12)$$

Further, by Condition 1.1(c), for $D^* \sim F^*$,

$$\mathbb{E}[D^* \log(D^*)_+] < \infty. \quad (1.13)$$

The CTBP defined above is a *splitting* process, with lifetime distribution G and offspring distribution $D^* - 1$. Denote by $N(t)$ the number of offspring of an individual at time t . Then

$\mu(t) = \mathbb{E}[N(t)] = \nu \cdot G(t)$, where $\nu = \mathbb{E}[D^* - 1]$. Furthermore, the Laplace-Stieltjes transform $\hat{\mu}$ is defined by

$$\hat{\mu}(s) = \nu \int_0^\infty e^{-st} dG(t).$$

The Malthusian parameter α of the branching process $\mathbf{BP}^*(\cdot)$ is the unique solution of the equation

$$\hat{\mu}(\alpha) = \nu \int_0^\infty e^{-\alpha t} dG(t) = 1. \quad (1.14)$$

Since $\nu > 1$, we obtain that $\alpha \in (0, \infty)$. We also let α_n be the solution to (1.14) with ν replaced with $\nu_n = \mathbb{E}[D_n(D_n - 1)]/\mathbb{E}[D_n]$. Clearly, $\alpha_n \rightarrow \alpha$, when Condition 1.1 holds, and $|\alpha_n - \alpha| = O(|\nu_n - \nu|)$.

Standard theory (see e.g., [6, 33, 34]) implies that under our assumptions of the model, namely (1.12) and (1.13), there exists a random variable \mathcal{W}^* such that

$$e^{-\alpha t} |\mathbf{BP}^*(t)| \xrightarrow{a.s., \mathbb{L}^1} \mathcal{W}^*. \quad (1.15)$$

Here the limiting random variable \mathcal{W}^* satisfies $\mathcal{W}^* > 0$ a.s. on the event of non-extinction of the branching process and is zero otherwise. Thus α measures the true rate of exponential growth of the branching process.

Define the distribution function \bar{G} , which is often referred to as the *stable-age distribution*, by

$$\bar{G}(x) = \nu \int_0^x e^{-\alpha y} dG(y), \quad (1.16)$$

where we recall that α is the Malthusian rate of growth parameter. Let $\bar{\nu}$ be the mean and $\bar{\sigma}^2$ the variance of \bar{G} . Then $\bar{\nu}, \bar{\sigma}^2 \in (0, \infty)$, since $\alpha > 0$. We also define \bar{G}_n to be the distribution function \bar{G}_n in (1.16) with ν and α replaced with ν_n and α_n , and we let $\bar{\nu}_n$ and $\bar{\sigma}_n^2$ be its mean and variance.

We need a small variation of the above branching process where the root of the branching process dies immediately giving birth to a D number of children where D has distribution F . The details for every other individual in this branching process remain unchanged from the original description, namely each individual survives for a random amount of time with distribution G giving rise to a $D^* - 1$ number of children where $D^* \sim F^*$, the size-biased distribution function F^* . Writing $|\mathbf{BP}(t)|$ for the number of alive individuals at time t , it is easy to see here as well that

$$e^{-\alpha t} |\mathbf{BP}(t)| \xrightarrow{a.s., \mathbb{L}^1} \tilde{\mathcal{W}}. \quad (1.17)$$

Here, $\tilde{\mathcal{W}}$ satisfies the stochastic equation,

$$\tilde{\mathcal{W}} = \sum_{i=1}^D \mathcal{W}^{*,(i)} e^{-\alpha \xi_i},$$

where $D \sim F$, and $\mathcal{W}^{*,(i)}$ are i.i.d. with the distribution of the limiting random variable in (1.15), and ξ_i are i.i.d. with distribution G . Let

$$\mathcal{W} = \tilde{\mathcal{W}} | \tilde{\mathcal{W}} > 0. \quad (1.18)$$

To simplify notation in the sequel, we will use $(\mathbf{BP}(t))_{t \geq 0}$ to denote a CTBP with the root having offspring either D or $D^* - 1$, which will be clear from the context.

We are now in a position to identify the limiting random variable Q as well as the parameters $\alpha, \beta, \alpha_n, \gamma_n, \gamma$:

Theorem 1.3 (Identification of the limiting variables) *The parameters $\alpha, \alpha_n, \beta, \gamma_n$ in Theorem 1.2 satisfy that α is the Malthusian rate of growth defined in (1.14) and α_n is the solution to (1.14) with ν_n replacing ν , while*

$$\gamma_n = \frac{1}{\alpha_n \bar{\nu}_n}, \quad \beta = \frac{\bar{\sigma}^2}{\bar{\nu}^3 \alpha}. \quad (1.19)$$

Further, Q can be identified as:

$$Q = \frac{1}{\alpha} (-\log \mathcal{W}^{(1)} - \log \mathcal{W}^{(2)} - \Lambda + c), \quad (1.20)$$

where $\mathbb{P}(\Lambda \leq x) = e^{-e^{-x}}$, so that Λ is a standard Gumbel random variable, $\mathcal{W}^{(1)}, \mathcal{W}^{(2)}$ are two independent copies of the variable \mathcal{W} in (1.18), also independent from Λ , and c is the constant

$$c = \log(\mu(\nu - 1)^2 / (\nu \alpha \bar{\nu})). \quad (1.21)$$

Remark 1.4 (Asymptotic mean) *We can replace α_n and γ_n by their limits α and $\gamma = 1/(\alpha \bar{\nu})$ in (1.10) precisely when $\gamma_n = \gamma + o(1/\sqrt{\log n})$ and $\alpha_n = \alpha + o(1/\log n)$. Since $|\alpha_n - \alpha| = O(|\nu_n - \nu|)$, $|\bar{\nu}_n - \bar{\nu}| = O(|\nu_n - \nu|)$, these conditions are equivalent to $\nu_n = \nu + o(1/\sqrt{\log n})$ and $\bar{\nu}_n = \bar{\nu} + o(1/\log n)$, respectively.*

Theorem 1.3 implies that also the random variable Q is remarkably universal, in the sense that it always involves two martingale limit variables corresponding to the flow problem, and a Gumbel distribution. While such results were known for the exponential distribution (see e.g., [9]), this is the first time that FPP on random graphs with general edge weights is studied.

Let $L_n(i)$ denote the weight of the i^{th} shortest path, so that $L_n = L_n(1)$, and let $H_n(i)$ denote its length. Further let $\bar{H}_n(i)$ and $\bar{L}_n(i)$ denote the re-centered and normalized quantities as in Theorem 1.2. The same proof for the optimal path easily extends to prove asymptotic results for the joint distribution of the weights and hopcount of these ranked paths. To keep the study to a manageable length, we shall skip a proof of this easy extension.

Theorem 1.5 (Multiple paths) *Under the conditions of Theorem 1.2, for every $m \geq 1$,*

$$((\bar{H}_n(i), \bar{L}_n(i)))_{i \in [m]} \xrightarrow{d} ((Z_i, Q_i))_{i \in [m]}, \quad (1.22)$$

as $n \rightarrow \infty$, where for $i \geq 1$, Z_i and Q_i are independent and Z_i has a standard normal distribution, while

$$Q_i = \frac{1}{\alpha} (-\log \mathcal{W}^{(1)} - \log \mathcal{W}^{(2)} - \Lambda_i + c), \quad (1.23)$$

where $(\Lambda_i)_{i \geq 1}$ are the ordered points of an inhomogeneous Poisson point process with intensity $\lambda(t) = e^t$.

1.4 Related random graph models

Uniform random graphs with a prescribed degree sequence. We call a graph *simple* when it contains no self-loops nor multiple edges. It is well known that the CM conditioned on being simple is a *uniform* random graph with the same degrees. As a result, our theorems extend to this setting:

Theorem 1.6 (Extension to uniform random graphs with prescribed degrees) *Under the conditions of Theorem 1.2, the results in Theorems 1.2 and the identification of the limiting variables in Theorem 1.3 apply to uniform random graphs with prescribed degree sequence \mathbf{d} satisfying Condition 1.1.*

The proof of Theorem 1.6 follows rather directly from that of Theorems 1.2-1.3, by conditioning on simplicity. By [13] or [37], under Condition 1.1,

$$\lim_{n \rightarrow \infty} \mathbb{P}(\text{CM}_n(\mathbf{d}) \text{ simple}) = e^{-\nu/2 - \nu^2/4}. \quad (1.24)$$

This proof of (1.24) follows by a Poisson approximation on the number of self-loops and the number of multiple edges, which are proved to converge to two independent Poisson random variables with means $\nu/2$ and $\nu^2/4$ respectively. We can interpret the probability in (1.24) as the probability that both these Poisson variables are equal to zero, which is equivalent to the graph being simple. Now the proof of the main theorem reveals that in order to find the minimal weight path between vertices U_1, U_2 , we only need to investigate of order \sqrt{n} edges. Therefore, the event of simplicity of the configuration model will be mainly determined by the *uninspected* edges, and is therefore asymptotically independent of (H_n, L_n) . This explains Theorem 1.6. We give a full proof of Theorem 1.6 in Section 6.

Rank-1 inhomogeneous random graphs. Fix a sequence of positive weights $(w_i)_{i \in [n]}$. We shall assume that there exists a distribution function F_W on \mathbb{R}^+ such that

$$F_{n,w}(x) = \frac{1}{n} \sum_{i \in [n]} \mathbf{1}_{\{w_i \leq x\}} \rightarrow F_W(x), \quad (1.25)$$

for each point of continuity of F_W . Here, $\mathbf{1}_A$ denotes the indicator of the set A . Let W_n denote the weight of a uniformly chosen vertex in $[n]$, i.e., $W_n = w_V$, where $V \in [n]$ is chosen uniformly. Let W_n^* denote the size-biased version of W_n , i.e.,

$$\mathbb{P}(W_n^* \leq x) = \mathbb{E}[W_n \mathbf{1}_{\{W_n \leq x\}}] / \mathbb{E}[W_n]. \quad (1.26)$$

Now given these weights, we construct a random graph by attaching an edge between vertex i and j with probability

$$p_{ij} = 1 - e^{-w_i w_j / \ell_n}, \quad (1.27)$$

where, with some abuse of notation,

$$\ell_n = \sum_{i \in [n]} w_i, \quad (1.28)$$

is the sum of the vertex weights, and the status of different edges are independent. Let $\nu = \mathbb{E}[W^2] / \mathbb{E}[W]$. We always assume $\nu > 1$ as this is necessary and sufficient for the existence of a giant component (see, e.g., [14]). Note that letting $w_i = \lambda$, we immediately get the Erdős-Rényi random graph with edge connection probability $1 - e^{-\lambda/n}$. Thus, this model is a natural generalization of the classical random graph model. Related models are the *generalized random graph* introduced by Britton, Deijfen and Martin-Löf in [16], for which

$$p_{ij} = \frac{w_i w_j}{\ell_n + w_i w_j}, \quad (1.29)$$

and the *random graph with given prescribed degrees* or Chung-Lu model, where instead

$$p_{ij} = \max(w_i w_j / \ell_n, 1), \quad (1.30)$$

and which has been studied intensively by Chung and Lu (see [18, 19, 20, 21, 22]). Let W denote a random variable with distribution F_W . By Janson [38], when $W_n \xrightarrow{d} W$ and $\mathbb{E}[W_n^2] \rightarrow \mathbb{E}[W^2]$, the three random graph models defined above are *asymptotically equivalent*, meaning that all events have asymptotically equal probabilities. By [14], with $N_k(n)$ denoting the number of vertices with degree k ,

$$N_k(n)/n \xrightarrow{\mathbb{P}} \mathbb{E} \left[e^{-W} \frac{W^k}{k!} \right]. \quad (1.31)$$

This proves that $D_n \xrightarrow{d} D$, where D has the mixed Poisson distribution given in (1.31), i.e., for this model

$$F(x) = \sum_{k \leq x} \mathbb{E} \left[e^{-W} \frac{W^k}{k!} \right]. \quad (1.32)$$

This is formulated in the following theorem:

Theorem 1.7 (Extension to rank-1 inhomogeneous random graphs) *For rank-1 inhomogeneous random graphs with edge probabilities in (1.27), (1.29) or (1.30), where the weight of a uniform vertex W_n satisfies that $W_n \xrightarrow{d} W$, $\mathbb{E}[W_n] \rightarrow \mathbb{E}[W]$, $\mathbb{E}[W_n^2] \rightarrow \mathbb{E}[W^2]$ and, for every $K_n \rightarrow \infty$,*

$$\mathbb{E}[W_n^2 \log(W_n/K_n)_+] = o(1),$$

the results in Theorems 1.2 and 1.3 hold with limiting degree distribution F in (1.32).

Theorem 1.7 can be understood as follows. By [16], the generalized random graph *conditioned on its degree sequence* is a uniform random graph with the same degree sequence. Therefore, Theorem 1.7 follows from Theorem 1.6 when the conditions on the degrees in Condition 1.1 hold in probability. By (1.31), $D_n \xrightarrow{d} D$, where D has the mixed Poisson distribution given in (1.31). Therefore, it suffices to show that $\mathbb{E}[W_n] \rightarrow \mathbb{E}[W]$, $\mathbb{E}[W_n^2] \rightarrow \mathbb{E}[W^2]$ and $\lim_{n \rightarrow \infty} \mathbb{E}[W_n^2 \log(W_n/K_n)_+] = 0$ imply that the same convergence holds for the degree of a random vertex. This is proved in Section 6.

1.5 Examples

In this section, we discuss a few special examples of the edge weights that have arisen in a number of different contexts in the literature and have been treated via distribution specific techniques.

Exponential weights. FPP on random graphs with exponential edge weights have received substantial attention in the literature (see e.g., [5, 9, 10, 11]). Let $G(x) = 1 - e^{-x}$, for $x \geq 0$, denote the distribution function of an exponential random variable with mean 1. This was one of the first models to be formulated and analyzed in the context of the integer lattice, see [50] and the complete analysis in [28]. For exponential weights, the Malthusian rate of growth parameter α satisfies

$$\nu \int_0^\infty e^{-\alpha x} dG(x) = \nu \int_0^\infty e^{-\alpha x} e^{-x} dx = 1, \quad (1.33)$$

so that $\alpha = \nu - 1$, $\alpha_n = \nu_n - 1$. Similarly, one can compute that $\bar{G}_n(x) = 1 - e^{-\nu_n x}$, $x \geq 0$, so that

$$\bar{\nu}_n = 1/\nu_n, \quad \bar{\nu} = 1/\nu, \quad \bar{\sigma}^2 = 1/\nu^2. \quad (1.34)$$

Using these values in Theorem 1.3 shows that H_n converges to a normal distribution, with asymptotic mean and asymptotic variance both equal to $\frac{\nu}{\nu-1} \log n$. Finally $c = \log \left(\mu(\nu-1)^2 / (\alpha \nu \bar{\nu}) \right) =$

$\log(\mu(\nu - 1))$, which is equal to the constant in [9, (3.7)].^{1 2} This result thus generalizes the hopcount result in [9], where FPP with exponential weights is considered on the configuration model with i.i.d. degrees with $D \geq 2$ a.s. In [9], also *infinite variance* degrees are studied, a case that we do not investigate here. In fact, some of the results we prove here do *not* extend to this setting, see Section 1.6 for more details.

Exponential weights plus a large constant. We next study what happens when $X_e = 1 + E_e/k$, where (E_e) are i.i.d. exponentials with mean 1, and k is a large constant. This setting is, apart from a trivial time-rescaling, identical to the setting where $X_e = k + E_e$. In this case, one would expect that for large k , H_n is close to the graph distance between a pair of uniformly chosen vertices in $[n]$, conditioned to be connected. This problem has attracted considerable attention, see, in particular, [25] for the Norros-Reittu model and [30] for the CM. In these works, it has been shown that $(H_n - \log_\nu(n))_{n \geq 1}$ is a tight sequence of random variables. This suggests (compare with Theorems 1.2-1.3) that, as $k \rightarrow \infty$,

$$\alpha \rightarrow \log \nu, \quad \bar{\nu} \rightarrow 1, \quad \frac{\bar{\sigma}^2}{\bar{\nu}^3 \alpha} \rightarrow 0. \quad (1.35)$$

We now check this intuitive argument. Indeed,

$$\nu \int_0^\infty e^{-\alpha x} dG(x) = \nu k \int_1^\infty e^{-\alpha x} e^{-k(x-1)} dx = \frac{\nu k}{\alpha + k} e^{-\alpha} = 1. \quad (1.36)$$

While solving this equation *explicitly* is hard, it is not too difficult to see that $k \rightarrow \infty$ implies that $\alpha \rightarrow \log \nu$.

We can compute the stable-age distribution as $1 + \text{Exp}(k + \alpha)$, so that $\bar{\nu} = 1 + \frac{1}{k + \alpha}$, while $\bar{\sigma}^2 = 1/(k + \alpha)^2 \rightarrow 0$. Therefore, $\bar{\nu} \sim 1$, which in turn also implies that $\alpha \bar{\nu} \rightarrow \log \nu$. Also,

$$\frac{\bar{\sigma}^2}{\bar{\nu}^3 \alpha} \sim k^{-2} (\log \nu)^{-1} \rightarrow 0. \quad (1.37)$$

This shows that the two settings of graph distances and FPP with weights $1 + \text{Exp}(1)/k$ match up nicely when $k \rightarrow \infty$.

Weak disorder on random regular graph with large degree. As a third example we consider the configuration model with fixed degrees r , and where each edge is given an edge weight E^s , $s > 0$ where $E \sim \text{Exp}(1)$. The parameter s plays the role of inverse temperature in statistical physics with $s \rightarrow \infty$ corresponding to the minimal spanning tree with exponential edge weights. This setting has been studied on the complete graph in [8], and here we make the connection to the results proved there.

In this case, $\nu = r - 1$. The Malthusian parameter α satisfies (compare (1.14)),

$$(r - 1)p \int_0^\infty e^{-\alpha x - x^p} x^{p-1} dx = 1, \quad (1.38)$$

where $p = 1/s$. We can not solve (1.38) explicitly, but when $r \rightarrow \infty$, we conclude that $\alpha \rightarrow \infty$, so that the above equation is close to

$$rp \int_0^\infty e^{-\alpha x} x^{p-1} dx = rp \alpha^{-p} \Gamma(p) = r \alpha^{-p} \Gamma(p + 1) = 1, \quad (1.39)$$

¹In [9, (3.7)], the Gumbel variable Λ , which appears as $\log M$ in [9, (C.19)], where M is an exponential random variable, should be replaced with $-\Lambda$.

²Also in [9, (5.4)] there is an error in the precise limiting random variable for FPP on the Erdős-Rényi random graph due to the fact that [9, (4.16)] is not correct.

so that

$$\alpha = (r\Gamma(1 + 1/s))^s. \quad (1.40)$$

The moments $\bar{\nu}$ and $\bar{\sigma}^2$ are then approximately given by

$$\bar{\nu} \approx rp \int_0^\infty e^{-\alpha x} x^p dx = rp\alpha^{-(p+1)}\Gamma(p+1) = p/\alpha, \quad (1.41)$$

and

$$\bar{\sigma}^2 \approx rp \int_0^\infty e^{-\alpha x} x^{p+1} dx - \bar{\nu}^2 = rp\alpha^{-(p+2)}\Gamma(p+2) - (p/\alpha)^2 = p/\alpha^2, \quad (1.42)$$

where we repeatedly use (1.40). As a result, we obtain

$$\gamma = \frac{1}{\alpha\bar{\nu}} \approx 1/p = s, \quad \beta = \frac{\bar{\sigma}^2}{\bar{\nu}^3\alpha} \approx 1/p^2 = s^2. \quad (1.43)$$

These results match up nicely with the result on the complete graph, obtained when $r = n - 1$, for which [8] show that a central limit theorem holds for H_n with asymptotic mean $s \log n$ and asymptotic variance $s^2 \log n$, while $n^s[L_n - \frac{1}{\lambda} \log n]$ converges in distribution, where $\lambda = \Gamma(1 + 1/s)^s$.

1.6 Discussion

In this section, we discuss our results, possible extensions and open problems.

(a) **Universality.** As Theorems 1.2-1.3 show, even second order asymptotics for the hopcount in the presence of disorder in the network depends only on the first two moments of the size-biased offspring distribution and on the edge-weight distribution, but not on any other property of the network model. Further, the limit distribution of the minimal weight between two uniform vertices conditioned on being connected has a universal shape, even though the martingale limit of the flow naturally strongly depends both on the graph topology as well as on the edge weight distribution.

(b) **Divergence from the mean-field setting.** One famous model that has witnessed an enormous amount of interest in probabilistic combinatorial optimization is the mean-field model, where one starts with the complete graph on n vertices and then attaches random edge weights and analyzes optimal path structure. See [36] for a study on the effect of exponential edge weights on the geometry of the graph, [27] for study of the minimal spanning tree and [3] for a discussion of a number of other problems. Branching process methods have been used to good effect in this setting as well to analyze the effect of random disorder on the geometry of the graph, see [8] for example and often give good heuristics for what one would expect in the sparse random graph setting. However, what the main theorems imply is that in a number of situations, the mean-field setting diverges markedly from the random graph setting. For example, when each edge has E^{-s} weight where E has an exponential distribution and $s > 0$, one can show that in fact the hopcount between typical vertices converges to a constant [9], while Theorem 1.3 implies that even in this case, for the CM, the hopcount scales as $\log n$ and satisfies a CLT.

(c) **Infinite variance configuration model.** In [10], we have also investigated the setting where the degrees are i.i.d. with $\mathbb{E}[D^2] = \infty$ and with exponential edge weights. In this case, the result for L_n is markedly different, in the sense that L_n converges in distribution without re-centering. Further, H_n satisfies a central limit with asymptotic mean and variance equal to a multiple of $\log n$. Now, when taking $X_e \sim 1 + \text{Exp}(1)$, by [31], there are paths of length $\log \log n$ connecting vertices U_1 and U_2 , conditionally on U_1 and U_2 being connected. Since the weight of such a path is of the same order, we conclude that $H_n, L_n = \Theta_{\mathbb{P}}(\log \log n)$. Thus, in such cases it is possible that H_n acts on a different scale, even though $H_n \xrightarrow{\mathbb{P}} \infty$. It would be of interest to

investigate whether H_n always satisfies a central limit theorem, and, if so, whether the order of magnitude of its variance is always equal to that of its mean.

(d) **The $X \log X$ -condition.** By Condition 1.1(c), we assume that the degrees satisfy a second moment condition with an additional logarithmic factor. This is equivalent to the CTBP satisfying an $X \log X$ condition (uniformly in the size n of the graph). It would be of interest to investigate what happens when this condition fails. It is well known that the branching process martingale limit is identically equal to 0 when $\mathbb{E}[X] < \infty$, but $\mathbb{E}[X \log(X)_+] = \infty$ (see e.g., [6] or [33], or [46]). Therefore, the limit in (1.20) does not exist. This suggests that (1.10) should be replaced with $L_n - \frac{2}{\alpha_n} \log(t_n)$, where t_n is such that $|\text{BP}^*(t_n)|n^{-1/2}$ has a non-trivial limit.

(e) **Flooding and diameter.** In [5], the *flooding time* and *diameter*, i.e., $\max_{j \in [n]: L_{U_1, j} < \infty} L_{U_1, j}$, respectively $\max_{i, j \in [n]: L_{i, j} < \infty} L_{i, j}$, where $L_{i, j}$ is the minimal weight between the vertices i and j and U_1 is, as before, a randomly selected vertex, is investigated in the context of the CM with exponential edge weights. It would be of interest to investigate the flooding time for general edge weights. We expect that the exponential distribution is special, since there the typical weight has the same order of magnitude as the maximum over the vertices of the minimal edge weight from that vertex. This fact is only true when the weight distribution has an exponential tail. For example, taking $X_e = E_e^s$ for $s > 1$, the maximal minimal weight from a vertex is of order $(\log n)^s$, which is much larger than the typical distance, which is $\Theta_{\mathbb{P}}(\log n)$ due to our main results. It would be interesting to investigate what the limit of the weight diameter is in this simple example.

(f) **Superconcentration and chaos.** In this study we have looked at some global functionals of the optimal path between randomly selected vertices and in particular have shown that the weight of the optimal path satisfies $L_n / \log n = O_{\mathbb{P}}(1)$. Analogous to various related problems in statistical physics such as random polymers, or FPP on the lattice, this suggests that the optimal path problem satisfies superconcentration. In particular, it suggests that this random combinatorial optimization problem is *chaotic* in the sense that there exists $\varepsilon_n \rightarrow 0$ such that refreshing a fraction ε_n of the edge weights with new random variables with the same distribution would entirely change the actual optimal path, in the sense that the new optimal path would be “almost” disjoint of the original optimal path, see e.g. [17]. Such questions have also arisen in computer science wherein one is interested in judging the “importance” and fair price of various edges in the optimal path; if an edge being deleted causes a large change in the cost of the new optimal path, then that edge is deemed very valuable. These form the basis of various “truth and auction mechanisms” in computer science (see e.g. [47], [26], [7]). It would be interesting to derive rigorous results in our present context.

(g) **Pandemics, gossip and other models of diffusion:** First passage percolation models as well as models using FPP as a building block have started to play an increasingly central role in the applied probability community in describing the flow of materials, ranging from viral epidemics ([23]), gossip algorithms ([4]) and more general finite Markov interchange processes ([2]). Models with more general edge distributions have also arisen in understanding the flow of information and reconstruction of such information networks in sociology and computer science, see e.g. [43], [44] for just some examples in this vast field.

2 Proof: construction of the flow clusters

We start with some central constructions that lay the ground work for the proofs of the main results. We denote by U_1 and U_2 two randomly selected vertices, conditioned on being connected. We think of the weights as edge lengths so that they induce a random metric on the graph $\text{CM}_n(\mathbf{d})$. For a half-edge y , we let P_y denote the half-edge to which it is paired, i.e., (y, P_y) forms an edge. Further, we let V_y be the vertex to which the half-edge y is incident.

2.1 Flow clusters from U_1 and U_2

To understand the shortest path between these vertices, think of water percolating through the network at rate one, started simultaneously from the two vertices. For any $t \geq 0$, the set of vertices first seen by the flow from U_i will often be referred to the flow cluster or the shortest weight graph of vertex U_i . When the two flows collide or create prospective collision edges, then these generate prospective shortest paths.

Let us now give a precise mathematical formulation to the above description. We grow two flow clusters (i.e. two stochastic processes in continuous time) from U_1 and U_2 , simultaneously. The main ingredients of the two flow clusters, namely the alive set $A(t)$ will only change at random times $T_0 = 0 < T_1 < T_2 < \dots$ and therefore the definition can be given recursively. At time $t = T_0 = 0$, the vertices U_1 and U_2 die instantaneously, and give rise to d_{U_1} and d_{U_2} children. These children correspond to half-edges incident to U_1 and U_2 . We start by testing whether any of the half-edges incident to U_1 are paired to one another. If so, then we remove both half-edges from the total set of d_{U_1} half-edges. We then define $X_0^{(1)}$ the number of unpaired half-edges after the self-loops incident to U_1 are removed. We next continue with the d_{U_2} half-edges incident to U_2 , and check whether they are paired to one of the $X_0^{(1)}$ remaining half-edges incident to U_1 or any of the d_{U_2} half-edges incident to U_2 . When such a half-edge is paired to one of the d_{U_2} sibling half-edges, a self-loop is formed. When such a half-edge is paired to one of the $X_0^{(1)}$ remaining half-edges incident to vertex U_1 , a so-called *collision edge* is formed. A collision possibly yields the path with minimal weight between U_1 and U_2 . We let $X_0^{(2)}$ denote the number of unpaired half-edges after the tests for collision edges and cycles have been performed. Note that, by construction, each of the $X_0^{(i)}$ half-edges incident to the vertices U_i , where $i \in \{1, 2\}$, are paired to *new* vertices, i.e., vertices distinct from U_1 and U_2 .

For the moment we collect the collision edges at time T_0 , together with the weights of the connecting edge between U_1 and U_2 , and continue with the description of the flow clusters. All edges that are not paired to one of the other $d_{U_1} + d_{U_2} - 1$ half-edges incident to either U_1 or U_2 together form the set $A(0)$, the set of active half-edges at time 0. For $y \in A(0)$, we define $I(y) = i$ if the half-edge y is connected to U_i , $i = 1, 2$, and we define $(R_0(y))_{y \in A(0)}$ as an i.i.d. sequence of life times having distribution function G .

We denote the set of half-edges at time t by $A(t)$. For $y \in A(t)$, we record $I(y)$, which is the index $i \in \{1, 2\}$ to which U_i the half-edge is connected, and we let $H(y)$ denote the height of y to $U_{I(y)}$. This height equals 0 for $y \in A(0)$. When we introduce new half-edges at $A(t)$ at later times we will specify the height of these half-edges. Now define $T_1 = \min_{y \in A(0)} R_0(y)$ and denote by y_0^* the half-edge equal to the argument of this minimum, hence $R_0(y_0^*) = \min_{y \in A(0)} R_0(y)$. Since life-times have a continuous distribution, y_0^* is a.s. *unique*. Now set $A(t) = A(0)$, $0 \leq t < T_1$, i.e., the active set remains the same during the interval $[0, T_1)$, and define the flow cluster $\text{SWG}(t)$, for $0 \leq t < T_1$, by

$$\text{SWG}(t) = \{y, I(y), H(y), R_t(y)\}_{y \in A(t)}, \quad (2.1)$$

where $I(y)$ and $H(y)$ are defined above and $R_t(y) = R_0(y) - t$, $0 \leq t \leq T_1$, denotes the remaining lifetime of half-edge y . This concludes the initial step in the recursion, where we defined $A(t)$ and $\text{SWG}(t)$ during the random interval $[T_0, T_1)$.

We continue using induction, by defining $A(t)$ and $\text{SWG}(t)$ during the random interval $[T_k, T_{k+1})$, given that the processes are defined on $[0, T_k)$. At time $t = T_k$, we remove y_{k-1}^* from the set $A(t-)$. By construction, we know that $z_k \equiv P_{y_{k-1}^*} \notin A(t-)$, so that V_{z_k} is not a vertex that has been reached by the flow at time t . Then, for each of the $d_{V_{z_k}} - 1$ other half-edges incident to vertex V_{z_k} we test whether it is part of a self-loop or paired to a half-edge from the set $A(t-)$. All half-edges incident to V_{z_k} that are part of a self-loop or incident to $A(t-)$ are removed from vertex V_{z_k} ; we also remove the involved half-edges from the set $A(t-)$. We will discuss the role

of the half-edges incident to V_{z_k} that are paired to half-edges in $\mathbf{A}(t-)$ in more detail below.

For all the remaining siblings of z_k we do the following: Let x be one such half-edge of V_{z_k} , then x is added to $\mathbf{A}(T_k)$, $I(x) = I(y_{k-1}^*)$, $H(x) = H(y_{k-1}^*) + 1$, while $R_{T_k}(x)$ is an i.i.d. life time with distribution G . We now set $\mathbf{A}(t) = \mathbf{A}(T_k)$, $T_k \leq t < T_{k+1}$, where $T_{k+1} = T_k + \min_{y \in \mathbf{A}(T_k)} R_{T_k}(y)$, and where the minimizing half-edge is called y_k^* . Furthermore, for $t \in [T_k, T_{k+1})$, we can define $\text{SWG}(t)$ by (2.1), where $R_t(y) = R_{T_k}(y) - (t - T_k)$. Finally, we denote the number of the $d_{V_{z_k}} - 1$ other half-edges incident to vertex V_{z_k} that do not form a self-loop and that are not paired to a half-edge from the set $\mathbf{A}(t-)$ by X_k . Later, it will also be convenient to introduce $B_k = d_{V_{z_k}} - 1$. Let $S_k = |\mathbf{A}(T_k)|$, so that $S_0 = X_0^{(1)} + X_0^{(2)}$, while S_k satisfies the recursion

$$S_k = S_{k-1} + X_k - 1. \quad (2.2)$$

This describes the evolution of $(\text{SWG}(t))_{t \geq 0}$.

Cycle edges and collision edges. At the times T_k , $k \geq 1$, we find the half-edge y_{k-1}^* which is paired to $z_k = P_{y_{k-1}^*}$, and for each of the other half-edges x incident to V_{z_k} , we check whether or not $P_x \in \mathbf{A}(T_k-)$. The half-edges paired to alive half-edges in $\mathbf{A}(T_k-)$ are special. Indeed, the edge (x, P_x) creates a cycle when $I(x) = I(P_x)$ while (x, P_x) completes a path between U_1 and U_2 , when $I(x) = 3 - I(P_x)$. Precisely the latter edges can create the shortest-weight path between U_1, U_2 . Let us describe these collision edges in more detail.

At time T_k and when we create a collision edge consisting of x_k and P_{x_k} , then we record

$$\left((T_k, I(z_k), H(z_k), H(P_{x_k}), R_{T_k}(P_{x_k})) \right)_{k \geq 0}. \quad (2.3)$$

It is possible that multiple half-edges incident to V_{z_k} create collision edges, and if so, we collect all of them in the list in (2.3). In this definition it is tempting to write $I(x_k)$ and $H(x_k)$, but note that $x_k \notin \mathbf{A}(T_k)$, whereas its sibling half-edge $z_k \in \mathbf{A}(T_k)$, and, moreover, x_k and z_k have the same ancestor and the same height. With some abuse of notation we denote the i th collision edge by (x_i, P_{x_i}) ; here P_{x_i} is an alive half-edge and x_i the half-edge which pairs to P_{x_i} ; further z_i is the sibling of x_i paired with the minimal edge y^* found by the flow. Let $T_i^{(\text{col})}$ be the time of creation of the i th collision edge. The weight of the (unique) path between U_1 and U_2 that passes through the edge consisting of x_i and P_{x_i} equals $2T_i^{(\text{col})} + R_{T_i^{(\text{col})}}(P_{x_i})$, so that the shortest weight equals:

$$L_n = \min_{i \geq 0} [2T_i^{(\text{col})} + R_{T_i^{(\text{col})}}(P_{x_i})]. \quad (2.4)$$

Let I^* denote the minimizer of $i \mapsto 2T_i^{(\text{col})} + R_{T_i^{(\text{col})}}(P_{x_i})$, then

$$H_n = H(z_{I^*}) + H(P_{x_{I^*}}) + 1. \quad (2.5)$$

Of course, (2.4) and (2.5) need a proof, which we give now.

Proof that L_n given by (2.4) yields the minimal weight. Observe that each path between U_1 and U_2 has a weight L that can be written in the form $2T_i + R_{T_i}(P_{x_i})$ for some $i \geq 0$. Indeed, let $(i_0 = U_1, i_1, i_2, \dots, i_k = U_2)$ form a path with weight L , and denote the weight on $i_{j-1}i_j$ by X_{e_j} for $1 \leq j \leq k$. For $k = 1$, we obviously find $X_{e_1} = 2T_0 + X_{e_1}$. For general $k \geq 1$, take the maximal $j \geq 0$ such that $X_{e_1} + \dots + X_{e_j} \leq L/2$. Then, we write

$$L = \begin{cases} 2 \sum_{s=1}^j X_{e_s} + [\sum_{s=j+1}^k X_{e_s} - \sum_{s=1}^j X_{e_s}], & \text{when } \sum_{s=1}^j X_{e_s} < \sum_{s=j+1}^k X_{e_s}, \\ 2 \sum_{s=j+1}^k X_{e_s} + [\sum_{s=1}^j X_{e_s} - \sum_{s=j+1}^k X_{e_s}], & \text{when } \sum_{s=1}^j X_{e_s} > \sum_{s=j+1}^k X_{e_s}, \end{cases}$$

which in either case is of the form $L = 2T_m + R_{T_m}(y)$, for some $m \geq 0$ and some half-edge y . Note that in the construction of the flow clusters, instead of putting weight on the edges, we have given weights to half-edges instead. In the representation (2.4) full edge weight is given to the active half-edges and weight 0 to the ones with which they are paired. When the collision edge has been found we give the full weight to the *parent*-edge P_x . So, in fact, by the redistribution of the weights in (2.4) is an equality in distribution. This completes the proof of the claim. \blacksquare

Remark 2.1 (On the number of collision edges) *We do not have to find all collision edges. The recursion can be stopped when $T_k > L/2$ for some $k \geq 1$, where L is the weight of one the collision edges found previously. This is immediately clear, since all collision edges found at T_k or later have weight exceeding $2T_k > L$.*

2.2 Main result: Poisson Point Process limit

In this section, we state our main result, which will imply Theorems 1.2- 1.3.

Basic constructions and properties. To state our main technical result concerning the appearance of collision edges, we need to define some new constructs. We start by defining a rescaled version of the point process corresponding to the points in (2.3). Let us first setup some notation. For $i \in \{1, 2\}$ and $t \geq 0$, we let

$$|\text{SWG}(t)| = \#\{y \in \mathbf{A}(t)\}, \quad |\text{SWG}^{(i)}(t)| = \#\{y \in \mathbf{A}(t) : I(y) = i\}, \quad (2.6)$$

be the number of alive half-edges at time t , as well as those that are closest to vertex i . By construction, since we check whether the half-edges form a cycle or a collision edge when the half-edges are born, $\text{SWG}^{(1)}(t)$ and $\text{SWG}^{(2)}(t)$ are *disjoint*. Consider the filtration $(\mathcal{F}_s)_{s \geq 0}$ with $\mathcal{F}_s = \sigma((\text{SWG}(t))_{t \in [0, s]})$ denoting the sigma-algebra generated by the shortest-weight graph up to time s .

Fix a deterministic sequence $s_n \rightarrow \infty$ that will be chosen later on. Now let

$$t_n = \frac{1}{2\alpha_n} \log n, \quad \bar{t}_n = \frac{1}{2\alpha_n} \log n - \frac{1}{2\alpha_n} \log (\mathcal{W}_{s_n}^{(1)} \mathcal{W}_{s_n}^{(2)}), \quad (2.7)$$

where, for $s \geq 0$,

$$\mathcal{W}_s^{(i)} = e^{-\alpha_n s} |\text{SWG}^{(i)}(s)|. \quad (2.8)$$

Note that $e^{\alpha_n t_n} = \sqrt{n}$, so that at time t_n , both $|\text{SWG}^{(i)}(s)|$ are of order \sqrt{n} ; consequently the variable t_n denotes the typical time at which collision edges start appearing, and the time \bar{t}_n incorporates for stochastic fluctuations in the size of the SWGs. The precise rate at which $s_n \rightarrow \infty$ for asymptotic properties of the construction to hold is determined in the proof of Proposition 2.4 below. In particular we choose $s_n \rightarrow \infty$ such that $\text{SWG}^{(i)}(t)$ for $t \leq s_n$ can be coupled with two independent two-stage branching processes $\text{BP}^{(i)}(t)$ such that with high probability $\{\text{BP}(t) = |\text{SWG}(t)|\} \forall t \leq s_n$.

Define the *residual life-time distribution* F_R to have density f_R given by

$$f_R(x) = \frac{\int_0^\infty e^{-\alpha y} g(x+y) dy}{\int_0^\infty e^{-\alpha y} [1 - G(y)] dy}. \quad (2.9)$$

Recall that the i th collision edge is given by (x_i, P_{x_i}) , where P_{x_i} is an alive half-edge and x_i the half-edge which pairs to P_{x_i} . In terms of the above definitions, we define

$$\bar{T}_i^{(\text{col})} = T_i^{(\text{col})} - \bar{t}_n, \quad \bar{H}_i^{(\text{or})} = \frac{H(x_i) - t_n/\bar{\nu}_n}{\sqrt{\bar{\sigma}^2 t_n/\bar{\nu}^3}}, \quad \bar{H}_i^{(\text{de})} = \frac{H(P_{x_i}) - t_n/\bar{\nu}_n}{\sqrt{\bar{\sigma}^2 t_n/\bar{\nu}^3}}, \quad (2.10)$$

and write the random variables $(\Xi_i)_{i \geq 1}$ with $\Xi_i \in \mathbb{R} \times \{1, 2\} \times \mathbb{R} \times \mathbb{R} \times [0, \infty)$, by

$$\Xi_i = (\bar{T}_i^{(\text{col})}, I(x_i), \bar{H}_i^{(\text{or})}, \bar{H}_i^{(\text{de})}, R_{T_i}(P_{x_i})). \quad (2.11)$$

Then, for sets A in the Borel σ -algebra of the space $\mathcal{S} := \mathbb{R} \times \{1, 2\} \times \mathbb{R} \times \mathbb{R} \times [0, \infty)$, we define the point process

$$\Pi_n(A) = \sum_{i \geq 1} \delta_{\Xi_i}(A), \quad (2.12)$$

where δ_x gives measure 1 to the point x . Let $\mathcal{M}(\mathcal{S})$ denote the space of all simple locally finite point processes on \mathcal{S} equipped with the vague topology (see e.g. [41]). On this space one can naturally define the notion of weak convergence of a sequence of random point processes $\Pi_n \in \mathcal{M}(\mathcal{S})$. This is the notion of convergence referred to in the following theorem. In the theorem, we let Φ denote the distribution function of a standard normal random variable.

Theorem 2.2 (PPP limit of collision edges) *Consider the distribution of the point process $\Pi_n \in \mathcal{M}(\mathcal{S})$ defined in (2.12) conditional on $(\text{SWG}(s))_{s \in [0, s_n]}$ such that $\mathcal{W}_{s_n}^{(1)} > 0$ and $\mathcal{W}_{s_n}^{(2)} > 0$. Then Π_n converges in distribution as $n \rightarrow \infty$ to a Poisson Point Process (PPP) Π with intensity measure*

$$\lambda(dt \times i \times dx \times dy \times dr) = \frac{2\nu f_R(0)}{\mu} e^{2\alpha t} dt \otimes \{1/2, 1/2\} \otimes \Phi(dx) \otimes \Phi(dy) \otimes F_R(dr). \quad (2.13)$$

Completion of the proof of Theorems 1.2, 1.3 and 1.5. Let us now prove Theorem 1.2 subject to Theorem 2.2. First of all, by (2.10), (2.4) and (2.5) and Remark 2.1,

$$\left(\frac{H_n - \frac{1}{\alpha_n \bar{\nu}} \log n}{\sqrt{\frac{\bar{\sigma}^2}{\bar{\nu}^3 \alpha} \log n}}, L_n - \frac{1}{\alpha_n} \log n \right), \quad (2.14)$$

is a continuous function of the point process Π_n , and, therefore, by the continuous mapping theorem, the above random variable converges in distribution to some limiting random variables (Z, Q) .

Recall that I^* denotes the minimizer of $i \mapsto 2T_i^{(\text{col})} + R_{T_i^{(\text{col})}}(P_{x_i})$. By (2.4), the weight L_n as well as the value of I^* , are functions of the first and the last coordinates of Π_n . The hopcount H_n is a function of the third and the fourth, instead. By the product form of the intensity in (2.13), we obtain that the limits (Z, Q) are independent. Therefore, it suffices to study their marginals. The same observation applies to the multiple path problem in Theorem 1.5.

We start with the limiting distribution of the hopcount. By (2.10),

$$\frac{H_n - \frac{1}{\alpha_n \bar{\nu}_n} \log n}{\sqrt{\frac{\bar{\sigma}^2}{\bar{\nu}^3 \alpha} \log n}} = \frac{1}{2} \sqrt{2} \bar{H}_{I^*}^{(\text{or})} + \frac{1}{2} \sqrt{2} \bar{H}_{I^*}^{(\text{de})} + o_{\mathbb{P}}(1). \quad (2.15)$$

By Theorem 2.2, the random variables $(\bar{H}_{I^*}^{(\text{or})}, \bar{H}_{I^*}^{(\text{de})})$, converge to two independent standard normals, so that also the left-hand side of (2.15) converges to a standard normal.

The limiting distribution of the weight L_n is slightly more involved. By (2.7), (2.4) and (2.10),

$$\begin{aligned} L_n - \frac{1}{\alpha_n} \log n &= L_n - 2t_n = L_n - 2\bar{t}_n - \frac{1}{\alpha_n} \log(\mathcal{W}_{s_n}^{(1)} \mathcal{W}_{s_n}^{(2)}) \\ &= -\frac{1}{\alpha_n} \log(\mathcal{W}_{s_n}^{(1)} \mathcal{W}_{s_n}^{(2)}) + \min_{i \geq 1} [2T_i^{(\text{col})} + R_{T_i^{(\text{col})}}(P_{x_i})] - 2\bar{t}_n \\ &= -\frac{1}{\alpha_n} \log(\mathcal{W}_{s_n}^{(1)} \mathcal{W}_{s_n}^{(2)}) + \min_{i \geq 1} [2\bar{T}_i^{(\text{col})} + R_{T_i^{(\text{col})}}(P_{x_i})]. \end{aligned} \quad (2.16)$$

By Proposition 2.4 below, $(\mathcal{W}_{s_n}^{(1)}, \mathcal{W}_{s_n}^{(2)}) \xrightarrow{d} (\mathcal{W}^{(1)}, \mathcal{W}^{(2)})$, which are two independent copies of the random variable in (1.18). Hence,

$$L_n - \frac{1}{\alpha_n} \log n \xrightarrow{d} -\frac{1}{\alpha} \log(\mathcal{W}^{(1)}\mathcal{W}^{(2)}) + \min_{i \geq 1} [2P_i + R_i], \quad (2.17)$$

where $(P_i)_{i \geq 1}$ form a PPP with intensity $\frac{2\nu f_R(0)}{\mu} e^{2\alpha t} dt$, and $(R_i)_{i \geq 1}$ are i.i.d. random variables with distribution function F_R independently of $(P_i)_{i \geq 1}$.

We next identify the distribution of $M = \min_{i \geq 1} [2P_i + R_i]$. First, $(2P_i)_{i \geq 1}$ forms a Poisson process with intensity $\frac{\nu f_R(0)}{\mu} e^{\alpha t} dt$. According to [52, Example 3.3 on page 137] the point process $(2P_i + R_i)_{i \geq 1}$ is a non-homogeneous Poisson process with mean-measure the convolution of $\mu(-\infty, x] = \int_{-\infty}^x \frac{\nu f_R(0)}{\mu} e^{\alpha t} dt$ and F_R . Hence $\mathbb{P}(M \geq x)$ equals the Poisson probability of 0, where the parameter of the Poisson distribution is $(\mu * F_R)(x)$, so that

$$\mathbb{P}(M \geq x) = \exp\left\{-\frac{\nu f_R(0)}{\mu} e^{\alpha x} \int_0^\infty F_R(z) e^{-\alpha z} dz\right\}. \quad (2.18)$$

Let Λ have a Gumbel distribution, i.e., $\mathbb{P}(\Lambda \leq x) = e^{-e^{-x}}$, $x \in \mathbb{R}$, then

$$\mathbb{P}(-a\Lambda + b \geq x) = e^{-e^{x/a} e^{-b/a}}. \quad (2.19)$$

From the identity:

$$\frac{\nu f_R(0)}{\mu} e^{\alpha x} \int_0^\infty F_R(z) e^{-\alpha z} dz = e^{x/a} e^{-b/a},$$

we conclude that if we take $a = 1/\alpha$ and $b = -\alpha^{-1} \log\left((\nu f_R(0)/\mu) \int_0^\infty F_R(z) e^{-\alpha z} dz\right)$, then

$$\min_{i \geq 1} (2P_i + R_i) \stackrel{d}{=} -\alpha^{-1} \Lambda - \alpha^{-1} \log(\nu f_R(0)B/\mu), \quad (2.20)$$

with $B = \int_0^\infty F_R(z) e^{-\alpha z} dz$. In the following lemma, we simplify these constants:

Lemma 2.3 (The constant) *The constants $B = \int_0^\infty F_R(z) e^{-\alpha z} dz$ and $f_R(0)$ are given by*

$$B = \bar{\nu}/(\nu - 1), \quad f_R(0) = \alpha/(\nu - 1). \quad (2.21)$$

Consequently, the constant c in the limit variable (1.20) equals

$$c = -\log(\nu f_R(0)B/\mu) = \log(\mu(\nu - 1)^2/(\alpha\nu\bar{\nu})). \quad (2.22)$$

Proof. We start by computing $f_R(0)$, for which we note that by (2.9) and (1.14),

$$f_R(0) = \frac{\int_0^\infty e^{-\alpha y} g(y) dy}{\int_0^\infty e^{-\alpha y} [1 - G(y)] dy} = \left(\nu \int_0^\infty e^{-\alpha y} [1 - G(y)] dy\right)^{-1}. \quad (2.23)$$

Further, by partial integration,

$$\int_0^\infty e^{-\alpha y} [1 - G(y)] dy = \left[-\frac{1}{\alpha} e^{-\alpha y} [1 - G(y)]\right]_{y=0}^\infty - \frac{1}{\alpha} \int_0^\infty e^{-\alpha y} g(y) dy = \frac{1}{\alpha} - \frac{1}{\alpha\nu} = \frac{\nu - 1}{\alpha\nu}, \quad (2.24)$$

where we again use (1.14). Combining both equalities yields $f_R(0) = \alpha/(\nu - 1)$.

For B , we again use partial integration, followed by the substitution of (2.9); this yields

$$\begin{aligned} B &= \int_0^\infty F_R(z) e^{-\alpha z} dz = \frac{1}{\alpha} \int_0^\infty f_R(z) e^{-\alpha z} dz \\ &= \frac{\nu}{\nu-1} \int_0^\infty e^{-\alpha z} \int_0^\infty e^{-\alpha y} g(y+z) dy dz, \end{aligned} \quad (2.25)$$

by (2.24). The final integral can be computed using

$$\begin{aligned} &\int_{-\infty}^\infty e^{-\alpha z} \mathbb{1}_{\{z \geq 0\}} \int_{-\infty}^\infty e^{-\alpha y} g(y+z) \mathbb{1}_{\{y \geq 0\}} dy dz \\ &= \int_0^\infty s g(s) e^{-\alpha s} ds = \frac{1}{\nu} \int_0^\infty s \bar{G}(ds) = \bar{\nu}/\nu. \end{aligned} \quad (2.26)$$

This completes the proof of both Theorem 1.2 and Theorem 1.3, given Theorem 2.2. \blacksquare

2.3 Overview of the proof of Theorem 2.2

In this section, we reduce the proof of Theorem 2.2 to two main propositions. Recall the shortest weight graph or flow cluster $\text{SWG}(t)$ defined in the previous section as well as the associated filtration $(\mathcal{F}_t)_{t \geq 0}$. We shall couple these flow clusters from two points with $(\text{BP}(t))_{t \geq 0}$ where $\text{BP}(t) = (\text{BP}^{(1)}(t), \text{BP}^{(2)}(t))$ are two independent CTBPs starting with offspring distribution D . For a prescribed such coupling of $(\text{SWG}(t))_{t \geq 0}$ and $(\text{BP}(t))_{t \geq 0}$, we let $\text{SWG}(t) \triangle \text{BP}(t)$ denote the set of miscoupled half-edges at time t . Then we prove the following limiting result:

Proposition 2.4 (Coupling the SWG to a BP) (a) *There exists $s_n \rightarrow \infty$ and a coupling of $(\text{SWG}(s))_{s \geq 0}$ and $(\text{BP}(s))_{s \geq 0}$ such that*

$$\mathbb{P}\left(\left|\text{SWG}(s)_{s \in [0, s_n]} = \text{BP}(s)_{s \in [0, s_n]}\right| \right) = 1 - o(1). \quad (2.27)$$

Consequently, with $\mathcal{W}_{s_n}^{(i)} = e^{-\alpha_n s_n} |\text{SWG}^{(i)}(s_n)|$,

$$\liminf_{\varepsilon \downarrow 0} \liminf_{n \rightarrow \infty} \mathbb{P}\left(\mathcal{W}_{s_n}^{(1)} \in [\varepsilon, 1/\varepsilon], \mathcal{W}_{s_n}^{(2)} \in [\varepsilon, 1/\varepsilon] \mid \mathcal{W}_{s_n}^{(1)} > 0, \mathcal{W}_{s_n}^{(2)} > 0\right) = 1. \quad (2.28)$$

(b) *There exists a coupling of $(\text{SWG}(s))_{s \geq 0}$ and $(\text{BP}_{(n)}(s))_{s \geq 0}$, and sequences $\varepsilon_n = o(1)$ and $B_n \rightarrow \infty$ such that, conditionally on \mathcal{F}_{s_n} ,*

$$\mathbb{P}\left(\left|\text{SWG}(t_n + B_n) \triangle \text{BP}_{(n)}(t_n + B_n)\right| \geq \varepsilon_n \sqrt{n} \mid \mathcal{F}_{s_n}\right) \xrightarrow{\mathbb{P}} 0, \quad (2.29)$$

where $(\text{BP}_{(n)}(t))_{t \geq 0} = (\text{BP}_{(n)}^{(1)}(t), \text{BP}_{(n)}^{(2)}(t))_{t \geq 0}$ and $(\text{BP}_{(n)}^{(1)}(s))_{s \geq s_n}, (\text{BP}_{(n)}^{(2)}(s))_{s \geq s_n}$ are two independent two-stage Bellman-Harris processes with offspring $D_n^* - 1$ (where D_n^* has the size-biased distribution F_n^* of F_n , see (1.11)) for every individual, and edge weights with continuous distribution function G , and starting at time s_n in $\text{BP}(s_n)$ from part (a), respectively.

The proof of Proposition 2.4 is deferred to Section 4. In the sequel, we shall assume that \mathbb{P} denotes the coupling measure from Proposition 2.4. In particular, this yields a coupling between $\text{CM}_n(\mathbf{d})$ for different $n \geq 1$, as well as a coupling between $\text{CM}_n(\mathbf{d})$ and the n -dependent branching processes $(\text{BP}(s))_{s \geq 0}$. Under this coupling law, we can speak of convergence in probability, and we shall frequently do this in the sequel.

For $i \in \{1, 2\}$, $k \geq 0$, and $t \geq 0$, we define

$$|\text{SWG}_k^{(i)}[t, t+s]| = \#\{y \in \mathbf{A}(t) : I(y) = i, H(y) = k, R_t(y) \in [0, s]\}, \quad (2.30)$$

as the number of alive half-edges at time t that (a) are in the SWG of vertex U_i , (b) have height k , and (c) have remaining lifetime at most s . We further write

$$|\text{SWG}_{\leq k}^{(i)}[t, t+s]| = \#\{y \in \mathbf{A}(t) : I(y) = i, H(y) \leq k, R_t(y) \in [0, s]\}, \quad (2.31)$$

for the number of vertices that have height at most k . To formulate the CLT for the height of vertices, we will choose

$$k_t(x) = \frac{t}{\bar{\nu}} + x\sqrt{t\frac{\bar{\sigma}^2}{\bar{\nu}^3}}. \quad (2.32)$$

Finally, for a half-edge $y \in \mathbf{A}(t)$, we let $X_y^* = d_{V_y} - 1$.

Proposition 2.5 (Ages and heights in SWG) *Fix $j \in \{1, 2\}$, $x, y, t \in \mathbb{R}$, $s_1, s_2 > 0$. Then, conditionally on $\mathcal{W}_{s_n}^{(1)}\mathcal{W}_{s_n}^{(2)} > 0$,*

(a)

$$\begin{aligned} e^{-2\alpha_n t_n} |\text{SWG}_{\leq k_{t_n}(x)}^{(j)}[\bar{t}_n + t, \bar{t}_n + t + s_1]| |\text{SWG}_{\leq k_{t_n}(y)}^{(3-j)}[\bar{t}_n + t, \bar{t}_n + t + s_2]| \\ \xrightarrow{\mathbb{P}} e^{2\alpha t} \Phi(x)\Phi(y)F_R(s_1)F_R(s_2), \end{aligned} \quad (2.33)$$

(b)

$$\begin{aligned} e^{-2\alpha_n t_n} |\text{SWG}_{\leq k_{t_n}(x)}^{(j)}[\bar{t}_n + t, \bar{t}_n + t + s_1]| \sum_v X_v^* \mathbb{1}_{\{v \in \text{SWG}_{\leq k_{t_n}(y)}^{(3-j)}[\bar{t}_n + t, \bar{t}_n + t + s_2]\}} \\ \xrightarrow{\mathbb{P}} \nu e^{2\alpha t} \Phi(x)\Phi(y)F_R(s_1)F_R(s_2). \end{aligned} \quad (2.34)$$

The first assertion in the above proposition follows from [53, Theorem 1(b)] in the case that our CTBP has finite-variance offspring. The proof of Proposition 2.5 is deferred to Section 5.

Completion of the proof of Theorem 2.2. Recall that $\mathcal{F}_t = \sigma((\text{SWG}(s))_{s \in [0, t]})$. We will investigate the number of collision edges (x_i, P_{x_i}) with $I(x_i) = j \in \{1, 2\}$, $H(x_i) \leq k_{t_n}(x)$, $H(P_{x_i}) \leq k_{t_n}(y)$ and $R_{T_i^{(\text{col})}}(P_{x_i}) \in [0, s)$ created in the time interval $[\bar{t}_n + t, \bar{t}_n + t + \varepsilon)$, where $\varepsilon > 0$ is small. We let $\mathcal{I} = [a, b) \times \{j\} \times (-\infty, x] \times (-\infty, y] \times [0, s]$ be a subset of \mathcal{S} , and we prove that

$$\mathbb{P}(\Pi_n(\mathcal{I}) = 0 \mid \mathcal{F}_{s_n}) \xrightarrow{\mathbb{P}} \exp\left\{-\int_a^b \frac{2\nu f_R(0)}{\mu} e^{2\alpha t} \Phi(x)\Phi(y)F_R(s)dt\right\}. \quad (2.35)$$

By [41, Theorem 4.7], this proves the claim.

We split

$$\mathcal{I} = \bigcup_{\ell=1}^N \mathcal{I}_\ell^{(\varepsilon)}, \quad (2.36)$$

where $\mathcal{I}_\ell^{(\varepsilon)} = [t_{\ell-1}^{(\varepsilon)}, t_\ell^{(\varepsilon)}) \times \{j\} \times (-\infty, x] \times (-\infty, y] \times [0, s)$, with $t_\ell^{(\varepsilon)} = a + \ell\varepsilon$ and $\varepsilon = (b-a)/N$, with $N \in \mathbb{N}$. We will let $\varepsilon \downarrow 0$ later on. For a fixed $\varepsilon > 0$, we say that a collision edge (x_i, P_{x_i}) is a *first round collision edge* when there exists $j \in [N]$ and a half-edge $y \in \mathbf{A}(t_{\ell-1}^{(\varepsilon)})$ such that y is found by the flow in the time interval $\mathcal{I}_\ell^{(\varepsilon)}$, y is paired to the half-edge P_y whose sibling half-edge x_i is paired to $P_{x_i} \in \mathbf{A}(t_{\ell-1}^{(\varepsilon)})$ with $I(y) = j \neq I(P_{x_i}) = 3-j$. We call all other collision edges *second round collision edges*. Denote the point processes of first and second round collision edges by $\Pi_n^{(\text{FR})}$ and $\Pi_n^{(\text{SR})}$, so that $\Pi_n = \Pi_n^{(\text{FR})} + \Pi_n^{(\text{SR})}$. The next two lemmas investigate the point processes $\Pi_n^{(\text{FR})}$ and $\Pi_n^{(\text{SR})}$:

Lemma 2.1 (PPP limit for the first round collision edges) For every $s \geq 0$, $x, y \in \mathbb{R}$, $j \in \{1, 2\}$, $\varepsilon > 0$ and $\ell \in [N]$, as $n \rightarrow \infty$,

$$\mathbb{P}(\Pi_n^{(\text{FR})}(\mathcal{I}_\ell^{(\varepsilon)}) = 0 \mid \mathcal{F}_{t_{\ell-1}^{(\varepsilon)}}) \xrightarrow{\mathbb{P}} \exp \left\{ -e^{2\alpha t_{\ell-1}^{(\varepsilon)}} \Phi(x) \Phi(y) F_R(s) F_R(\varepsilon) \right\}. \quad (2.37)$$

Proof. The number of half-edges $z \in \mathbf{A}(\bar{t}_n + t_{\ell-1}^{(\varepsilon)})$ that are found by the flow having $I(z) = j$ and $H(z) \leq k_{t_n}(x)$ is equal to

$$|\text{SWG}_{\leq k_{t_n}(x)}^{(j)}[\bar{t}_n + t_{\ell-1}^{(\varepsilon)}, \bar{t}_n + t_{\ell-1}^{(\varepsilon)} + \varepsilon]|. \quad (2.38)$$

Fix such a half-edge z , and note that it is paired to P_z that has $X_z^* = d_{V_{P_z}} - 1$ sibling half-edges. For each of these half-edges we test whether it is paired to a half-edge in $\mathbf{A}(\bar{t}_n + t_{\ell-1}^{(\varepsilon)})$ or not. Therefore, the total number of tests performed in the time interval $[t_{\ell-1}^{(\varepsilon)}, t_\ell^{(\varepsilon)})$ is equal to

$$\sum_z X_z^* \mathbb{1}_{\{z \in \text{SWG}_{\leq k_{t_n}(x)}^{(j)}[\bar{t}_n + t_{\ell-1}^{(\varepsilon)}, \bar{t}_n + t_\ell^{(\varepsilon)}]\}}. \quad (2.39)$$

By construction, we test whether these half-edges are paired to half-edges that are incident to the SWG or not. Each of these edges is paired to a half-edge $w \in \mathbf{A}(\bar{t}_n + t_{\ell-1}^{(\varepsilon)})$ with $I(w) = 3 - j$ (and thus creating a collision edge) and $H(w) \leq k_{t_n}(y)$ and $R_{\bar{t}_n + t_{\ell-1}^{(\varepsilon)}}(w) \in [0, s)$ with probability equal to

$$\frac{1}{\ell_n - o(n)} |\text{SWG}_{\leq k_{t_n}(y)}^{(3-j)}[\bar{t}_n + t_{\ell-1}^{(\varepsilon)}, \bar{t}_n + t_{\ell-1}^{(\varepsilon)} + s]|. \quad (2.40)$$

Therefore, the expected number of first round collision edges (x_i, P_{x_i}) with $I(x_i) = j \in \{1, 2\}$, $H(x_i) \leq k_{t_n}(x)$, $H(P_{x_i}) \leq k_{t_n}(y)$ and $R_{\bar{t}_n + t_{\ell-1}^{(\varepsilon)}}(P_{x_i}) \in [0, s)$ created in the time interval $[\bar{t}_n + t_{\ell-1}^{(\varepsilon)}, \bar{t}_n + t_\ell^{(\varepsilon)})$ equals the product of the expressions in (2.39) and (2.40), and can be rewritten as

$$\begin{aligned} & \frac{1}{\ell_n - o(n)} e^{2\alpha n t_n} \left(e^{-\alpha n t_n} \sum_z X_z^* \mathbb{1}_{\{z \in \text{SWG}_{\leq k_{t_n}(x)}^{(j)}[\bar{t}_n + t_{\ell-1}^{(\varepsilon)}, \bar{t}_n + t_\ell^{(\varepsilon)}]\}} \right) \\ & \times \left(e^{-\alpha n t_n} |\text{SWG}_{\leq k_{t_n}(y)}^{(3-j)}[\bar{t}_n + t_{\ell-1}^{(\varepsilon)}, \bar{t}_n + t_{\ell-1}^{(\varepsilon)} + s]| \right). \end{aligned} \quad (2.41)$$

By Proposition 2.5, conditionally on $\mathcal{F}_{\bar{t}_n + t_{\ell-1}^{(\varepsilon)}}$, and using that $(\ell_n - o(n))^{-1} e^{2\alpha n t_n} \rightarrow \mu^{-1}$, we find that (2.41), which represents the expected number of collision edges x_i with $I(x_i) = j \in \{1, 2\}$, $H(x_i) \leq k_{t_n}(x)$, $H(P_{x_i}) \leq k_{t_n}(y)$ and $R_{\bar{t}_n + t_{\ell-1}^{(\varepsilon)}}(P_{x_i}) \in [0, s)$ created in the time interval $[\bar{t}_n + t_{\ell-1}^{(\varepsilon)}, \bar{t}_n + t_\ell^{(\varepsilon)})$, converges in probability to:

$$\frac{\nu}{\mu} e^{2\alpha t} \Phi(x) \Phi(y) F_R(s) F_R(\varepsilon). \quad (2.42)$$

Further, for $\varepsilon > 0$, conditionally on $\mathcal{F}_{\bar{t}_n + t_{\ell-1}^{(\varepsilon)}}$, the probability that none of the half-edges found in the time interval in between $[\bar{t}_n + t_{\ell-1}^{(\varepsilon)}, \bar{t}_n + t_\ell^{(\varepsilon)})$ creates a collision edge is asymptotically equal to

$$\begin{aligned} & \prod_{v \in \text{SWG}_{\leq k_{\bar{t}_n}(x)}^{(j)}[\bar{t}_n + t_{\ell-1}^{(\varepsilon)}, \bar{t}_n + t_\ell^{(\varepsilon)}]} \left(1 - \frac{1}{\ell_n - o(n)} |\text{SWG}_{\leq k_{t_n}(y)}^{(3-j)}[\bar{t}_n + t_{\ell-1}^{(\varepsilon)}, \bar{t}_n + t_{\ell-1}^{(\varepsilon)} + s]| \right)^{X_v^*} \\ & \xrightarrow{\mathbb{P}} \exp \left\{ -\frac{\nu}{\mu} e^{2\alpha t_{\ell-1}^{(\varepsilon)}} \Phi(x) \Phi(y) F_R(s) F_R(\varepsilon) \right\}. \end{aligned} \quad (2.43)$$

■

Lemma 2.2 (A bound on the second round collision edges) For $x, y \in \mathbb{R}$, $j \in \{1, 2\}$, $\varepsilon > 0$ and $\ell \in [N]$, as $n \rightarrow \infty$,

$$\mathbb{P}(\Pi_n^{(\text{SR})}(\mathcal{I}_\ell^{(\varepsilon)}) \geq 1 \mid \mathcal{F}_{t_{\ell-1}^{(\varepsilon)}}) = O_{\mathbb{P}}(1)F_R(\varepsilon)G(\varepsilon). \quad (2.44)$$

Proof. By analogous arguments as above, the expected number of second round collision edges is of order

$$O_{\mathbb{P}}(1)e^{2\alpha t}\Phi(x)\Phi(y)F_R(s)F_R(\varepsilon)G(\varepsilon), \quad (2.45)$$

since one of the half-edges z that is found by the flow in the time interval $[\bar{t}_n + t_{\ell-1}^{(\varepsilon)}, \bar{t}_n + t_\ell^{(\varepsilon)}]$ needs to satisfy that one of the $d_{V_{P_z}} - 1$ half-edges has weight at most ε , and which, upon being found, needs to create a collision edge. \blacksquare

Now we are ready to complete the proof of Theorem 2.2. We use that

$$\mathbb{P}(\Pi_n(\mathcal{I}) = 0 \mid \mathcal{F}_{s_n}) = \mathbb{E}\left[\prod_{\ell=1}^N \mathbb{P}(\Pi_n(\mathcal{I}_\ell^{(\varepsilon)}) = 0 \mid \mathcal{F}_{t_{\ell-1}^{(\varepsilon)}}) \mid \mathcal{F}_{s_n}\right]. \quad (2.46)$$

We start with the upper bound, for which we use that

$$\begin{aligned} \mathbb{P}(\Pi_n(\mathcal{I}_\ell^{(\varepsilon)}) = 0 \mid \mathcal{F}_{t_{\ell-1}^{(\varepsilon)}}) &\leq \mathbb{P}(\Pi_n^{(\text{FR})}(\mathcal{I}_\ell^{(\varepsilon)}) = 0 \mid \mathcal{F}_{t_{\ell-1}^{(\varepsilon)}}) \\ &\xrightarrow{\mathbb{P}} \exp\left\{-e^{2\alpha t_{\ell-1}^{(\varepsilon)}}\Phi(x)\Phi(y)F_R(s)F_R(\varepsilon)\right\}, \end{aligned} \quad (2.47)$$

by Lemma 2.1. We conclude that

$$\begin{aligned} \mathbb{P}(\Pi_n(\mathcal{I}) = 0 \mid \mathcal{F}_{s_n}) &\leq \mathbb{E}\left[\prod_{\ell=1}^N \exp\left\{-e^{2\alpha t_{\ell-1}^{(\varepsilon)}}\Phi(x)\Phi(y)F_R(s)F_R(\varepsilon)\right\} \mid \mathcal{F}_{s_n}\right] \\ &= \exp\left\{-\sum_{\ell=1}^N e^{2\alpha t_{\ell-1}^{(\varepsilon)}}\Phi(x)\Phi(y)F_R(s)F_R(\varepsilon)\right\} \\ &\rightarrow \exp\left\{-f_R(0)\int_a^b e^{2\alpha t}\Phi(x)\Phi(y)F_R(s)dt\right\}, \end{aligned} \quad (2.48)$$

since $\lim_{\varepsilon \downarrow 0} F_R(\varepsilon)/\varepsilon = f_R(0)$, and the Riemann approximation

$$\varepsilon \sum_{\ell=1}^N e^{2\alpha t_{\ell-1}^{(\varepsilon)}} \rightarrow \int_a^b e^{2\alpha t} dt. \quad (2.49)$$

This proves the upper bound.

For the lower bound, we instead bound

$$\begin{aligned} \mathbb{P}(\Pi_n(\mathcal{I}) = 0 \mid \mathcal{F}_{s_n}) &\geq \mathbb{E}\left[\prod_{\ell=1}^N \mathbb{P}(\Pi_n^{(\text{FR})}(\mathcal{I}_\ell^{(\varepsilon)}) = 0 \mid \mathcal{F}_{t_{\ell-1}^{(\varepsilon)}}) \mid \mathcal{F}_{s_n}\right] \\ &\quad - \mathbb{E}\left[\left(\sum_{\ell=1}^N \mathbb{P}(\Pi_n^{(\text{SR})}(\mathcal{I}_\ell^{(\varepsilon)}) \geq 1 \mid \mathcal{F}_{t_{\ell-1}^{(\varepsilon)}})\right) \wedge 1 \mid \mathcal{F}_{s_n}\right]. \end{aligned} \quad (2.50)$$

The first term has already been dealt with, the second term is, by Lemma 2.2, bounded by

$$\mathbb{E}\left[\left(O_{\mathbb{P}}(1)\sum_{\ell=1}^N F_R(\varepsilon)G(\varepsilon)\right) \wedge 1 \mid \mathcal{F}_{s_n}\right] = o_{\mathbb{P}}(1), \quad (2.51)$$

as $\varepsilon \downarrow 0$, by dominated convergence, since $F_R(\varepsilon) = \varepsilon f_R(0)(1 + o(1))$ and $G(\varepsilon) = o(1)$. \blacksquare

3 Height CLT and stable age for CTBP

In this section, we set the stage for the proof of Proposition 2.5 for CTBPs, by investigating the first and second moment of particles of several types. We will make use of second moment methods similar to the ones in [53], but with a suitable truncation argument to circumvent the problem of infinite-variance offspring distributions.

We take $K \geq 1$ large and define, for an appropriate $\eta \in (0, 1)$ that will be determined later on,

$$m_i = K\eta^{-i}, \quad (3.1)$$

and investigate the Bellman-Harris process where each individual in generation i has offspring distribution $(X \wedge m_i)$ instead of X , where X denotes the offspring of our CTBP.

We denote the number of alive individuals in generation k at time t in the original branching process by $|\text{BP}_k(t)|$, and let $|\text{BP}_k[t, t+s]|$ denote the number of alive individuals in generation k with residual lifetime at most s . We let $|\text{BP}_k^{(\bar{m})}[t, t+s]|$ denote the number of individuals in generation k at time t and with remaining lifetime at most s of the *truncated* branching process. Define

$$|\text{BP}_{\leq k}^{(\bar{m})}[t, t+s]| = \sum_{j=0}^k |\text{BP}_j^{(\bar{m})}[t, t+s]|, \quad |\text{BP}^{(\bar{m})}[t, t+s]| = \sum_{j=0}^{\infty} |\text{BP}_j^{(\bar{m})}[t, t+s]|. \quad (3.2)$$

We also write $|\text{BP}^{(\bar{m})}(t)| = \lim_{s \rightarrow \infty} |\text{BP}^{(\bar{m})}[t, t+s]|$. A key ingredient to the proof of Proposition 2.5 is Proposition 3.1 below. In its statement, we also use $\eta = \nu \int_0^{\infty} e^{-2\alpha s} dG(s)$, so that $\eta < 1$ since α is such that $\nu \int_0^{\infty} e^{-\alpha s} dG(s) = 1$.

Proposition 3.1 (First and second moment CLT) *Choose $m_i = K\eta^{-i}$ as in (3.1). Assume that the $X \log X$ condition holds, i.e., $\mathbb{E}[X \log(X)_+] < \infty$ where X is the random amount of offspring of our CTBP. Then with $A = (\nu - 1)/\alpha\nu\bar{\nu}$,*

(a)

$$\lim_{t \rightarrow \infty} e^{-\alpha t} \mathbb{E}[|\text{BP}(t)|] = A, \quad \lim_{t \rightarrow \infty} e^{-\alpha t} \mathbb{E}[|\text{BP}(t)| - |\text{BP}^{(\bar{m})}(t)|] = 0, \quad (3.3)$$

(b) *there exists a $C > 0$ such that uniformly in $t \rightarrow \infty$,*

$$e^{-2\alpha t} \mathbb{E}[|\text{BP}^{(\bar{m})}(t)|^2] \leq CK, \quad (3.4)$$

(c)

$$\lim_{t \rightarrow \infty} e^{-\alpha t} \mathbb{E}[|\text{BP}_{\leq k_t(x)}^{(\bar{m})}[t, t+s]|] = A\Phi(x)F_R(s), \quad (3.5)$$

where $k_t(x)$ is defined in (2.32).

(d) *The same results hold uniformly in n when $t = t_n = \frac{1}{2\alpha_n} \log n$ as in (2.7), α is replaced by α_n , K by K_n and the branching process offspring distribution X_n depends on n in such a way that $X_n \xrightarrow{d} X$, $\mathbb{E}[X_n] \rightarrow \mathbb{E}[X]$ and $\limsup_n \mathbb{E}[X_n \log(X_n/K_n)_+] = 0$, for any $K_n \rightarrow \infty$.*

Proof. We start by proving Proposition 3.1(a). The first claim is proved in [33, 35]. We bound the first moment of the difference between the truncated and the original branching process. Let ν be the expected offspring of the Bellman-Harris process, and let $\nu^{(i)} = \mathbb{E}[X \mathbb{1}_{\{X \leq m_i\}}]$, where $m_i = K\eta^{-i}$. We compute that

$$e^{-\alpha t} \mathbb{E}\left[\sum_{k=0}^{\infty} [|\text{BP}_k(t)| - |\text{BP}_k^{(\bar{m})}(t)|]\right] = e^{-\alpha t} \sum_{k=0}^{\infty} [\nu^k - \prod_{i=1}^k \nu^{(i)}] [G^{*k}(t) - G^{*(k+1)}(t)], \quad (3.6)$$

where G is the distribution function of the edge weights. In order to bound the differences $\nu^k - \prod_{i=1}^k \nu^{(i)}$, we rely on the following lemma:

Lemma 3.2 (Effect of truncation on expectation CTBP) Fix $\eta \in (0, 1)$ and $m_i = K\eta^{-i}$, if $\mathbb{E}[X \log(X)_+] < \infty$, then

$$\left[1 - \prod_{i=1}^{\infty} \frac{\nu^{(i)}}{\nu}\right] \leq (\log(1/\eta))^{-1} \mathbb{E}[X \log(X/K)_+] = o_K(1), \quad (3.7)$$

where $o_K(1)$ denotes a quantity that converges to zero as $K \rightarrow \infty$.

Proof. Since $\frac{\nu^{(i)}}{\nu} \leq 1$ for all $i \geq 1$, it is easily shown by induction that

$$1 - \prod_{i=1}^k \frac{\nu^{(i)}}{\nu} \leq \sum_{i=1}^k \left(1 - \frac{\nu^{(i)}}{\nu}\right) \leq \sum_{i=1}^{\infty} \left(1 - \frac{\nu^{(i)}}{\nu}\right). \quad (3.8)$$

Now, using that $\nu > 1$,

$$\sum_{i=1}^{\infty} \left(1 - \frac{\nu^{(i)}}{\nu}\right) \leq \sum_{i=1}^{\infty} \mathbb{E}[X \mathbb{1}_{\{X > m_i\}}] = \mathbb{E}\left[X \sum_{i=1}^{\infty} \mathbb{1}_{\{m_i < X\}}\right], \quad (3.9)$$

and we note that the number of i for which $m_i = K\eta^{-i} \leq x$ is at most $\lceil \log(x/K)/\log(1/\eta) \rceil \vee 0$. Therefore

$$1 - \prod_{i=1}^k \frac{\nu^{(i)}}{\nu} \leq \sum_{i=1}^{\infty} \left(1 - \frac{\nu^{(i)}}{\nu}\right) \leq (\log(1/\eta))^{-1} \mathbb{E}[X \log(X/K)_+], \quad (3.10)$$

which converges to zero when $K \rightarrow \infty$. ■

By Lemma 3.2 and (3.6),

$$e^{-\alpha t} \mathbb{E}\left[\sum_{k=0}^{\infty} \left[|\mathbf{BP}_k(t)| - |\mathbf{BP}_k^{(\vec{m})}(t)|\right]\right] = o_K(1) e^{-\alpha t} \mathbb{E}\left[\sum_{k=0}^{\infty} |\mathbf{BP}_k(t)|\right] = o_K(1), \quad (3.11)$$

which completes the proof of Proposition 3.1(a).

We continue with the proof of the second moment estimate in Proposition 3.1(b). We follow the proof in [53], keeping attention to the truncation. We introduce h as the generating function of X , and h_j as the generating function of $(X \wedge m_j)$, i.e.,

$$h(s) = \mathbb{E}[s^X], \quad h_j(s) = \mathbb{E}[s^{(X \wedge m_j)}], \quad (3.12)$$

where m_j is given by (3.1). Parallel to calculations in the proof of [53, Lemma 4],

$$\mathbb{E}[|\mathbf{BP}^{(\vec{m})}|^2] = h_1''(1) (\mathbb{E}[|\mathbf{BP}^{(\vec{m}_1)}|])^2 * G + h_1'(1) \mathbb{E}[|\mathbf{BP}^{(\vec{m}_1)}|^2] * G, \quad (3.13)$$

where $\vec{m}_1 = (m_2, m_3, \dots)$, is \vec{m} with the first element removed. Transforming to

$$|\overline{\mathbf{BP}}^{(\vec{m})}(t)| = e^{-\alpha t} |\mathbf{BP}^{(\vec{m})}(t)|, \quad (3.14)$$

we obtain, by multiplying both sides of (3.13) by $e^{-2\alpha t}$,

$$\mathbb{E}[|\overline{\mathbf{BP}}^{(\vec{m})}|^2] = \frac{\eta h_1''(1)}{\nu} (\mathbb{E}[|\overline{\mathbf{BP}}^{(\vec{m}_1)}|])^2 * Q + \frac{\eta h_1'(1)}{\nu} \mathbb{E}[|\overline{\mathbf{BP}}^{(\vec{m}_1)}|^2] * \bar{G}, \quad (3.15)$$

where

$$\bar{G}(t) = \nu \int_0^t e^{-\alpha y} dG(y), \quad Q(t) = \eta^{-1} \int_0^t e^{-\alpha y} d\bar{G}(y) = \eta^{-1} \nu \int_0^t e^{-2\alpha y} dG(y), \quad (3.16)$$

and where we recall that $\eta = \int_0^\infty e^{-\alpha y} d\bar{G}(y) < 1$ and $\nu = h'(1)$. Iteration of (3.15) yields

$$\mathbb{E}[|\overline{\text{BP}}^{(\vec{m})}|^2] = \sum_{j=1}^{\infty} b_1 \cdots b_{j-1} a_j \mathbb{E}[|\overline{\text{BP}}^{(\vec{m}_j)}|^2] * \bar{G}_{j-1} * Q, \quad (3.17)$$

where

$$a_j = \frac{\eta h_j''(1)}{\nu}, \quad b_j = \frac{\eta h_j'(1)}{\nu}, \quad (3.18)$$

and where $\vec{m}_j = (m_{j+1}, m_{j+2}, \dots)$. According to (3.11) the expectation $\mathbb{E}[|\overline{\text{BP}}^{(\vec{m}_j)}(t_n)|]$ has the same asymptotic behavior as $\mathbb{E}[|\overline{\text{BP}}(t_n)|]$. Hence, by [53, Lemma 1(a)] or alternatively by part (a),

$$\lim_{n \rightarrow \infty} \mathbb{E}[|\overline{\text{BP}}^{(\vec{m}_j)}(t_n)|] = \lim_{n \rightarrow \infty} \mathbb{E}[|\overline{\text{BP}}(t_n)|] = A. \quad (3.19)$$

Since $b_1 b_2 \dots b_j \leq \eta^j$ tends to zero exponentially, this leads to

$$\lim_{n \rightarrow \infty} \mathbb{E}[|\overline{\text{BP}}^{(\vec{m})}(t_n)|^2] = A^2 \sum_{j=1}^{\infty} b_1 \cdots b_{j-1} a_j. \quad (3.20)$$

We bound the arising sum in the following lemma:

Lemma 3.3 (Effect of truncation on variance CTBP) *For $m_i = K\eta^{-i}$, and with $\nu = \mathbb{E}[X]$,*

$$\sum_{j=1}^{\infty} b_1 \cdots b_{j-1} a_j \leq \frac{2\nu K}{1-\eta}. \quad (3.21)$$

Proof. We bound $b_j \leq \eta$, and

$$a_j \leq \eta \mathbb{E}[(X \wedge m_j)^2] = \eta \left(m_j^2 \mathbb{P}(X > m_j) + \mathbb{E}[X^2 \mathbb{1}_{\{X \leq m_j\}}] \right), \quad (3.22)$$

so that

$$\sum_{j=1}^{\infty} b_1 \cdots b_{j-1} a_j \leq \sum_{j=1}^{\infty} m_j^2 \mathbb{P}(X > m_j) \eta^j + \sum_{j=1}^{\infty} \mathbb{E}[X^2 \mathbb{1}_{\{X \leq m_j\}}] \eta^j. \quad (3.23)$$

We bound both terms separately. The first contribution equals

$$K^2 \sum_{j=1}^{\infty} \mathbb{P}(X > K\eta^{-j}) \eta^{-j} = K^2 \mathbb{E} \left[\sum_{j=1}^{\infty} \eta^{-j} \mathbb{1}_{\{K\eta^{-j} < X\}} \right] = K^2 \mathbb{E} \left[\frac{\eta^{-a(X)} - 1}{1 - \eta} \right], \quad (3.24)$$

where $a(x) = \max\{j: K\eta^{-j} < x\} = \lceil \log(x/K) / \log(1/\eta) \rceil$. Therefore, $\eta^{-a(X)} \leq X/K$, so that

$$\sum_{j=1}^{\infty} m_j^2 \mathbb{P}(X > m_j) \eta^j \leq \frac{K^2}{1-\eta} \mathbb{E}[X/K] = \frac{K\nu}{1-\eta}. \quad (3.25)$$

The second contribution is bounded in a similar way as

$$\sum_{j=1}^{\infty} \mathbb{E}[X^2 \mathbb{1}_{\{X \leq m_j\}}] \eta^j = \mathbb{E} \left[\sum_{j=1}^{\infty} X^2 \eta^j \mathbb{1}_{\{X \leq K\eta^{-j}\}} \right] = \mathbb{E} \left[\frac{X^2 \eta^{b(X)}}{1-\eta} \right], \quad (3.26)$$

where $b(x) = \min\{i: K\eta^{-i} \geq x\} = \lceil \log(x/K) / \log(1/\eta) \rceil \geq \log(x/K) / \log(1/\eta)$, so that $\eta^{b(X)} \leq K/X$. Therefore,

$$\sum_{j=1}^{\infty} \mathbb{E}[X^2 \mathbb{1}_{\{X \leq m_j\}}] \eta^j \leq \frac{1}{1-\eta} \mathbb{E}[KX] = \frac{K\nu}{1-\eta}, \quad (3.27)$$

as required. ■

Combining (3.20) with Lemma 3.3 yields:

$$\lim_{n \rightarrow \infty} \mathbb{E}[|\overline{\text{BP}}^{(n)}(t_n)|^2] \leq \frac{2A^2 K \nu}{(1-\eta)},$$

so that (b) follows with $C = \frac{2A^2 \nu}{(1-\eta)}$.

For Proposition 3.1(c), we start by showing that, for the original branching process $(|\text{BP}(t)|)_{t \geq 0}$,

$$e^{-\alpha t} \sum_{j=0}^{k_t(x)} \mathbb{E}[|\text{BP}_j[t, t+s]|] \rightarrow A\Phi(x)F_R(s). \quad (3.28)$$

Conditioning on the lifetime (with c.d.f. equal to G) of the first individual, after which the individual dies and splits in a random number offspring with mean ν ,

$$\mathbb{E}[|\text{BP}_j[t, t+s]|] = \nu \int_0^t \mathbb{E}[|\text{BP}_{j-1}[t-y, t+s-y]|] dG(y). \quad (3.29)$$

As before,

$$|\overline{\text{BP}}_j[t, t+s]| = e^{-\alpha t} |\text{BP}_j[t, t+s]|. \quad (3.30)$$

Rewriting (3.29) we obtain the recursion

$$\mathbb{E}[|\overline{\text{BP}}_j[t, t+s]|] = \int_0^t \mathbb{E}[|\overline{\text{BP}}_{j-1}[t-y, t+s-y]|] d\bar{G}(y). \quad (3.31)$$

Hence, if we continue to iterate, we get

$$\mathbb{E}[|\overline{\text{BP}}_j[t, t+s]|] = \int_0^t \mathbb{E}[|\overline{\text{BP}}[t-y, t+s-y]|] d\bar{G}^{*j}(y), \quad (3.32)$$

where \bar{G}^{*j} is the j -fold convolution of \bar{G} , and hence the distribution function of the independent sum of j copies of a random variable each having c.d.f. \bar{G} . This is the point where we will use the CLT. For fixed $s > 0$, we define

$$|\overline{\text{BP}}_{>m}[t, t+s]| = \sum_{j=m+1}^{\infty} |\overline{\text{BP}}_j[t, t+s]|. \quad (3.33)$$

Observe that $|\text{BP}[t, t+s]| = \sum_{j=1}^{\infty} |\text{BP}_j[t, t+s]|$ is the total number of alive individuals of residual lifetime at most s , so that by [29, Theorem 24.1], (since G admits a density and $1 < \nu < \infty$, the conditions of this theorem are fulfilled),

$$\lim_{t \rightarrow \infty} \mathbb{E}[|\overline{\text{BP}}[t, t+s]|] = \lim_{t \rightarrow \infty} \sum_{j=0}^{\infty} \mathbb{E}[|\overline{\text{BP}}_j[t, t+s]|] = AF_R(s),$$

where

$$A = \frac{\nu - 1}{\alpha \nu^2 \int_0^{\infty} y e^{-\alpha y} dG(y)} = \frac{\nu - 1}{\alpha \nu \bar{\nu}}. \quad (3.34)$$

Hence, (3.28) follows if we show that

$$\mathbb{E}[|\overline{\text{BP}}_{>k_t(x)}[t, t+s]|] \rightarrow AF_R(s) - AF_R(s)\Phi(x) = AF_R(s)\Phi(-x). \quad (3.35)$$

Note that

$$\mathbb{E}[\overline{\text{BP}}_{>k_t(x)}[t, t+s]] = \int_0^t \mathbb{E}[\overline{\text{BP}}[t-u, t-u+s]] d\bar{G}^{*k_t(y)}(u). \quad (3.36)$$

Take an arbitrary $\varepsilon > 0$ and take t_0 so large so that for $t > t_0$,

$$|\mathbb{E}[\overline{\text{BP}}[t, t+s]] - AF_R(s)| \leq \varepsilon. \quad (3.37)$$

Then,

$$\begin{aligned} & \left| \mathbb{E}[\overline{\text{BP}}_{>k_t(x)}[t, t+s]] - AF_R(s)\Phi(-x) \right| \\ & \leq \varepsilon \bar{G}^{*k_t(x)}(t) + AF_R(s) |\bar{G}^{*k_t(x)}(t) - \Phi(-x)| + \int_{t-t_0}^t \mathbb{E}[\overline{\text{BP}}[t-u, t-u+s]] d\bar{G}^{*k_t(y)}(u). \end{aligned} \quad (3.38)$$

The last term vanishes since $\mathbb{E}[\overline{\text{BP}}[t, t+s]]$ is uniformly bounded and $\bar{G}^{*k_t(y)}(t) - \bar{G}^{*k_t(y)}(t-t_0) = o(1)$. Furthermore, with $m = k_t(x) \rightarrow \infty$,

$$k_t(x) \sim \frac{t}{\bar{\nu}} + x\sqrt{t\frac{\bar{\sigma}^2}{\bar{\nu}^3}} \iff t \sim m\bar{\nu} - x\bar{\sigma}\sqrt{m}. \quad (3.39)$$

As a result, by the CLT and the fact that $\bar{\nu}$ and $\bar{\sigma}^2$ are the mean and the variance of the distribution function \bar{G} ,

$$\lim_{t \rightarrow \infty} \bar{G}^{*k_t(x)}(t) = \Phi(-x). \quad (3.40)$$

Together with (3.38), this proves the claim in (3.35), and hence Proposition 3.1(c).

We continue with the proof of Proposition 3.1(a) for the n -dependent CTBP. We denote the number of alive individuals at time t in the n -dependent CTBP by $|\text{BP}_{(n)}(t)|$. We then have to show that uniformly in n ,

$$e^{-\alpha_n t_n} \mathbb{E}[|\text{BP}_{(n)}(t_n)|] \rightarrow A, \quad (3.41)$$

where A is given in (3.34). Denote by $\varphi(s) = \int_0^\infty e^{-sy} g(y) dy$, the Laplace transform of the lifetime distribution (g is the density of G). Then

$$\int_0^\infty e^{-st} \mathbb{E}[|\text{BP}_{(n)}(t)|] dt = \frac{1 - \varphi(s)}{s(1 - \nu_n \varphi(s))}. \quad (3.42)$$

This equation follows directly from [29, Equation 16.1], with m replaced by ν_n and is valid when the real part of s satisfies $\text{Re}(s) > \alpha_n$, where $\alpha_n > 0$ is defined as the unique value with $\nu_n \varphi(\alpha_n) = 1$. From the inversion formula for Laplace transforms, we obtain:

$$\mathbb{E}[|\text{BP}_{(n)}(t)|] = \frac{1}{2\pi i} \int_\Gamma e^{st} \frac{1 - \varphi(s)}{s(1 - \nu_n \varphi(s))} dt, \quad (3.43)$$

where Γ is the path $(c - i\infty, c + i\infty)$, with $c > \alpha_n$. Since $\alpha_n \rightarrow \alpha$ and $\nu_n \rightarrow \nu > 1$ and $\varphi(s)$ is the Laplace transform of a *probability density*, the function $s(1 - \nu_n \varphi(s))$ has a simple zero $s = \alpha_n$, but no other zeros in a small strip $|s - \alpha_n| < \varepsilon$. It is now easy to conclude from Cauchy's theorem, calculating the residue at $s = \alpha_n$, that

$$\begin{aligned} \mathbb{E}[|\text{BP}_{(n)}(t_n)|] &= e^{\alpha_n t_n} \frac{1 - \varphi(\alpha_n)}{\alpha_n \cdot (-\nu_n \varphi'(\alpha_n))} \left(1 + O(e^{-\varepsilon t_n})\right) \\ &= A_n e^{\alpha_n t_n} \left(1 + O(n^{-\varepsilon/(2\alpha_n)})\right), \end{aligned} \quad (3.44)$$

where

$$A_n = \frac{\nu_n - 1}{\alpha_n \nu_n^2 \int_0^\infty y e^{-\alpha_n y} dG(y)} = \frac{\nu_n - 1}{\alpha_n \nu_n \bar{\nu}_n}. \quad (3.45)$$

Since $A_n \rightarrow A$, by Condition 1.1(b), the claim (3.41) follows.

For the second statement in Proposition 3.1(a) for the n -dependent CTBP, we replace (3.10) by the equivalent n -dependent statement:

$$1 - \prod_{i=1}^k \frac{\nu_n^{(i)}}{\nu_n} \leq (\log(1/\eta))^{-1} \mathbb{E}[X_n [\log(X_n/K_n)_+]], \quad (3.46)$$

Since $X_n \xrightarrow{d} X$, $\mathbb{E}[X_n] \rightarrow \mathbb{E}[X]$ and $\limsup_{n \rightarrow \infty} \mathbb{E}[X_n \log(X_n/K_n)_+] = 0$ the statement follows.

For the n -dependent case of Proposition 3.1(b), we need to show that uniformly in n ,

$$e^{-2\alpha_n t_n} \mathbb{E}[|\mathbf{BP}_{(n)}^{(\bar{m})}(t_n)|^2] \leq CK_n, \quad (3.47)$$

for some constant C and where K is defined through $m_i = K_n \eta_n^i$ where $\eta_n = \nu_n \int_0^\infty e^{-2\alpha_n y} dG_n(y)$ and $\nu_n = \mathbb{E}[X_n]$. Copying the derivation which leads to (3.17), we obtain:

$$\mathbb{E}[|\overline{\mathbf{BP}}_{(n)}^{(\bar{m})}|^2] = \sum_{j=1}^{\infty} b_1^{(n)} \cdots b_{j-1}^{(n)} a_j^{(n)} \mathbb{E}[|\overline{\mathbf{BP}}_{(n)}^{(\bar{m}_j)}|^2] * \bar{G}_{j-1}^{(n)} * Q^{(n)}, \quad (3.48)$$

where

$$a_j^{(n)} = \frac{\eta_n \mathbb{E}[(X_n \wedge m_j)^2]}{\nu_n}, \quad b_j^{(n)} = \frac{\eta_n \mathbb{E}[(X_n \wedge m_j)]}{\nu_n}, \quad (3.49)$$

$$\bar{G}_n(t) = \nu_n \int_0^t e^{-\alpha_n y} dG(y), \quad Q_n(t) = \eta_n^{-1} \nu_n \int_0^t e^{-2\alpha_n y} dG(y). \quad (3.50)$$

From the proof of Lemma 3.3, we readily obtain that:

$$\sum_{j=1}^{\infty} b_1^{(n)} \cdots b_{j-1}^{(n)} a_j^{(n)} \leq \sum_{j=1}^{\infty} m_j^2 \mathbb{P}(X_n > m_j) \eta_n^j + \sum_{j=1}^{\infty} \mathbb{E}[X_n^2 \mathbb{1}_{\{X_n \leq m_j\}}] \eta_n^j \leq 2 \frac{K_n \nu_n}{1 - \eta_n}. \quad (3.51)$$

Since, $\nu_n \rightarrow \nu$ and $\eta_n \rightarrow \eta$ as $n \rightarrow \infty$, we find, by combining (3.48) with (3.51), that given $\varepsilon > 0$, there is an n_0 so that for $n > n_0$,

$$e^{-2\alpha_n t_n} \mathbb{E}[|\mathbf{BP}_{(n)}^{(\bar{m})}(t_n)|^2] \leq \frac{2K_n(\nu + \varepsilon)}{(1 - \eta - \varepsilon)(\bar{\nu} - \varepsilon)^2} \leq CK_n. \quad (3.52)$$

By (if necessary) enlarging the constant C we see that (3.47) holds for all n . This proves Proposition 3.1(b) for the n -dependent CTBP.

Finally, we consider Proposition 3.1(c) for the n -dependent CTBP. We denote by $|\mathbf{BP}_{(n),j}[t, t+s]|$ the number of individuals in generation j having residual lifetime at most s at time t of the CTBP with offspring given by X_n . Then, we obtain, compare (3.32),

$$\mathbb{E}[|\overline{\mathbf{BP}}_{(n),>k}[t, t+s]|] = \int_0^t \mathbb{E}[|\overline{\mathbf{BP}}_{(n)}[t-y, t+s-y]|] d\bar{G}_n^{*k}(y). \quad (3.53)$$

As in the proof of Proposition 3.1(c),

$$\lim_{t \rightarrow \infty} \mathbb{E}[|\overline{\mathbf{BP}}_{(n)}(t)|] = A_n F_R(s),$$

where A_n was defined in (3.45). A small extension of the CLT yields that

$$\bar{G}_{k t_n(x)}^{(n)}(t_n) \rightarrow \Phi(x).$$

Since $\nu_n \rightarrow \nu$, $\alpha_n \rightarrow \alpha$, $A_n \rightarrow A$, as $n \rightarrow \infty$, it follows that

$$\mathbb{E}[|\overline{\mathbf{BP}}_{(n),>k t_n(x)}[t, t+s]|] = \int_0^{t_n} \mathbb{E}[|\overline{\mathbf{BP}}_{(n)}[t_n-y, t_n-y+s]|] d\bar{G}_n^{*k t_n(x)}(y) \rightarrow A F_R(s) \Phi(x). \quad (3.54)$$

This completes the proof of Proposition 3.1(c) for n -dependent CTBPs. \blacksquare

4 Coupling to CTBP: Proof of Proposition 2.4

4.1 The coupling

The exploration of the total progeny of a branching process satisfies the same recurrence relation as in (2.2), apart from the fact that the random variables $(X_k)_{k \geq 1}$ are i.i.d. for a CTBP. For $\text{CM}_n(\mathbf{d})$, clearly, $(X_k)_{k \geq 1}$ are not i.i.d. We now describe stochastic relations between $(X_k)_{k \geq 1}$ given in the CM and an i.i.d. sequence $(Y_k)_{k \geq 1}$ with distribution equal to that of $D_n^* - 1$ for all $k \geq 1$, where the distribution of $D_n^* - 1$ has probability mass function $(g_k^{(n)})_{k \geq 0}$ defined by

$$\mathbb{P}(D_n^* - 1 = k) = g_k^{(n)} = \frac{k+1}{\ell_n} \sum_{i=1}^n \mathbb{1}_{\{d_i=k+1\}}, \quad k \geq 0. \quad (4.1)$$

Recall that U_1, U_2 are chosen uniformly at random from $[n]$. We continue with the definition of the *size-biased reordering* of $[n] \setminus \{U_1, U_2\}$:

Definition 4.1 (Size-biased reordering) *Given the set $[n] = \{1, 2, \dots, n\}$, vertices U_1, U_2 , and degree sequence \mathbf{d} , so that element $i \in [n]$ has degree d_i , a size-biased reordering of $[n] \setminus \{U_1, U_2\}$ of size m_n is a random sequence of (different) elements V_1, \dots, V_{m_n} , where we select V_i , $1 \leq i \leq m_n$ at random from the set $[n] \setminus \{U_1, U_2, V_1, \dots, V_{i-1}\}$ with probability proportional to the remaining degrees:*

$$\{d_1, \dots, d_n\} \setminus \{d_{U_1}, d_{U_2}, d_{V_1}, \dots, d_{V_{i-1}}\}.$$

Let $B_i + 1 = d_{V_i}$. We let X_i be the number of sibling half-edges of V_i that do not create cycles, i.e., are connected to vertices unequal to $\{U_1, U_2, V_1, \dots, V_{i-1}\}$. Thus, clearly, $X_i \leq B_i$. The above set-up allows us to define the coupling between $(B_i)_{i \geq 1}$ and $(Y_i)_{i \geq 1}$, where $(Y_i)_{i \geq 1}$, is an i.i.d. sequence with distribution (4.1).

Construction 4.2 (Coupling of size-biased reordering) *We couple $(B_i)_{i \geq 1}$ and $(Y_i)_{i \geq 1}$ in the following way:*

(a) *Draw Y_i as an independent copy of the distribution in (4.1). This can be achieved by drawing a uniform half-edge y from the total of ℓ_n half-edges. Let $V'_i = V_y$ denote the vertex to which the chosen half-edge is incident, and let $Y_i = d_{V'_i} - 1$.*

(b) *If $V'_i \notin \{U_1, U_2, V_1, \dots, V_{i-1}\}$, then $B_i = Y_i$ and $V_i = V'_i$, and we say that V_i is successfully coupled with V'_i .*

(c) *If $V'_i \in \{U_1, U_2, V_1, \dots, V_{i-1}\}$, so that we draw a half-edge incident to the set $\{U_1, U_2, V_1, \dots, V_{i-1}\}$, then we redraw a half-edge from the set of half-edges incident to $[n] \setminus \{U_1, U_2, V_1, \dots, V_{i-1}\}$ with probability proportional to their degree, we let V_i denote the vertex incident to the half-edge drawn, $B_i = d_{V_i} - 1$, and we say that both V_i and V'_i are miscoupled.*

(d) *We define X_i as the number of the $d_{V_i} - 1$ half-edges incident to vertex V_i that are not paired to a half-edge incident to $\{U_1, U_2, V_1, \dots, V_{i-1}\}$.*

We next investigate the above coupling. For this, it will be useful to note that when D_n , with distribution function F_n given in (1.2), satisfies Condition 1.1(c), then the maximal degree $\Delta_n = \max_{i \in [n]} d_i$ satisfies

$$\Delta_n = o(\sqrt{n/\log n}). \quad (4.2)$$

Indeed, suppose that $\Delta_n \geq \varepsilon \sqrt{n/\log n}$. Then, pick $K_n = n^{1/4}$ to obtain that

$$\begin{aligned} \mathbb{E}[D_n^2 \log(D_n/K_n)_+] &= \frac{1}{n} \sum_{k=1}^n d_k^2 \log(d_k/n^{1/4})_+ \geq \frac{\Delta_n^2}{n} \log(\Delta_n/n^{1/4}) \\ &\geq n^{-1} (\varepsilon \sqrt{n/\log n})^2 \log(n^{1/4}/(\log n)^{1/2}) \geq \varepsilon^2/8. \end{aligned} \quad (4.3)$$

This is in contradiction to Condition 1.1(c), so we conclude that (4.2) holds.

We define the sigma-algebra \mathcal{G}_i by $\mathcal{G}_i = \sigma(d_{U_1}, d_{U_2}, X_0^{(1)}, X_0^{(2)}, X_j, B_j)_{j \in [i]}$, see Section 2 for the definition of $X_0^{(i)}$, $i = 1, 2$.

Lemma 4.3 (Coupling to an i.i.d. sequence) *Assume that Condition 1.1(c) holds. For all $i \leq m_n$, and assuming that $m_n \leq \sqrt{n \log n}$,*

$$\mathbb{P}(B_i \neq Y_i \mid \mathcal{G}_{i-1}) \leq \frac{1}{\ell_n} \left(S_0 + \sum_{s=1}^{i-1} (B_s + 1) \right) = o(1), \quad (4.4)$$

where, as before, $S_0 = X_0^{(1)} + X_0^{(2)}$.

Proof. In Construction 4.2, $B_i \neq Y_i$ precisely when $V_i' \in \{U_1, U_2, V_1, \dots, V_{i-1}\}$, which, given \mathcal{G}_{i-1} , has probability at most

$$\mathbb{P}(B_i \neq Y_i \mid \mathcal{G}_{i-1}) \leq \frac{1}{\ell_n} \left(S_0 + \sum_{s=1}^{i-1} (B_s + 1) \right), \quad (4.5)$$

since Y_i draws uniformly from a total of ℓ_n half-edges, whereas in the previous draws at most $X_0^{(1)} + X_0^{(2)} + \sum_{s=1}^{i-1} (B_s + 1)$ half-edges are incident to the vertices $\{U_1, U_2, V_1, \dots, V_{i-1}\}$.

By (4.2), $\Delta_n = o(\sqrt{n/\log n})$ so that

$$X_0^{(1)} + X_0^{(2)} + \sum_{s=1}^{i-1} (B_s + 1) \leq m_n \Delta_n = o(n).$$

The final statement in (4.4) follows from $\ell_n = n\mu(1 + o(1))$. ■

Lemma 4.4 (Probability of drawing a half-edge incident to a previously found vertex)

Assume that Condition 1.1(c) holds. For all $i \leq m_n$, and assuming that $m_n \leq \sqrt{n \log n}$,

$$\mathbb{P}(X_i < B_i \mid \mathcal{G}_{i-1}) \leq \frac{B_i}{\ell_n(1 - o(1))} \left(S_0 + \sum_{s=1}^{i-1} B_s \right). \quad (4.6)$$

Proof. Recall the definition of S_i in (2.2). We have $X_i < B_i$ precisely when we pair at least one of the B_i half-edges to a half-edge incident to $\{U_1, U_2, V_1, \dots, V_{i-1}\}$. Since there are precisely B_i half-edges that need to be paired, and the number of half-edges incident to $\{U_1, U_2, V_1, \dots, V_{i-1}\}$, given \mathcal{G}_{i-1} , equals S_{i-1} , we find

$$\mathbb{P}(X_i < B_i \mid \mathcal{G}_{i-1}) \leq \frac{B_i \cdot S_{i-1}}{\ell_n - \sum_{s=1}^{i-1} (B_s - 1) - S_0 - 1}. \quad (4.7)$$

Clearly, $S_{i-1} \leq S_0 + \sum_{s=1}^{i-1} B_s$, which completes the proof. As before, $\sum_{s=1}^{i-1} (B_s - 1) \leq m_n \Delta_n = o(n)$ a.s. when $m_n \leq \sqrt{n \log n}$, which explains the $\ell_n(1 - o(1))$ in the denominator of (4.6). ■

Coupling the flows. In the above, we have described the coupling of our size-biased reordering. We now extend this to a coupling between the SWGs and CTBPs. We will couple in such a way that vertices that are successfully coupled (and thus have the same number of offspring half-edges, respectively, individuals), also receive the *same* weight along these half-edges. Therefore, the subtrees of successfully coupled half-edges in the SWG and individuals in the CTBP are found at precisely the same times. We will consistently refer to alive objects in the SWG as alive half-edges, and as alive individuals in the CTBP.

More specifically, offspring half-edges or individuals of miscoupled vertices are by definition miscoupled. The weights assigned to miscoupled half-edges in the SWG and individuals in the CTBP will be independent. Recall that T_k denotes the time at which the k th half-edge is found by the flow in the SWG. When the half-edge found is incident to a successfully coupled vertex, then we use Construction 4.2 to couple the number of offspring half-edges in the SWG to the offspring individuals in the CTBP. When the half-edge is incident to a miscoupled vertex, it is only present in the SWG, and we draw a half-edge from the set of available half-edges as in Construction 4.2(c), ignoring Y_k in Construction 4.2. We define $(T'_k)_{k \geq 0}$ as the times where an individual dies in the CTBP, but no half-edge is found by the flow in the SWG. These events result from one or more miscouplings between the CTBP and the SWG. At such times we draw a half-edge y uniformly at random from the total number of half-edges, and let $Y_i = d_{V_y} - 1$ denote the number of sibling half-edges. Note that in this case, we do not rely on Construction 4.2, and thus V_y is not part of the size-biased reordering.

Because of the above construction, differences arising in the coupling are due to two effects:

- (1) a *miscoupling* occurs: miscoupling between the size-biased reordering B_i and the i.i.d. draw from the degree distribution Y_i ; and
- (2) a *cycle-creating event* occurs: Here we refer to the occurrence of cycles, which makes $X_i < B_i$, and, by our construction of the collision edges, removes the $B_i - X_i$ half-edges incident to vertex V_i , as well as the $B_i - X_i$ half-edges to which they are paired from SWG.

Recall that offspring of miscoupled vertices are also miscoupled, so any miscoupling gives rise to a tress of miscoupled children half-edges in the SWG, respectively, individuals in the CTBP.

In order to be able to show convergence in probability to a random variable, we assume that all couplings are defined on one and the same probability space.

4.2 Coupling the SWG to a CTBP: proof of Proposition 2.4(a)

Consider an age-dependent branching process with lifetimes having a distribution admitting density g , and offspring distribution given by $(f_k)_{k \geq 1}$ in the first generation and offspring distribution $(g_k)_{k \geq 0}$ in the second and all further generations, and let D have probability mass function (p.m.f.) $(f_k)_{k \geq 1}$.

Let D_n^* (D^*) be a random variable such that $D_n^* - 1$ ($D^* - 1$) has p.m.f. $g_k^{(n)}$ (g_k), i.e.,

$$\mathbb{P}(D_n^* - 1 = k) = \frac{k+1}{\ell_n} \sum_{i=1}^n \mathbb{1}_{\{d_i=k+1\}}, \quad \mathbb{P}(D^* - 1 = k) = \frac{(k+1)\mathbb{P}(D-1=k)}{\mu}. \quad (4.8)$$

By Condition 1.1, $D_n^* \xrightarrow{d} D^*$, so that, since D_n^* and D^* are discrete distributions,

$$d_{\text{TV}}(D_n^*, D^*) \rightarrow 0, \quad (4.9)$$

where d_{TV} denotes the total variation distance, see for instance [55, Theorem 6.1].

Proof of Proposition 2.4(a). Take s_n maximal such that

$$e^{2\alpha s_n} d_{\text{TV}}(D_n^*, D^*) \rightarrow 0. \quad (4.10)$$

According to (1.17), and with $i \in \{1, 2\}$,

$$e^{-\alpha s_n} |\text{BP}^{(i)}(s_n)| \xrightarrow{a.s.} \mathcal{W}^{(i)}, \quad (4.11)$$

where $\mathcal{W}^{(i)}$ are two independent copies of \mathcal{W} . Since $\mathbb{P}(\mathcal{W}^{(i)} < \infty) = 1$ and $e^{\alpha s_n} \rightarrow \infty$, we conclude that $|\text{BP}(s_n)| \leq k_n$ whp if we take $k_n = \lfloor e^{2\alpha s_n} \rfloor$. If this k_n does not satisfy $k_n = o(\sqrt{n})$ then lower s_n so that the corresponding value of $k_n = \lfloor e^{2\alpha s_n} \rfloor$ does satisfy $k_n = o(\sqrt{n})$.

Recall that $\mathcal{G}_i = \sigma(d_{U_1}, d_{U_2}, X_0^{(1)}, X_0^{(2)}, X_j, B_j)_{j \in [i]}$. By Boole's inequality,

$$\mathbb{P}(X_i \neq Y_i \mid \mathcal{G}_{i-1}) \leq \mathbb{P}(X_i < B_i \mid \mathcal{G}_{i-1}) + \mathbb{P}(B_i \neq Y_i \mid \mathcal{G}_{i-1}). \quad (4.12)$$

Consequently, by Lemmas 4.3–4.4, a lower bound for the probability of coupling successfully during the first $k_n = o(\sqrt{n})$ pairings is

$$\begin{aligned} \mathbb{P}\left(\bigcap_{i=1}^{k_n} \{X_i = Y_i\}\right) &= 1 - \mathbb{P}\left(\bigcup_{i=1}^{k_n} \{X_i \neq Y_i\}\right) \\ &\geq 1 - \frac{1}{n(1 - o(1))} \sum_{i=1}^{k_n} \left(\mathbb{E}[B_i S_0] + 1 + \sum_{s=1}^{i-1} (\mathbb{E}[B_i B_s] + 1) \right) \geq 1 - ck_n^2/n \rightarrow 1. \end{aligned} \quad (4.13)$$

Here we rely on the fact that

$$\mathbb{E}[B_i \mid \mathcal{G}_{i-1}] \leq \sum_{j \in [n]} \frac{d_j(d_j - 1)}{\ell_n - 2k_n \Delta_n} = \nu_n(1 + o(1)), \quad (4.14)$$

whenever $k_n \Delta_n = o(n)$, which follows from (4.2).

The lower bound (4.13) implies that, whp, the shortest weight graph $(\text{SWG}(s))_{s \leq s_n}$ is perfectly coupled to the CTBP $(\text{BP}(s))_{s \leq s_n}$. This proves Proposition 2.4(a). \blacksquare

We close this section by investigating moments of the size-biased variables $(B_i)_{i \geq 1}$ arising in the size-biased reordering. These moments play a crucial role throughout the remainder of this paper, and allow us to compare $(B_i)_{i \geq 1}$ to an i.i.d. sample of random variables having the size-biased random distribution.

Lemma 4.5 (Moments of the size-biased reordering) *Assume that Condition 1.1(a-c) holds. For all $i \leq m_n$, and assuming that $m_n \leq \sqrt{n \log n}$, and for any $K_n \rightarrow \infty$ such that $K_n^2 = o(n/m_n)$,*

$$\mathbb{E}[B_i \mathbb{1}_{\{B_i \leq K_n\}} \mid \mathcal{G}_{i-1}] = (1 + o_{\mathbb{P}}(1)) \nu_n, \quad (4.15)$$

$$\mathbb{E}[B_i \mathbb{1}_{\{B_i > K_n\}} \mid \mathcal{G}_{i-1}] = o_{\mathbb{P}}(1). \quad (4.16)$$

Proof. We use the upper bound

$$\mathbb{E}[B_i \mathbb{1}_{\{B_i \geq a\}} \mid \mathcal{G}_{i-1}] \leq \frac{1}{\ell_n(1 - o(1))} \sum_{l \in [n]} d_l(d_l - 1) \mathbb{1}_{\{d_l \geq a+1\}}, \quad (4.17)$$

where we again use that, since $m_n \leq \sqrt{n \log n}$,

$$\ell_n - S_0 - \sum_{j=1}^{i-1} B_j \geq \ell_n - m_n \Delta_n = \ell_n - o(n). \quad (4.18)$$

This provides the necessary upper bound in (4.15) by taking $a = 0$ and from the identity $\nu_n = \sum_{l \in [n]} d_l(d_l - 1)/\ell_n$. For (4.16), this also proves the necessary bound, since

$$\frac{1}{\ell_n} \sum_{l \in [n]} d_l(d_l - 1) \mathbb{1}_{\{d_l \geq K_n\}} = o(1). \quad (4.19)$$

For the lower bound in (4.15), we bound, instead,

$$\mathbb{E}[B_i \mathbb{1}_{\{B_i \leq K_n\}} \mid \mathcal{G}_{i-1}] \geq \frac{1}{\ell_n} \sum_{l \in [n]} d_l(d_l - 1) \mathbb{1}_{\{d_l \leq K_n + 1\}} \mathbb{1}_{\{l \text{ not chosen yet}\}}, \quad (4.20)$$

where the event “ l is not chosen yet” means that the vertex l has not been chosen in the size-biased reordering until time $i - 1$. We now bound

$$\mathbb{E}[B_i \mathbb{1}_{\{B_i \leq K_n\}} \mid \mathcal{G}_{i-1}] \geq \frac{1}{\ell_n} \sum_{l \in [n]} d_l(d_l - 1) \mathbb{1}_{\{d_l \leq K_n + 1\}} - \frac{1}{\ell_n} \sum_{l \in [n]} d_l(d_l - 1) \mathbb{1}_{\{d_l \leq K_n + 1\}} \mathbb{1}_{\{l \text{ is chosen}\}}. \quad (4.21)$$

The first term equals $\nu_n(1 + o(1))$. The second term is a.s. bounded by $m_n K_n^2 / \ell_n = o(1)$, since $K_n^2 = o(n/m_n)$. \blacksquare

4.3 Completing the coupling: Proof of Proposition 2.4(b)

In this section, we use Proposition 3.1 to prove Proposition 2.4(b). In order to bound the difference between $\text{BP}(t)$ and $\text{SWG}(t)$, we will introduce several events. Let $B_n, C_n, \varepsilon_n, \overline{m}_n, \underline{m}_n$ denote sequences of constants for which $B_n, C_n \rightarrow \infty$ and $\varepsilon_n \rightarrow 0$ arbitrarily slowly, and $\overline{m}_n \gg \sqrt{n}, \underline{m}_n \ll \sqrt{n}$. Later in this proof, we will formulate restrictions on these sequences.

Define the event \mathcal{A}_n as follows:

$$\mathcal{A}_n = \{|\text{SWG}(t_n + B_n) \Delta \text{BP}_{(n)}(t_n + B_n)| < \varepsilon_n \sqrt{n}\}, \quad (4.22)$$

where $|\text{SWG}(t_n + B_n) \Delta \text{BP}_{(n)}(t_n + B_n)|$ is the number of miscoupled half-edges plus the number of miscoupled individuals. Then Proposition 2.4(b) can be reformulated as

$$\mathbb{P}(\mathcal{A}_n^c \mid \mathcal{F}_{s_n}) = o_{\mathbb{P}}(1). \quad (4.23)$$

In order to prove (4.23), we introduce the following events:

$$\begin{aligned} \mathcal{B}_n &= \{Y^{(\text{BP})}(t_n + B_n) \leq \overline{m}_n\} \cap \{Y^{(\text{SWG})}(t_n + B_n) \leq \overline{m}_n\} \\ &\quad \cap \{Y^{(\text{BP})}(t_n - B_n) \leq \underline{m}_n\} \cap \{Y^{(\text{SWG})}(t_n - B_n) \leq \underline{m}_n\}, \end{aligned} \quad (4.24)$$

$$\mathcal{C}_n = \{\text{SWG}(t) = \text{BP}(t), \forall t \leq t_n - B_n\}, \quad (4.25)$$

$$\mathcal{D}_n = \{\exists i \text{ such that } T_i \leq t_n + B_n, V_i \text{ miscoupled}, d_{V_i} \geq C_n\}, \quad (4.26)$$

where

$$Y^{(\text{BP})}(t) = |\{v: v \in \text{BP}_{(n)}(s) \text{ for some } s \leq t\}|, \quad (4.27)$$

denotes the total number of individuals ever born into the BP before time t and

$$Y^{(\text{SWG})}(t) = |\{v: v \in \text{SWG}(s) \text{ for some } s \leq t\}|, \quad (4.28)$$

denotes the number of half-edges in the SWG that have ever been alive before time t . Informally, on \mathcal{B}_n , the total number of half-edges in SWG and individuals in the CTBP are not too large. On

\mathcal{C}_n there is no early miscoupling, while on \mathcal{D}_n , there is no miscoupled vertex having high degree until a late stage.

Obviously

$$\begin{aligned} & \mathbb{P}(\mathcal{A}_n^c \mid \mathcal{F}_{s_n}) \\ & \leq \mathbb{P}(\mathcal{B}_n^c \mid \mathcal{F}_{s_n}) + \mathbb{P}(\mathcal{C}_n^c \cap \mathcal{B}_n \mid \mathcal{F}_{s_n}) + \mathbb{P}(\mathcal{D}_n^c \cap \mathcal{B}_n \cap \mathcal{C}_n \mid \mathcal{F}_{s_n}) + \mathbb{P}(\mathcal{A}_n^c \cap \mathcal{B}_n \cap \mathcal{C}_n \cap \mathcal{D}_n \mid \mathcal{F}_{s_n}). \end{aligned} \quad (4.29)$$

To bound conditional probabilities of the form $\mathbb{P}(\mathcal{E}^c \mid \mathcal{F}_{s_n})$, we start by noting that it suffices to prove that $\mathbb{P}(\mathcal{E}^c) = o(1)$, since then, by the Markov inequality and for every $\varepsilon > 0$,

$$\mathbb{P}\left(\mathbb{P}(\mathcal{E}^c \mid \mathcal{F}_{s_n}) \geq \varepsilon\right) \leq \mathbb{E}[\mathbb{P}(\mathcal{E}^c \mid \mathcal{F}_{s_n})]/\varepsilon = \mathbb{P}(\mathcal{E}^c)/\varepsilon = o(1). \quad (4.30)$$

Thus, we are left to prove that

$$\mathbb{P}(\mathcal{B}_n^c) = o(1), \quad \mathbb{P}(\mathcal{C}_n^c \cap \mathcal{B}_n) = o(1), \quad \mathbb{P}(\mathcal{D}_n^c \cap \mathcal{B}_n \cap \mathcal{C}_n) = o(1), \quad \mathbb{P}(\mathcal{A}_n^c \cap \mathcal{B}_n \cap \mathcal{C}_n \cap \mathcal{D}_n) = o(1). \quad (4.31)$$

We will do so in the above order.

Lemma 4.6 (Expected number of particles born) *For all $t \geq 0$,*

$$\mathbb{E}[Y^{(\text{BP})}(t)] = 2\left(1 - \frac{\mu_n G(t)}{\nu_n - 1}\right) + \frac{\nu_n}{\nu_n - 1} \mathbb{E}[|\text{BP}_{(n)}(t)|]. \quad (4.32)$$

Consequently, when $e^{\alpha_n(t_n + B_n)} = o(\overline{m}_n)$, $e^{\alpha_n(t_n - B_n)} = o(\underline{m}_n)$,

$$\mathbb{P}(\mathcal{B}_n^c) = o(1). \quad (4.33)$$

Proof. Note that we grow two SWGs and two BPs, which explains the factor 2 in (4.32). As is well known the expected number of descendants in generation k of a BP equals ν_n^k , where ν_n denotes the mean offspring. Here, we deal with a delayed BP where in the first generation the mean number of offspring equals $\mu_n = \mathbb{E}[D_n]$; the factor $G^{*k}(t) - G^{*(k+1)}(t)$ represents the probability that an individual of generation k is alive at time t ; together this yields:

$$\mathbb{E}[|\text{BP}_{(n)}(t)|] = \sum_{k=1}^{\infty} 2\mu_n \nu_n^{k-1} [G^{*k}(t) - G^{*(k+1)}(t)], \quad \mathbb{E}[Y^{(\text{BP})}(t)] = 2 + \sum_{k=1}^{\infty} 2\mu_n \nu_n^{k-1} G^{*k}(t). \quad (4.34)$$

We can rewrite the equality for $\mathbb{E}[|\text{BP}_{(n)}(t)|]$ to obtain

$$\begin{aligned} \mathbb{E}[|\text{BP}_{(n)}(t)|] &= 2\mu_n G(t)/\nu_n + \sum_{k=1}^{\infty} 2\mu_n [\nu_n^{k-1} - \nu_n^{k-2}] G^{*k}(t) \\ &= 2\mu_n G(t)/\nu_n - 2(1 - \nu_n^{-1}) + (1 - \nu_n^{-1}) \mathbb{E}[Y^{(\text{BP})}(t)]. \end{aligned} \quad (4.35)$$

Solving for $\mathbb{E}[Y^{(\text{BP})}(t)]$ yields the proof of (4.32).

To bound $\mathbb{P}(\mathcal{B}_n^c)$, we note that we have to bound events of the form $\mathbb{P}(Y^{(\text{BP})}(t) \geq m)$ and $\mathbb{P}(Y^{(\text{SWG})}(t) \geq m)$ for various choices of m and t . We use the Markov inequality and (4.32) to bound

$$\mathbb{P}(Y^{(\text{BP})}(t) \geq m) \leq \mathbb{E}[Y^{(\text{BP})}(t)]/m \leq \frac{\nu_n}{m(\nu_n - 1)} \mathbb{E}[|\text{BP}_{(n)}(t)|] + \frac{2}{m}. \quad (4.36)$$

By Proposition 3.1(d), $\mathbb{E}[|\text{BP}_{(n)}(t_n)|] = A_n e^{\alpha_n t_n} (1 + o(1))$, so that

$$\mathbb{P}(Y^{(\text{BP})}(t_n) \geq m) = \Theta(e^{\alpha_n t_n} / m). \quad (4.37)$$

The conditions on t and m in Lemma 4.6 have been chosen precisely so that $e^{\alpha_n(t_n - B_n)}/\underline{m}_n \rightarrow 0$, and $e^{\alpha_n(t_n + B_n)}/\overline{m}_n \rightarrow 0$.

We continue with $\mathbb{P}(Y^{(\text{SWG})}(t) \geq m)$. We use the same steps as above, and start by computing

$$\mathbb{E}[Y^{(\text{SWG})}(t)] = 2 + 2 \sum_{k=0}^{\infty} G^{**k}(t) \mathbb{E}[P_k^*], \quad (4.38)$$

where $P_0^* = \ell_n/n$ and

$$P_k^* = \sum_{|\pi|=k, \pi \subseteq \text{CM}_n(\mathbf{d})} (d_{\pi_k} - 1)/n,$$

is the sum of the number of half-edges at the ends of paths of lengths k in $\text{CM}_n(\mathbf{d})$, from a uniformly selected starting point. See [39, Section 5] for more details on paths in $\text{CM}_n(\mathbf{d})$. We compute that

$$\mathbb{E}[P_k^*] = \frac{1}{n} \sum_{v_0, \dots, v_k} d_{v_0} \prod_{i=1}^k \frac{d_{v_i} (d_{v_i} - 1)}{\ell_n - 2i + 1}, \quad (4.39)$$

where the sum is over distinct vertices in $[n]$. By [39, Proof of Lemma 5.1], the latter sum is bounded by

$$\mathbb{E}[P_k^*] \leq \mathbb{E}[D_n] \nu_n^k / n. \quad (4.40)$$

As a result, we have that $\mathbb{E}[Y^{(\text{SWG})}(t)] \leq \mathbb{E}[Y^{(\text{BP})}(t)]$, and we can repeat our arguments for $\mathbb{E}[Y^{(\text{BP})}(t)]$. ■

Lemma 4.7 (No early miscoupling) *When $e^{\alpha_n(t_n - B_n)} = o(\underline{m}_n)$ and $\underline{m}_n = o(\sqrt{n})$, then:*

$$\mathbb{P}(\mathcal{C}_n^c \cap \mathcal{B}_n) = o(1). \quad (4.41)$$

Proof. By the proof of Lemma 4.6, whp $Y^{(\text{BP})}(t_n - B_n) \leq \underline{m}_n$. By (4.13), the probability that there exists a miscoupling before the draw of the \underline{m}_n th half-edge is $o(1)$ when $\underline{m}_n = o(\sqrt{n})$. ■

Lemma 4.8 (No late miscouplings of high degree) *If $\overline{m}_n \leq \sqrt{n \log n}$, and C_n satisfies*

$$\frac{\overline{m}_n^2}{\ell_n} \sum_{i \in [n]} d_i^2 \mathbb{1}_{\{d_i \geq C_n\}} = o(n), \quad (4.42)$$

then

$$\mathbb{P}(\mathcal{D}_n^c \cap \mathcal{B}_n \cap \mathcal{C}_n) = o(1). \quad (4.43)$$

Proof. On the event \mathcal{B}_n : $Y^{(\text{BP})}(t_n + B_n) \leq \overline{m}_n$. An upper bound for the probability of miscoupling a vertex of degree at least C_n during the first \overline{m}_n pairings is thus

$$\sum_{i \in [n]} \left(\frac{d_i}{\ell_n(1 - o(1))} \right)^2 \mathbb{1}_{\{d_i \geq C_n\}} \leq \frac{1}{\ell_n^2(1 - o(1))} \sum_{i \in [n]} d_i^2 \mathbb{1}_{\{d_i \geq C_n\}} = o(1). \quad (4.44)$$

Proposition 4.9 (Miscoupled vertices have small offspring) *If $\overline{m}_n \leq \sqrt{n \log n}$ and $e^{2\alpha_n B_n} C_n \overline{m}_n^2 / \ell_n = o(\sqrt{n})$, then*

$$\mathbb{P}(\mathcal{A}_n^c \cap \mathcal{B}_n \cap \mathcal{C}_n \cap \mathcal{D}_n) = o(1). \quad (4.45)$$

Proof. We split the proof into three contributions, namely, a bound on $|\text{SWG}(t) \setminus \text{BP}_{(n)}(t)|$, a bound on the contribution to $|\text{BP}_{(n)}(t) \setminus \text{SWG}(t)|$ due to cycle-creating events, and a bound on the contribution to $|\text{BP}_{(n)}(t) \setminus \text{SWG}(t)|$ due to miscouplings. We start with the first bound:

A bound on $|\text{SWG}(t) \setminus \text{BP}_{(n)}(t)|$. By construction, the number of miscoupled half-edges in $\text{SWG}(t)$ at any time t is bounded from above by

$$\sum_{j=1}^{\text{MIS}(t)} Y_j^{(\text{SWG})}(t - \tilde{T}_j), \quad (4.46)$$

where $\text{MIS}(t)$ denotes the number of miscoupled vertices, \tilde{T}_j is the birth of the j th miscoupled vertex, and for the j th miscoupled vertex \tilde{V}_j , $Y_j^{(\text{SWG})}(t)$ is the number of half-edges at flow distance (total edge weight) at most t from \tilde{V}_j . On the event \mathcal{C}_n , $\tilde{T}_1 \geq t_n - B_n$. Therefore, for every $t \leq t_n + B_n$, on the event \mathcal{C}_n ,

$$|\text{SWG}(t) \setminus \text{BP}_{(n)}(t)| \leq \sum_{j=1}^{\text{MIS}(t_n+B_n)} Y_j^{(\text{SWG})}(2B_n). \quad (4.47)$$

By the Markov inequality,

$$\begin{aligned} & \mathbb{P}\left(\{|\text{SWG}(t) \setminus \text{BP}_{(n)}(t)| \geq \varepsilon\sqrt{n}\} \cap \mathcal{B}_n \cap \mathcal{C}_n \cap \mathcal{D}_n\right) \\ & \leq (\varepsilon\sqrt{n})^{-1} \mathbb{E}\left[\mathbb{1}_{\mathcal{B}_n \cap \mathcal{C}_n \cap \mathcal{D}_n} \sum_{j=1}^{\text{MIS}(t_n+B_n)} Y_j^{(\text{SWG})}(2B_n)\right]. \end{aligned} \quad (4.48)$$

We rewrite

$$\begin{aligned} & \mathbb{E}\left[\mathbb{1}_{\mathcal{B}_n \cap \mathcal{C}_n \cap \mathcal{D}_n} \sum_{j=1}^{\text{MIS}(t_n+B_n)} Y_j^{(\text{SWG})}(2B_n)\right] \\ & = (1 + o(1)) \sum_{i \in [n]} \mathbb{P}(i \text{ miscoupled}, \mathcal{B}_n \cap \mathcal{C}_n \cap \mathcal{D}_n) \mathbb{E}[Y^{(\text{SWG})}(2B_n)] \\ & \leq \sum_{i \in [n]} \left(\frac{d_i \bar{m}_n}{\ell_n}\right)^2 \mathbb{E}[Y^{(\text{SWG})}(2B_n)], \end{aligned} \quad (4.49)$$

where we use that, upon miscoupling of vertex i , we redraw a vertex from the size-biased distribution, for which the number of half-edges found before time $2B_n$ is equal to $\mathbb{E}[Y^{(\text{SWG})}(2B_n)](1+o(1))$ since $\bar{m}_n \leq \sqrt{n \log n}$ and \mathcal{B}_n occurs. Since $\mathbb{E}[Y^{(\text{SWG})}(t)] \leq \mathbb{E}[Y^{(\text{BP})}(t)]$, we obtain that

$$\mathbb{E}[Y_j^{(\text{SWG})}(2B_n)] \leq A_n e^{2\alpha_n B_n} (1 + o(1)). \quad (4.50)$$

Therefore, we arrive at

$$\mathbb{E}\left[\mathbb{1}_{\mathcal{B}_n \cap \mathcal{C}_n \cap \mathcal{D}_n} \sum_{j=1}^{\text{MIS}(t_n+B_n)} Y_j^{(\text{SWG})}(2B_n)\right] \leq O(1) e^{2\alpha_n B_n} \bar{m}_n^2 / \ell_n. \quad (4.51)$$

Combining (4.48)-(4.51) proves that $|\text{SWG}(t) \setminus \text{BP}_{(n)}(t)| = o_{\mathbb{P}}(\sqrt{n})$, since $(\varepsilon\sqrt{n})^{-1} \bar{m}_n^2 / \ell_n = o(1)$.

Bounding the contribution to $|\text{BP}_{(n)}(t) \setminus \text{SWG}(t)|$ due to cycle-creating events. On the event \mathcal{C}_n , $\tilde{T}_1 \geq t_n - B_n$. Recall that the j th miscoupled vertex is denoted by \tilde{V}_j , and that the time of the occurrence of the j th miscoupled vertex is \tilde{T}_j . On the event \mathcal{D}_n , $d_{\tilde{V}_j} \leq C_n$ for every j for which $\tilde{T}_j \leq t_n + B_n$. When a cycle-creating event occurs, the two half-edges that form the last

edge in the cycle are removed from $\text{SWG}(t)$, but they are kept in $\text{BP}(t)$. On the event $\mathcal{B}_n \cap \mathcal{C}_n$, the expected number of cycle-creating events is bounded by

$$O\left(\frac{\bar{m}_n^2}{\ell_n}\right). \quad (4.52)$$

Furthermore, on the event $\mathcal{B}_n \cap \mathcal{C}_n$, the expected offspring of the half-edges involved in cycle-creating events is at most

$$O\left(\frac{\bar{m}_n^2}{\ell_n}\right)\mathbb{E}[\tilde{Y}(2B_n)], \quad (4.53)$$

where $(\tilde{Y}(t))_{t \geq 0}$ is the total number of individuals that have ever been alive in a CTBP where all individuals have i.i.d. offspring with law $D_n^* - 1$, starting from $D_n^* - 1$ individuals. Indeed, we have no information about the remaining lifetime of the half-edge involved in the cycle-creating event. As a result, we *instantaneously* pair it to an i.i.d. draw of a half-edge, and start the $\text{BP}_{(n)}(t)$ from there. The total number of individuals ever alive only increases by this change. On the event \mathcal{D}_n , we have: $\mathbb{E}[\tilde{Y}(2B_n)] \leq C_n A_n e^{2\alpha_n B_n} (1 + o(1))$. By assumption $\bar{m}_n^2 C_n A_n e^{2\alpha_n B_n} / \ell_n = o(\sqrt{n})$. Therefore, the contribution to $|\text{BP}_{(n)}(t) \setminus \text{SWG}(t)|$ due to cycle-creating events is $o_{\mathbb{P}}(\sqrt{n})$, as required.

Bounding the contribution to $|\text{BP}_{(n)}(t) \setminus \text{SWG}(t)|$ due to miscouplings. We complete the proof of Proposition 4.9 by dealing with the contribution to $|\text{BP}_{(n)}(t) \setminus \text{SWG}(t)|$ due to miscouplings. Now, for $\text{BP}_{(n)}(t)$, we do *not* redraw the random variable \tilde{V}_j . We can give an upper bound on the contribution to $\text{BP}_{(n)}(t)$ of these miscouplings by instantaneously pairing the half-edges to an i.i.d. draw of a half-edge. As a result, the contribution to $|\text{BP}_{(n)}(t) \setminus \text{SWG}(t)|$ due to miscouplings can be bounded above by

$$\sum_{j: \tilde{T}_j \leq t_n + B_n} Y_j^{(\text{BP})}(t_n + B_n - \tilde{T}_j). \quad (4.54)$$

We use that, on the event \mathcal{C}_n , $\tilde{T}_j \geq t_n - B_n$, and on the event \mathcal{B}_n , the expected number of miscoupling is at most $O(\bar{m}_n^2/\ell_n)$. Finally, on the event \mathcal{D}_n , $d_{\tilde{V}_j} \leq C_n$ for every j for which $\tilde{T}_j \leq t_n + B_n$. Therefore,

$$\mathbb{E}\left[Y_j^{(\text{BP})}(t_n + B_n - \tilde{T}_j) \mathbb{1}_{\mathcal{B}_n \cap \mathcal{C}_n \cap \mathcal{D}_n}\right] \leq C_n A_n e^{2\alpha_n B_n} (1 + o(1)). \quad (4.55)$$

We conclude that, on the event $\mathcal{B}_n \cap \mathcal{C}_n \cap \mathcal{D}_n$,

$$\mathbb{E}\left[\sum_{j: \tilde{T}_j \leq t_n + B_n} Y_j^{(\text{BP})}(t_n + B_n - \tilde{T}_j) \mathbb{1}_{\mathcal{B}_n \cap \mathcal{C}_n \cap \mathcal{D}_n}\right] \leq O\left(\frac{\bar{m}_n^2}{\ell_n}\right) C_n e^{2\alpha_n B_n}. \quad (4.56)$$

By assumption the r.h.s. is $o(\sqrt{n})$. Therefore, the contribution to $|\text{BP}(t) \setminus \text{SWG}(t)|$ due to miscouplings is $o_{\mathbb{P}}(\sqrt{n})$, as required. This completes the proof of Proposition 4.9. \blacksquare

Proof of Proposition 2.4(b). It suffices to show that we can choose the sequences $B_n, C_n, \varepsilon_n, \bar{m}_n, \underline{m}_n$ such that all conditions in Lemmas 4.6–4.8 and Proposition 4.9 apply. It is readily verified that we can take:

$$\underline{m}_n = \sqrt{n}/(\log \log n)^{\alpha/2}, \quad \bar{m}_n = \sqrt{n}(\log n)^{1/4}, \quad (4.57)$$

and

$$B_n = \log \log \log n, \quad C_n = n^{1/4}, \quad \varepsilon_n = 1/\log n. \quad (4.58)$$

By Condition 1.1(c)

$$\frac{1}{n} \sum_{i \in [n]} d_i^2 \mathbb{1}_{\{d_i \geq C_n\}} = \mathbb{E}[D_n^2 \mathbb{1}_{\{D_n \geq C_n\}}] \leq \frac{\mathbb{E}[D_n^2 (\log(e \cdot D_n / C_n))_+]}{\log n} = o((\log n)^{-1}), \quad (4.59)$$

all the conditions in Lemmas 4.6–4.8 as well as Proposition 4.9 hold. Therefore, the claim in (4.23) follows, which completes the proof of Proposition 2.4(b). \blacksquare

5 Height CLT and stable age: Proof of Proposition 2.5

We first prove Proposition 2.5(a). By Proposition 2.4(a), at time s_n , whp, $(\text{SWG}(s))_{s \leq s_n}$ is perfectly coupled to the two independent CTBPs $(\text{BP}(s))_{s \leq s_n}$. The proof contains several key steps.

Reduction to a single BP. We start by showing that, in order for Proposition 2.5(a) to hold, it suffices to prove that for $j \in \{1, 2\}$, $x, t \in \mathbb{R}$ and $s > 0$,

$$e^{-\alpha_n t_n} |\text{BP}_{\leq k_{t_n}(x)}^{(j)}[\bar{t}_n + t, \bar{t}_n + t + s]| \xrightarrow{\mathbb{P}} e^{\alpha t} \Phi(x) F_R(s) \sqrt{\mathcal{W}^{(j)} / \mathcal{W}^{(3-j)}}, \quad (5.1)$$

where we use (2.28) in Proposition 2.4(a) to see that $\sqrt{\mathcal{W}^{(j)} / \mathcal{W}^{(3-j)}} \in [\varepsilon, 1/\varepsilon]$ whp. Indeed, by Proposition 2.4(b) and the fact that $e^{-\alpha_n t_n} = n^{-1/2}$, (5.1) immediately implies that

$$e^{-\alpha_n t_n} |\text{SWG}_{\leq k_{t_n}(x)}^{(j)}[\bar{t}_n + t, \bar{t}_n + t + s]| = e^{-\alpha_n t_n} |\text{BP}_{\leq k_{t_n}(x)}^{(j)}[\bar{t}_n + t, \bar{t}_n + t + s]| + e^{-\alpha_n t_n} o_{\mathbb{P}}(\varepsilon_n \sqrt{n}) \xrightarrow{\mathbb{P}} e^{\alpha t} \Phi(x) F_R(s) \sqrt{\mathcal{W}^{(j)} / \mathcal{W}^{(3-j)}}, \quad (5.2)$$

Therefore, independence and (5.1) also proves that for $j \in \{1, 2\}$, $x, y, t \in \mathbb{R}$ and s_1, s_2 ,

$$e^{-2\alpha_n t_n} |\text{SWG}_{\leq k_{t_n}(x)}^{(j)}[\bar{t}_n + t, \bar{t}_n + t + s_1]| |\text{SWG}_{\leq k_{t_n}(y)}^{(3-j)}[\bar{t}_n + t, \bar{t}_n + t + s_2]| \xrightarrow{\mathbb{P}} e^{2\alpha t} \Phi(x) \Phi(y) F_R(s_1) F_R(s_2), \quad (5.3)$$

which is the statement in Proposition 2.5(a).

Using the branching property. To prove (5.1), we note that $(\text{BP}^{(j)}(s))_{s \leq s_n}$ is the collection of alive individuals in the different generations of a CTBP, starting from the alive particles in $(\text{BP}^{(j)}(s_n))$. We condition on $(\text{BP}(s)^{(j)})_{s \in [0, s_n]}$. Then

$$|\text{BP}_{\leq k_{t_n}(x)}^{(j)}[\bar{t}_n + t, \bar{t}_n + t + s]| = \sum_{i \in \text{BP}^{(j)}(s_n)} \sum_{k=1}^{k_{t_n}(x) - G_i^{(j)}} |\text{BP}_k^{(i,j)}[\bar{t}_n + t - s_n - R_i, \bar{t}_n + t + s - s_n - R_i]|, \quad (5.4)$$

where $G_i^{(j)}$ is the generation of $i \in \text{BP}^{(j)}(s_n)$, while R_i is its remaining lifetime, and $(\text{BP}^{(i,j)}(t))_{t \geq 0}$ are i.i.d. CTBPs for different i , for which the offspring for each individual has distribution D_n^* .

Truncating the branching process. We continue by proving that we can truncate the branching process at the expense of an error term that converges to zero in probability. We let $\text{BP}^{(i,j,\bar{m})}$ denote the branching process $\text{BP}^{(i,j)}$ obtained by truncating particles in generation l (measured from the root i) by $m_l = K_n \eta^{-l}$. We take $K_n \rightarrow \infty$ such that $K_n e^{-\alpha_n s_n} = o(1)$. We first show

that, as $t \rightarrow \infty$, we can replace $e^{-\alpha_n t_n} |\text{BP}_{\leq k_{t_n}(x)}^{(i,j)}[\bar{t}_n, \bar{t}_n + s]|$ by $e^{-\alpha_n t_n} |\text{BP}_{\leq k_{t_n}(x)}^{(i,j,\bar{m})}[\bar{t}_n, \bar{t}_n + s]|$, at the expense of a $o_{\mathbb{P}}(1)$ -term. Indeed, with

$$|\text{BP}^{(i,j)}(t)| = \sum_{k=1}^{\infty} |\text{BP}_k^{(i,j)}(t)|, \quad |\text{BP}^{(i,j,\bar{m})}(t)| = \sum_{k=1}^{\infty} |\text{BP}_k^{(i,j,\bar{m})}(t)|, \quad (5.5)$$

we have, uniformly in $t \geq 0$ and $k \geq 0$, by the n -dependent version of Proposition 3.1(a) in Proposition 3.1(d),

$$e^{-\alpha_n t} \mathbb{E} \left[\left| \text{BP}_{\leq k}^{(i,j)}[t, t+s] - \text{BP}_{\leq k}^{(i,j,\bar{m})}[t, t+s] \right| \right] \leq e^{-\alpha_n t} \mathbb{E} \left[\left| \text{BP}^{(i,j)}(t) - \text{BP}^{(i,j,\bar{m})}(t) \right| \right] = o(1).$$

Therefore, using that the law of $\text{BP}_{\leq k_{t_n}(x)}^{(i,j)}$ only depends on \mathcal{F}_{s_n} through R_i, \bar{t}_n ,

$$\begin{aligned} & e^{-\alpha_n t_n} \sum_{i \in \text{BP}^{(j)}(s_n)} \sum_{k=1}^{k_{t_n}(x) - G_i^{(j)}} \mathbb{E} \left[\left| \text{BP}_k^{(i,j)}(\bar{t}_n + t - s_n - R_i) - \text{BP}_k^{(i,j,\bar{m})}(\bar{t}_n + t - s_n - R_i) \right| \mid \mathcal{F}_{s_n} \right] \\ & \leq e^{-\alpha_n t_n} \sum_{i \in \text{BP}^{(j)}(s_n)} \mathbb{E} \left[\left| \text{BP}_{\leq k_{t_n}(x)}^{(i,j)}(\bar{t}_n + t - s_n - R_i) - \text{BP}_{\leq k_{t_n}(x)}^{(i,j,\bar{m})}(\bar{t}_n + t - s_n - R_i) \right| \mid R_i, \bar{t}_n \right] \\ & = o(1) \sum_{i \in \text{BP}^{(j)}(s_n)} e^{\alpha_n(\bar{t}_n - t_n + t - s_n - R_i)} = o_{\mathbb{P}}(1) e^{-\alpha_n s_n} \sum_{i \in \text{BP}^{(j)}(s_n)} e^{-\alpha R_i}, \end{aligned} \quad (5.6)$$

since the random variable $|\bar{t}_n - t_n|$ is tight, and assuming that $s_n \rightarrow \infty$ so slowly that $s_n |\alpha_n - \alpha| = o(1)$. By [33, 35] and with $\sigma_i = s - R_i$, the birth-time of individual i ,

$$M_{s_n}^{(j)} = e^{-\alpha_n s_n} \sum_{i \in \text{BP}^{(j)}(s_n)} e^{-\alpha_n R_i} \xrightarrow{\mathbb{P}} \mathcal{W}^{(j)} / A, \quad (5.7)$$

where we use the fact that $(M_s^{(j)})_{s \geq 0}$ is an n -independent martingale by the remark on [34, p. 234 line 7]. We conclude that

$$\begin{aligned} & |\text{BP}_{\leq k_{t_n}(x)}^{(j)}[\bar{t}_n + t, \bar{t}_n + t + s]| \quad (5.8) \\ & = \sum_{i \in \text{BP}^{(j)}(s_n)} \sum_{k=1}^{k_{t_n}(x) - G_i^{(j)}} |\text{BP}_k^{(i,j,\bar{m})}[\bar{t}_n + t - s_n - R_i, \bar{t}_n + t + s - s_n - R_i]| + o_{\mathbb{P}}(1). \end{aligned}$$

A conditional second moment method: first moment. We next use a conditional second moment estimate on the sum on the right-hand side of (5.8), conditionally on \mathcal{F}_{s_n} . By the n -dependent version of Proposition 3.1(c) in Proposition 3.1(d), uniformly in n and for each $i \in \text{BP}^{(j)}(s_n)$ and $k_n = o(\sqrt{\log n})$,

$$e^{-\alpha_n t_n} \mathbb{E} \left[\left| \text{BP}_{\leq k_{t_n}(x) - k_n}^{(i,j,\bar{m})}[t_n, t_n + s] \right| \right] \rightarrow A \Phi(x) F_R(s). \quad (5.9)$$

As a result, when $\bar{t}_n + t - s_n - R_i \xrightarrow{\mathbb{P}} \infty$ and $G_i^{(j)} = o_{\mathbb{P}}(\sqrt{\log n})$,

$$\begin{aligned} & e^{-\alpha_n(\bar{t}_n + t - s_n - R_i)} \mathbb{E} \left[\left| \text{BP}_{\leq k_{t_n}(x)}^{(i,j,\bar{m})}[\bar{t}_n + t - s_n - R_i, \bar{t}_n + t + s - s_n - R_i] \right| \mid \mathcal{F}_{s_n} \right] \quad (5.10) \\ & = e^{-\alpha_n(\bar{t}_n + t - s_n - R_i)} \mathbb{E} \left[\left| \text{BP}_{\leq k_{t_n}(x)}^{(i,j,\bar{m})}[\bar{t}_n + t - s_n - R_i, \bar{t}_n + t + s - s_n - R_i] \right| \mid R_i, \bar{t}_n \right] \\ & = A \Phi(x) F_R(s) [1 + o_{\mathbb{P}}(1)]. \end{aligned}$$

This yields that, when $G_i^{(j)} = o_{\mathbb{P}}(\sqrt{\log n})$ (which happens whp when s_n is sufficiently small),

$$\begin{aligned} e^{-\alpha_n t_n} & \sum_{i \in \text{BP}^{(j)}(s_n)} \mathbb{E} \left[\sum_{k=1}^{k_{t_n}(x) - G_i^{(j)}} |\text{BP}_k^{(i,j,\bar{m})}[\bar{t}_n + t - s_n - R_i, \bar{t}_n + t + s - s_n - R_i]| \mid \mathcal{F}_{s_n} \right] \quad (5.11) \\ & = A e^{\alpha t} \Phi(x) F_R(s) [1 + o_{\mathbb{P}}(1)] \sum_{i \in \text{BP}^{(j)}(s_n)} e^{\alpha_n(\bar{t}_n - t_n - s_n - R_i)} \\ & \xrightarrow{\mathbb{P}} e^{\alpha t} \Phi(x) F_R(s) \sqrt{\mathcal{W}^{(j)}/\mathcal{W}^{(3-j)}}, \end{aligned}$$

since $e^{\alpha_n(t_n - \bar{t}_n)} = \sqrt{\mathcal{W}_{s_n}^{(j)} \mathcal{W}_{s_n}^{(3-j)}} \xrightarrow{\mathbb{P}} \sqrt{\mathcal{W}^{(j)} \mathcal{W}^{(3-j)}}$, whereas by (5.7), $\sum_{i \in \text{BP}^{(j)}(s_n)} e^{-\alpha_n(s_n + R_i)} \xrightarrow{\mathbb{P}} \mathcal{W}^{(j)}/A$.

A conditional second moment method: second moment. We next bound, conditionally on \mathcal{F}_{s_n} , the variance of the sum on the right-hand side of (5.8). By conditional independence of $(\text{BP}^{(i,j)})_{i \geq 1}$,

$$\begin{aligned} e^{-2\alpha_n t_n} \text{Var} \left(\sum_{i \in \text{BP}^{(j)}(s_n)} \sum_{k=1}^{k_{t_n}(x) - G_i^{(j)}} |\text{BP}_k^{(i,j,\bar{m})}[\bar{t}_n + t - s_n - R_i, \bar{t}_n + t + s - s_n - R_i]| \mid \mathcal{F}_{s_n} \right) \quad (5.12) \\ = e^{-2\alpha_n t_n} \sum_{i \in \text{BP}^{(j)}(s_n)} \text{Var} \left(\sum_{k=1}^{k_{t_n}(x) - G_i^{(j)}} |\text{BP}_k^{(i,j,\bar{m})}[\bar{t}_n + t - s_n - R_i, \bar{t}_n + t + s - s_n - R_i]| \mid \mathcal{F}_{s_n} \right). \end{aligned}$$

Since adding individuals and enlarging the time-frame does not reduce the variance,

$$\begin{aligned} \text{Var} \left(\sum_{k=1}^{k_{t_n}(x) - G_i^{(j)}} |\text{BP}_k^{(i,j,\bar{m})}[\bar{t}_n + t - s_n - R_i, \bar{t}_n + t + s - s_n - R_i]| \mid \mathcal{F}_{s_n} \right) \quad (5.13) \\ \leq \mathbb{E} \left[|\text{BP}^{(i,j,\bar{m})}(\bar{t}_n - s_n - R_i)|^2 \mid \mathcal{F}_{s_n} \right] = \mathbb{E} \left[|\text{BP}^{(i,j,\bar{m})}(\bar{t}_n - s_n - R_i)|^2 \mid R_i, \bar{t}_n \right]. \end{aligned}$$

By the n -dependent version of Proposition 3.1(b) in Proposition 3.1(d), uniformly in n and t ,

$$e^{-2\alpha_n t} \mathbb{E} \left[|\text{BP}^{(i,j,\bar{m})}(t)|^2 \right] \leq CK_n. \quad (5.14)$$

As a result,

$$\begin{aligned} e^{-2\alpha_n t_n} \text{Var} \left(\sum_{i \in \text{BP}^{(j)}(s_n)} \sum_{k=1}^{k_{t_n}(x) - G_i^{(j)}} |\text{BP}_k^{(i,j,\bar{m})}[\bar{t}_n + t - s_n - R_i, \bar{t}_n + t + s - s_n - R_i]| \mid \mathcal{F}_{s_n} \right) \quad (5.15) \\ \leq e^{-2\alpha_n t_n} \sum_{i \in \text{BP}^{(j)}(s_n)} \mathbb{E} \left[|\text{BP}^{(i,j,\bar{m})}(\bar{t}_n - s_n - R_i)|^2 \mid R_i, \bar{t}_n \right] \\ \leq CK_n e^{-2\alpha_n t_n} \sum_{i \in \text{BP}^{(j)}(s_n)} e^{2\alpha_n(\bar{t}_n - s_n - R_i)} \\ = CK_n e^{2\alpha_n(\bar{t}_n - t_n)} e^{-2\alpha_n s_n} \sum_{i \in \text{BP}^{(j)}(s_n)} e^{-2\alpha_n R_i} = O_{\mathbb{P}}(1) K_n e^{-2\alpha_n s_n} \sum_{i \in \text{BP}^{(j)}(s_n)} e^{-2\alpha_n R_i}, \end{aligned}$$

since $e^{2\alpha_n(\bar{t}_n - t_n)} = O_{\mathbb{P}}(1)$. We can bound this further as

$$K_n e^{-2\alpha_n s_n} \sum_{i \in \text{BP}^{(j)}(s_n)} e^{-2\alpha_n R_i} \leq K_n e^{-\alpha_n s_n} \left(e^{-\alpha_n s_n} \sum_{i \in \text{BP}^{(j)}(s_n)} e^{-\alpha_n R_i} \right) = O_{\mathbb{P}}(1) K_n e^{-\alpha_n s_n} = o_{\mathbb{P}}(1), \quad (5.16)$$

precisely when $K_n e^{-\alpha_n s_n} = o(1)$. By (5.11) and (5.15), the sum on the right-hand side of (5.8) is, conditionally on \mathcal{F}_{s_n} , concentrated around its asymptotic conditional mean given in (5.11). As a result, (5.1) follows. This completes the proof of Proposition 2.5(a). \blacksquare

In order to prove Proposition 2.5(b), we need to investigate the asymptotics of the sum $\sum_{i=1}^m X_i^*$, where $m = |\text{SWG}_{\leq k_{\bar{t}_n}(y)}^{(j)}[\bar{t}_n + t, \bar{t}_n + t + s_2]| \xrightarrow{\mathbb{P}} \infty$ on the event that $\mathcal{W}_{s_n}^{(1)} \mathcal{W}_{s_n}^{(2)} > 0$, and $(X_i^*)_{i \geq 1}$ are $X_i^* = d_{V_i} - 1$ with $(V_i)_{i \geq 1}$ the size-biased reordering of $(d_i)_{i \in [n] \setminus \mathcal{S}_m}$, where \mathcal{S}_m is the set of vertices found in $\text{SWG}(\bar{t}_n + t)$. We will prove that, conditionally on $\mathcal{F}_{\bar{t}_n + t}$,

$$\frac{1}{m\nu_n} \sum_{i=1}^m X_i^* \xrightarrow{\mathbb{P}} 1, \quad (5.17)$$

and then the proof of Proposition 2.5(b) follows from the proof of Proposition 2.5(a) and the fact that $\nu_n \rightarrow \nu$.

By the proof of Proposition 4.9, $|\mathcal{S}_m| \leq \bar{m}_n$, whp, where $\bar{m}_n = \sqrt{n}(\log n)^{1/4}$ as in (4.57). As a result the sequence $(d_i)_{i \in [n] \setminus \mathcal{S}_m}$ satisfies Condition 1.1 when $(d_i)_{i \in [n]}$ does. Hence Lemma 4.5 holds with B_i replaced by X_i^* , so that in particular, from the Markov inequality, conditionally on $\mathcal{F}_{\bar{t}_n + t}$,

$$\frac{1}{m} \sum_{i=1}^m X_i^* \mathbb{1}_{\{X_i^* > K_n\}} \xrightarrow{\mathbb{P}} 0. \quad (5.18)$$

We use a conditional second moment method on $\sum_{i=1}^m X_i^* \mathbb{1}_{\{X_i^* \leq K_n\}}$, conditionally on $\mathcal{F}_{\bar{t}_n + t}$. By (4.15) in Lemma 4.5,

$$\mathbb{E} \left[\sum_{i=1}^m X_i^* \mathbb{1}_{\{X_i^* \leq K_n\}} \mid \mathcal{F}_{\bar{t}_n + t} \right] = m\nu_n(1 + o_{\mathbb{P}}(1)). \quad (5.19)$$

This gives the asymptotics of the first conditional moment of $\sum_{i=1}^m X_i^* \mathbb{1}_{\{X_i^* \leq K_n\}}$. For the second moment, we start by bounding the covariances. We note that, for $1 \leq i < j \leq m$,

$$\begin{aligned} & \text{Cov} \left(X_i^* \mathbb{1}_{\{X_i^* \leq K_n\}}, X_j^* \mathbb{1}_{\{X_j^* \leq K_n\}} \mid \mathcal{F}_{\bar{t}_n + t} \right) \\ &= \mathbb{E} \left[X_i^* \mathbb{1}_{\{X_i^* \leq K_n\}} \left(\mathbb{E} [X_j^* \mathbb{1}_{\{X_j^* \leq K_n\}} \mid \mathcal{F}_{\bar{t}_n + t}, X_1^*, \dots, X_i^*] - \mathbb{E} [X_j^* \mathbb{1}_{\{X_j^* \leq K_n\}} \mid \mathcal{F}_{\bar{t}_n + t}] \right) \mid \mathcal{F}_{\bar{t}_n + t} \right]. \end{aligned} \quad (5.20)$$

By (4.15) in Lemma 4.5, as well as the fact that $i \leq \bar{m}_n = o(n)$,

$$\mathbb{E} [X_j^* \mathbb{1}_{\{X_j^* \leq K_n\}} \mid \mathcal{F}_{\bar{t}_n + t}, X_1^*, \dots, X_i^*] - \mathbb{E} [X_j^* \mathbb{1}_{\{X_j^* \leq K_n\}} \mid \mathcal{F}_{\bar{t}_n + t}] = o_{\mathbb{P}}(1), \quad (5.21)$$

so that also

$$\text{Cov} \left(X_i^* \mathbb{1}_{\{X_i^* \leq K_n\}}, X_j^* \mathbb{1}_{\{X_j^* \leq K_n\}} \mid \mathcal{F}_{\bar{t}_n + t} \right) = o_{\mathbb{P}}(1). \quad (5.22)$$

Further, a trivial bound on the second moment together with (4.15) in Lemma 4.5 yields that

$$\text{Var} \left(X_i^* \mathbb{1}_{\{X_i^* \leq K_n\}} \mid \mathcal{F}_{\bar{t}_n + t} \right) \leq K_n \mathbb{E} [X_i^* \mid \mathcal{F}_{\bar{t}_n + t}] = K_n \nu_n (1 + o_{\mathbb{P}}(1)). \quad (5.23)$$

As a result, whenever $K_n m = o(m^2)$,

$$\text{Var} \left(\sum_{i=1}^m X_i^* \mathbb{1}_{\{X_i^* \leq K_n\}} \mid \mathcal{F}_{\bar{t}_n + t} \right) = o_{\mathbb{P}}(m^2), \quad (5.24)$$

which together with (5.19) proves that, conditionally on $\mathcal{F}_{\bar{t}_n + t}$,

$$\frac{1}{m\nu_n} \sum_{i=1}^m X_i^* \mathbb{1}_{\{X_i^* \leq K_n\}} \xrightarrow{\mathbb{P}} 1. \quad (5.25)$$

Together with (5.18), this proves (5.17), as required. \blacksquare

6 Extensions to other random graphs

Proof of Theorem 1.6. Let $\text{UG}_n(\mathbf{d})$ be a uniform random graph with degree sequence \mathbf{d} . By [12] (see also [13]), we have that the law of $\text{UG}_n(\mathbf{d})$ is the same as that of $\text{CM}_n(\mathbf{d})$ conditioned on being simple, i.e., for every sequence of events \mathcal{H}_n defined on graphs with vertex set $[n]$,

$$\mathbb{P}(\text{UG}_n(\mathbf{d}) \in \mathcal{H}_n) = \mathbb{P}(\text{CM}_n(\mathbf{d}) \in \mathcal{H}_n \mid \text{CM}_n(\mathbf{d}) \text{ simple}) = \frac{\mathbb{P}(\text{CM}_n(\mathbf{d}) \in \mathcal{H}_n, \text{CM}_n(\mathbf{d}) \text{ simple})}{\mathbb{P}(\text{CM}_n(\mathbf{d}) \text{ simple})}. \quad (6.1)$$

By (1.24), it suffices to investigate $\mathbb{P}(\text{CM}_n(\mathbf{d}) \in \mathcal{H}_n, \text{CM}_n(\mathbf{d}) \text{ simple})$. We take

$$\mathcal{H}_n = \left\{ \frac{H_n - \gamma_n \log n}{\sqrt{\beta \log n}} \leq x, L_n - \frac{1}{\alpha_n} \log n \leq y \right\}, \quad (6.2)$$

where L_n and H_n are the hopcount and weight of the optimal path between two uniformly selected vertices conditioned on being connected.

By the results in Section 2, and with $\tilde{t}_n = t_n + B_n$, where $B_n = \log \log \log n$ is defined in (4.58), whp, we have found the minimal weight path before time \tilde{t}_n . The probability that we have found a self-loop or multiple edge at time \tilde{t}_n is negligible, since, by that time we have found of order $\bar{m}_n = \sqrt{n}(\log n)^{1/4}$ vertices and paired of order \bar{m}_n edges, see Lemma 4.6. Let $\tilde{d}_i(\tilde{t}_n)$ denote the number of unpaired half-edges incident to vertex i at time \tilde{t}_n . Since $\text{CM}_n(\mathbf{d})$ is created by matching the half-edges uniformly at random, in order to create $\text{CM}_n(\mathbf{d})$ after time \tilde{t}_n , we need to match the half-edges corresponding to $(\tilde{d}_i(\tilde{t}_n))_{i \in [n]}$. This corresponds to the configuration model on $[n]$ with degrees $(\tilde{d}_i(\tilde{t}_n))_{i \in [n]}$. Since we have found of order $\bar{m}_n = \sqrt{n}(\log n)^{1/4}$ vertices and paired of order \bar{m}_n edges at time \tilde{t}_n , when \mathbf{d} satisfies Condition 1.1, then so does $(\tilde{d}_i(\tilde{t}_n))_{i \in [n]}$ with the *same* limiting degree distribution D . As result, the probability that the configuration model on $[n]$ with degrees $(\tilde{d}_i(\tilde{t}_n))_{i \in [n]}$ is simple is asymptotically equal to $e^{-\nu/2 - \nu^2/4}(1 + o(1))$, and we obtain that the event that $\text{CM}_n(\mathbf{d})$ is simple is asymptotically independent of the event \mathcal{H}_n in (6.2). Therefore, Theorem 1.6 follows from Theorems 1.2-1.3. \blacksquare

Proof of Theorem 1.7. By Janson [38], when $W_n \xrightarrow{d} W$ and $\mathbb{E}[W_n^2] \rightarrow \mathbb{E}[W^2]$, the inhomogeneous random graphs with edge probabilities in (1.27), (1.29) or (1.30) are asymptotically equivalent, so it suffices to prove the claim for the *generalized random graph* for which $p_{ij} = w_i w_j / (\ell_n + w_i w_j)$. As explained in Section 1.4, conditionally on the degrees in the generalized random graph being equal to \mathbf{d} , the distribution of the resulting random graph is uniform over all random graphs with these degrees. Therefore, Theorem 1.7 follows from Theorem 1.6 if we prove that Condition 1.1 follows from the statements that $W_n \xrightarrow{d} W$, $\mathbb{E}[W_n] \rightarrow \mathbb{E}[W]$, $\mathbb{E}[W_n^2] \rightarrow \mathbb{E}[W^2]$ and $\lim_n \mathbb{E}[W_n^2 \log(W_n/K_n)_+] = 0$. We denote by $\mathbf{d} = (d_i)_{i \in [n]}$ the degree sequence in the generalized random graph, and note that \mathbf{d} now is a *random* sequence. We work conditionally on \mathbf{d} , and let $\mathbb{P}_n, \mathbb{E}_n$ denote the conditional probability and expectation given \mathbf{d} . Then, we prove that $\mathbb{P}_n(D_n = k) \xrightarrow{\mathbb{P}} \mathbb{P}(D = k)$, $\mathbb{E}_n[D_n] \xrightarrow{\mathbb{P}} \mathbb{E}[D]$, $\mathbb{E}_n[D_n^2] \xrightarrow{\mathbb{P}} \mathbb{E}[D^2]$ and $\mathbb{E}_n[D_n^2 \log(D_n/K_n)_+] \xrightarrow{\mathbb{P}} 0$ for every $K_n \rightarrow \infty$.

We let $D_n = d_V$, where $V \in [n]$ is a uniformly chosen vertex. First, by (1.31), $D_n \xrightarrow{d} D$, where D is a Poisson random variable with random intensity W . Further,

$$\mathbb{E}_n[D_n] = \frac{1}{n} \sum_{i \in [n]} d_i, \quad \mathbb{E}_n[D_n^2] = \frac{1}{n} \sum_{i \in [n]} d_i^2, \quad (6.3)$$

where $d_i = \sum_{j \in [n], j \neq i} I_{ij}$ and I_{ij} are independent Bernoulli variables with parameter $p_{ij} = w_i w_j / (\ell_n + w_i w_j)$. It tedious, but not difficult, to show that the above sums are concentrated

around their means, for example by computing their means and variances and showing that the variances are of smaller order than their means squared. We omit the details.

In order to show that $\mathbb{E}_n[D_n^2 \log(D_n/K_n)_+] = o_p(1)$, we note that

$$\mathbb{E}_n[D_n^2 \log(D_n/K_n)_+] = \frac{1}{n} \sum_{i \in [n]} d_i^2 \log(d_i/K_n)_+. \quad (6.4)$$

As before, $d_i = \sum_{j \in [n], j \neq i} I_{ij}$ and I_{ij} are independent Bernoulli variables with parameter $p_{ij} = w_i w_j / (\ell_n + w_i w_j)$. By standard Chernoff bounds, there exists a constant $a > 0$ such that, for every $\lambda > 2$,

$$\mathbb{P}(d_i \geq \lambda \mathbb{E}[d_i]) \leq e^{-a\lambda \mathbb{E}[d_i]}. \quad (6.5)$$

Here,

$$\mathbb{E}[d_i] = \sum_{j \neq i} w_i w_j / (\ell_n + w_i w_j) \in w_i \left(\frac{\ell_n - \sqrt{n}}{\ell_n(1 + o(1))}, 1 \right), \quad (6.6)$$

since $\max_i w_i = o(\sqrt{n})$. As a result,

$$\mathbb{E}_n[D_n^2 \log(D_n/K_n)_+] \leq \frac{4}{n} \sum_{i \in [n]} w_i^2 \log(2w_i/K_n)_+ + \frac{1}{n} \sum_{i \in [n]} \mathbb{1}_{\{d_i \geq 2w_i\}} d_i^2 \log(d_i/K_n)_+. \quad (6.7)$$

The first term vanishes by the fact that $\lim_{n \rightarrow \infty} \mathbb{E}[W_n^2 \log(W_n/K_n)_+] = 0$. The second term can be split as

$$\begin{aligned} \frac{1}{n} \sum_{i \in [n]} \mathbb{1}_{\{d_i \geq 2w_i\}} d_i^2 \log(d_i/K_n)_+ &\leq \frac{1}{n} \sum_{k=1}^{\infty} \sum_{i \in [n]} \mathbb{1}_{\{d_i \in [2^k w_i, 2^{k+1} w_i)\}} d_i^2 \log(d_i/K_n)_+ \\ &\leq \frac{1}{n} \sum_{i \in [n]} \sum_{k=1}^{\infty} 4^{k+1} \mathbb{1}_{\{d_i \geq 2^k w_i\}} w_i^2 \log(2^{k+1} w_i/K_n)_+. \end{aligned} \quad (6.8)$$

By (6.5)-(6.6) with $\lambda = 2^k$, the mean of the above random variable vanishes, which, by Markov's inequality, implies that it converges to zero in probability. \blacksquare

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