

THE AUGMENTED MULTIPLICATIVE COALESCENT, BOUNDED SIZE RULES AND CRITICAL DYNAMICS OF RANDOM GRAPHS.

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ABSTRACT. Random graph models with limited choice have been studied extensively with the goal of understanding the mechanism of the emergence of the giant component. One of the standard models are the Achlioptas random graph processes on a fixed set of n vertices. Here at each step, one chooses two edges uniformly at random and then decides which one to add to the existing configuration according to some criterion. An important class of such rules are the *bounded-size rules* where for a fixed $K \geq 1$, all components of size greater than K are treated equally. While a great deal of work has gone into analyzing the subcritical and supercritical regimes, the nature of the critical scaling window, the size and complexity (deviation from trees) of the components in the critical regime and nature of the merging dynamics has not been well understood. In this work we study such questions for general bounded-size rules. Our first main contribution is the construction of an extension of Aldous's standard multiplicative coalescent process which describes the asymptotic evolution of the vector of sizes and surplus of all components. We show that this process, referred to as the *standard augmented multiplicative coalescent* (AMC) is 'nearly' Feller with a suitable topology on the state space. Our second main result proves the convergence of suitably scaled component size and surplus vector, for any bounded-size rule, to the standard AMC. This result is new even for the classical Erdős-Rényi setting. The key ingredients here are a precise analysis of the asymptotic behavior of various susceptibility functions near criticality and certain bounds from [8], on the size of the largest component in the barely subcritical regime.

1. INTRODUCTION

The profusion of empirical data for real-world networks has provided an impetus for research into mathematical models which explain the various observed statistics such as scale-free degree distributions, small world properties and clustering. A range of mathematical models has been proposed, both static and dynamic, with the aim of understanding the structural properties of these real-world networks and their evolution over time. One particular direction of significant research is focused on understanding the effect of choice in the evolution of random network models (see [27] and references therein). More precisely, suppose that at time $t = 0$ we start with the empty configuration on $[n] := \{1, 2, \dots, n\}$ vertices. At each discrete step $k = 0, 1, 2, \dots$, we choose two edges $(e_1(k), e_2(k))$ uniformly at random amongst all $\binom{n}{2}$ edges and decide whether the graph at instant $(k+1)$, denoted by $\mathbf{G}_n(k+1)$, is $\mathbf{G}_n(k) \cup e_1(k)$ or $\mathbf{G}_n(k) \cup e_2(k)$ according to some pre-specified rule that takes into account suitable properties

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Date: October 16, 2013.

2000 Mathematics Subject Classification. Primary: 60C05, 05C80, 90B15; Secondary: ;

Key words and phrases. bounded-size rules, surplus, critical random graphs, scaling window, multiplicative coalescent, entrance boundary, giant component, branching processes, inhomogeneous random graphs, differential equation method, dynamic random graph models.

of the chosen edges with respect to the present configuration $\mathbf{G}_n(k)$. Speeding up time by a factor of n and abusing notation, for $t \geq 0$ write, $\mathbf{G}_n(t) = \mathbf{G}_n(\lfloor nt/2 \rfloor)$. Then the basic goal is to understand the effect of the rule governing the edge formations in the evolution of various characteristics of the network such as, the size of the largest component, the vector of sizes of all components, component complexities, etc. Three prototypical examples to keep in mind are as follows:

(a) **Erdős-Rényi random graph:** At each stage include edge e_1 and ignore e_2 . This results in the classical Erdős-Rényi random graph evolution. For a component \mathcal{C} , define $|\mathcal{C}|$ for the size (number of vertices) of the component. Well known results [12, 16] say that the critical time for the emergence of a giant component for this model is 1, namely for $t < 1$ the size of the largest component $|\mathcal{C}_1^{\text{ER}}(t)| = O(\log n)$ while for $t > 1$, the size of the largest component $|\mathcal{C}_1^{\text{ER}}(t)| = \Theta(n)$.

(b) **Bohman-Frieze (BF) process:** This was the first rigorously analyzed example of a rule that delayed the emergence of the giant component through limited choice [11]. Here the rule is to use the first edge if it connects two singletons (vertices which have no connections at the present time), otherwise use the second edge. It has been shown ([19, 27]) that there is a critical time $t_c^{\text{BF}} \approx 1.176$ when the largest component transitions from $O(\log n)$ to $\Theta(n)$.

(c) **General bounded-size rules (BSR):** The BF process corresponds to a choice rule which treats all components with size greater than one in an identical fashion. It is a special case of the general family of models, referred to as bounded-size rules. Here one fixes $K \geq 1$ and then the rule for attachment is invariant on components of size greater than K . We postpone a precise description to Section 2.2. General bounded-size rules were analyzed in [27] where it was shown that there exists a (rule dependent) critical time t_c such that for $t < t_c$, the largest component $|\mathcal{C}_1^{\text{BSR}}(t)| = O(\log n)$ when $t < t_c$ while $|\mathcal{C}_1^{\text{BSR}}(t)| = \Theta(n)$ for $t > t_c$.

Thus as time transitions from below to above t_c , a giant component (of the same order as the network) emerges. Motivated by recent results on the Erdős-Rényi random graph components at criticality [2] as well as general rules such as the (unbounded-size) product rule [1], there has been a renewed interest in understanding the precise nature of the emergence of the giant component as well as structural properties of components near t_c for classes of rules which incorporate limited choice in their evolution. Define the surplus or complexity of a component $\mathbf{spls}(\mathcal{C})$ as

$$\mathbf{spls}(\mathcal{C}) = \text{number of edges} - (|\mathcal{C}| - 1). \quad (1.1)$$

If a component were a tree, its surplus would be zero, thus this is a measure of the deviation of the component from a tree. Write $\mathcal{C}_i(t)$ for the i -th largest component and $\xi_i(t) := \mathbf{spls}(\mathcal{C}_i(t))$ for the surplus of the component $\mathcal{C}_i(t)$. For any of the rules above and a fixed $t \geq 0$, consider the vector of component sizes and associated surplus $(|\mathcal{C}_i(t)|, \xi_i(t) : i \geq 1)$. In the context of the Erdős-Rényi random graph process, precise fine-scale results are known about the nature of the emergence of the giant component as time t transitions through the scaling window around $t_c^{\text{ER}} = 1$. More precisely, for fixed $\lambda \in \mathbb{R}$ write

$$\bar{\mathcal{C}}^{\text{ER}}(\lambda) := \left(\frac{1}{n^{2/3}} \left| \mathcal{C}_i^{\text{ER}} \left(1 + \frac{1}{n^{1/3}} \lambda \right) \right| : i \geq 1 \right) \text{ and } \bar{\mathbf{Y}}^{\text{ER}}(\lambda) := \left(\xi_i^{\text{ER}} \left(1 + \frac{1}{n^{1/3}} \lambda \right) : i \geq 1 \right).$$

Then Aldous in [4] showed:

- (a) The process $(\bar{\mathcal{C}}^{\text{ER}}(\lambda) : -\infty < \lambda < \infty)$ converges to a Markov process called the standard multiplicative coalescent.
- (b) For fixed $\lambda \in \mathbb{R}$, the rescaled component sizes and the corresponding surplus $(\bar{\mathcal{C}}^{\text{ER}}(\lambda), \bar{\mathbf{Y}}^{\text{ER}}(\lambda))$ converge jointly to a limiting random process described by excursions

above zero of an inhomogeneous reflected Brownian motion \hat{W}_λ and a counting process \hat{N}_λ with intensity function $\hat{W}_\lambda(\cdot)$.

We give a precise description of these results in Section 2.3.1. Obtaining similar results on critical asymptotics for general inhomogeneous Markovian models such as the bounded-size rules requires new ideas. These rules lack a simple description for the dependence between edges making the direct use of the component exploration and associated random walk construction, the major workhorse in understanding random graph models at criticality ([4, 9, 10, 21, 24]), intractable. Thus it is nontrivial to identify the critical scaling window for such processes, let alone distributional asymptotics for the component sizes and surplus. In the current work we develop a different machinery that allows us to identify the critical scaling window for all bounded-size rules. Furthermore, denoting the suitably scaled component sizes and surplus processes as $(\bar{\mathbf{C}}^{(n)}(\lambda), \bar{\mathbf{Y}}^{(n)}(\lambda))$, our results describe the joint asymptotic behavior of

$$\left((\bar{\mathbf{C}}^{(n)}(\lambda_1), \bar{\mathbf{Y}}^{(n)}(\lambda_1)), \dots, (\bar{\mathbf{C}}^{(n)}(\lambda_m), \bar{\mathbf{Y}}^{(n)}(\lambda_m)) \right)$$

for $-\infty < \lambda_1 < \lambda_2 < \dots < \lambda_m < \infty$. The starting point of our work is the construction of a Markov process that is associated with the inhomogeneous reflected Brownian motion $\{\hat{W}_\lambda\}_{\lambda \in \mathbb{R}}$ and the associated counting process $\{\hat{N}_\lambda\}_{\lambda \in \mathbb{R}}$ which we refer to as the augmented multiplicative coalescent (AMC). The main result of this work shows that AMC is the characterizing process for the universality class that includes, in addition to critically scaled components and surplus vectors for Erdős-Rényi graphs, analogous processes for all bounded-size rules. More precisely, our contributions are as follows.

- (a) In Theorem 3.1 we show the existence and “near” Feller property of a Markov process $\mathbf{Z}(\lambda)$, $-\infty < \lambda < \infty$, called the augmented multiplicative coalescent, which tracks the evolution of both component sizes and surplus edges over the critical window. Aldous’s standard multiplicative coalescent corresponds to the first coordinate of this process. Identifying the correct state space and topology that is suitable for obtaining the Feller property for this process turns out to be particularly delicate (see Remark 4.15). The (near) Feller property plays a key role in analyzing the joint distribution, at multiple time instants, of the component sizes and surplus for bounded-size rules in the critical scaling window. In proving the existence of the standard augmented process, a key role is played by Theorem 5.1 which is a generalization of a result of Aldous for the component sizes of an inhomogeneous random graph, to a setting where one considers joint distributions of component sizes and surplus. We believe that this result is of broader significance and can be used to analyze the distribution of surplus in the critical regime for various other inhomogeneous random graph models, e.g. the rank-1 inhomogeneous random graphs ([13]).
- (b) In Theorems 3.2 we analyze susceptibility functions (sums of moments of component sizes) associated with a general bounded-size rule. Spencer and Wormald [27] showed that these susceptibility functions converge to limiting monotonically increasing deterministic functions which are finite only for $t < t_c$ and explode for $t > t_c$. Theorem 3.2(i) uses a dynamic random graph process with immigration and attachment to show that these limiting functions for **all bounded-size rules** have the same critical exponents as the Erdős-Rényi random graph process. Theorem 3.2(ii) and (iii) shows that the susceptibility functions are close to their deterministic analogs in a strong sense even as $t \uparrow t_c$ when the limiting functions explode.
- (c) The analysis of the susceptibility functions gives rise to (rule dependent) constants $\alpha, \beta > 0$ which describe the nature of the explosion of the limiting susceptibility functions as $t \uparrow t_c$.

For a given bounded-size rule we consider the rescaled process $\{\bar{\mathbf{Z}}^{(n)}(\lambda) : -\infty < \lambda < \infty\}$ where $\bar{\mathbf{Z}}^{(n)}(\lambda) = (\bar{\mathbf{C}}^{(n)}(\lambda), \bar{\mathbf{Y}}^{(n)}(\lambda))$ with $\bar{\mathbf{C}}^{(n)}(\lambda)$ denoting the rescaled component sizes and $\bar{\mathbf{Y}}^{(n)}(\lambda)$ denoting the surplus of these components, namely

$$\bar{\mathbf{C}}^{(n)}(\lambda) := \left(\frac{\beta^{1/3}}{n^{2/3}} \left| C_i \left(t_c + \frac{\alpha\beta^{2/3}}{n^{1/3}} \lambda \right) \right| : i \geq 1 \right) \text{ and } \bar{\mathbf{Y}}^{(n)}(\lambda) = \left(\xi_i \left(t_c + \frac{\alpha\beta^{2/3}}{n^{1/3}} \lambda \right) : i \geq 1 \right).$$

Using the Feller property proved in Theorem 3.1, Theorem 5.1, results on the susceptibility functions, and bounds on the size of the largest component in the barely subcritical regime from [8], we show the convergence of the finite-dimensional distributions of this process to that of the augmented multiplicative coalescent $\{\mathbf{Z}(\lambda) : -\infty < \lambda < \infty\}$, namely for any set of times $-\infty < \lambda_1 < \lambda_2 < \dots < \lambda_m < \infty$,

$$\left(\bar{\mathbf{Z}}^{(n)}(\lambda_1), \dots, \bar{\mathbf{Z}}^{(n)}(\lambda_m) \right) \xrightarrow{d} (\mathbf{Z}(\lambda_1), \dots, \mathbf{Z}(\lambda_m)).$$

The result in particular identifies the critical scaling window **for all bounded-size rules** as well as the asymptotic joint distributions of component sizes and surplus for any fixed λ , implying that such rules belong to the same universality class as the Erdős-Rényi random graph process. The convergence for the joint distribution of the surplus and the component sizes for multiple time points λ in the critical scaling window for Erdős-Rényi random graph process is regarded as a ‘‘folk theorem’’ – our results, in particular, give a rigorous proof of this statement.

The paper is organized as follows. In Section 2 we introduce some common notation, give a precise description of bounded-size rules, and give an informal description of the augmented multiplicative coalescent. Section 3 contains the statements of our main results. Sections 4 and 5 are devoted to proving the existence and near Feller property of the AMC. In particular Section 5 contains the proof of Theorem 3.1. Section 6 studies the asymptotics of the susceptibility functions associated with general bounded-size rules and proves Theorems 3.2. Finally in Section 7 we complete the proof of Theorem 3.3.

2. DEFINITIONS AND NOTATION

2.1. Notation. We collect some common notation and conventions used in this work. A graph $\mathbf{G} = (\mathcal{V}, \mathcal{E})$ consists of a vertex set \mathcal{V} and an edge set \mathcal{E} , where \mathcal{V} is a subset of some type space \mathcal{X} . For a finite set A write $|A|$ for its cardinality. A graph \mathbf{G} with no vertices and edges will be called a **null graph**. For graphs $\mathbf{G}_1, \mathbf{G}_2$, if \mathbf{G}_1 is a subgraph of \mathbf{G}_2 we shall write this as $\mathbf{G}_1 \subset \mathbf{G}_2$. The number of vertices in a connected component \mathcal{C} of a graph \mathbf{G} will be called the size of the component and will be denoted by $|\mathcal{C}|$. Let \mathcal{G} be the set of all possible graphs $(\mathcal{V}, \mathcal{E})$ on a given type space \mathcal{X} . When \mathcal{V} is finite, we will consider \mathcal{G} to be endowed with the discrete topology and the corresponding Borel sigma field and refer to a random element of \mathcal{G} as a random graph.

We use $\xrightarrow{\mathbb{P}}$ and \xrightarrow{d} to denote convergence in probability and in distribution respectively. All the unspecified limits are taken as $n \rightarrow \infty$. Given a sequence of events $\{E_n\}_{n \geq 1}$, we say E_n occurs with high probability (whp) if $\mathbb{P}\{E_n\} \rightarrow 1$. The notation O, Ω, Θ and o are defined in the usual manner. More precisely, given a sequence of random variables $\{\xi_n\}_{n \geq 1}$ and a function $f(n)$, we say $\xi_n = O(f)$ if there is a constant C such that $\xi_n \leq Cf(n)$ whp, and we say $\xi_n = \Omega(f)$ if there is a constant C such that $\xi_n \geq Cf(n)$ whp. Say that $\xi_n = \Theta(f)$ if $\xi_n = O(f)$ and $\xi_n = \Omega(f)$. We say $\xi_n = o(f)$ if $\xi_n/f(n) \xrightarrow{\mathbb{P}} 0$.

For a Polish space S , $\mathcal{D}([0, T] : S)$ (resp. $\mathcal{D}([0, \infty) : S)$) denote the space of right continuous functions with left limits (RCLL) from $[0, T]$ (resp. $[0, \infty)$) equipped with the usual Skorohod topology. For a RCLL function $f : [0, \infty) \rightarrow \mathbb{R}$, we write $\Delta f(t) = f(t) - f(t-)$, $t > 0$. Suppose that (S, \mathcal{S}) is a measurable space and we are given a partial ordering on S . We say the S valued random variable ξ **stochastically dominates** $\tilde{\xi}$, and write $\xi \geq_d \tilde{\xi}$ if there exists a coupling between the two random variables on a common probability space such that $\xi^* \geq \tilde{\xi}^*$ a.s., where $\xi^* =_d \xi$ and $\tilde{\xi}^* =_d \tilde{\xi}$. For probability measures $\mu, \tilde{\mu}$ on S , we say μ stochastically dominates $\tilde{\mu}$, and write $\mu \geq_d \tilde{\mu}$ if $\xi \geq_d \tilde{\xi}$ where ξ has distribution μ and $\tilde{\xi}$ has distribution $\tilde{\mu}$. Two examples of S relevant to this work are $\mathcal{D}([0, T] : \mathbb{R})$ and $\mathcal{D}([0, T] : \mathcal{G})$ with the natural associated partial ordering. Given a metric space S , we denote by $\mathcal{B}(S)$ the Borel σ -field on S and by $\text{BM}(S), C_b(S), \mathcal{P}(S)$, the space of bounded (Borel) measurable functions, continuous and bounded function, and probability measures, on S , respectively. The set of nonnegative integers will be denoted by \mathbb{N}_0 .

2.2. Bounded-size rules (BSR). We now define the general class of rules that will be analyzed in this paper. Much of the notation follows [27] which provides a comprehensive analysis of the sub and supercritical regime. Fix $K \in \mathbb{N}$ and let $\Omega_0 = \{\varpi\}$ and $\Omega_K = \{1, 2, \dots, K, \varpi\}$ for $K \geq 1$, where ϖ will represent components of size greater than K . Given a graph \mathbf{G} and a vertex $v \in \mathbf{G}$, write $\mathcal{C}_v(\mathbf{G})$ for the component that contains v . Let

$$c(v) = \begin{cases} |\mathcal{C}_v(\mathbf{G})| & \text{if } |\mathcal{C}_v(\mathbf{G})| \leq K \\ \varpi & \text{if } |\mathcal{C}_v(\mathbf{G})| > K. \end{cases} \quad (2.1)$$

For a quadruple of vertices v_1, v_2, v_3, v_4 , write $\vec{v} = (v_1, v_2, v_3, v_4)$ and let $c(\vec{v}) = (c(v_1), c(v_2), c(v_3), c(v_4))$. Fix $F \subseteq \Omega_K^4$. We now define the random graph process $\{\mathbf{BSR}^{(n)}(k)\}_{k \geq 0}$ on the vertex set $[n]$ evolving through a F -bounded-size rule (F -BSR) as follows. Define $\mathbf{BSR}^{(n)}(0) = \mathbf{0}_n$, the graph on $[n]$ with no edges. Having defined $\mathbf{BSR}^{(n)}(k)$ for $k \geq 0$, $\mathbf{BSR}^{(n)}(k+1)$ is constructed as follows: Choose four vertices $(v_1(k), v_2(k), v_3(k), v_4(k))$ uniformly at random amongst all possible n^4 vertices uniformly at random and let

$$\vec{v}_k = (v_1(k), v_2(k), v_3(k), v_4(k)).$$

Denote the function $c(\vec{v})$ associated with $\mathbf{BSR}^{(n)}(k)$ as $c_k(\vec{v})$. Define

$$\mathbf{BSR}^{(n)}(k+1) = \begin{cases} \mathbf{BSR}^{(n)}(k) \cup (v_1(k), v_2(k)) & \text{if } c_k(\vec{v}_k) \in F \\ \mathbf{BSR}^{(n)}(k) \cup (v_3(k), v_4(k)) & \text{otherwise.} \end{cases} \quad (2.2)$$

These rules are called bounded-size rules since they treat all components of size greater than K identically. Concrete examples of such rules include Erdős-Rényi random graph (here $K = 0$, $F = \Omega_0^4 = \{\varpi, \varpi, \varpi, \varpi\}$) and Bohman-Frieze process (here $K = 1$, $F = \{(1, 1, \alpha, \beta) : \alpha, \beta \in \Omega_1\}$).

Continuous time formulation $\{\mathbf{BSR}^{(n)}(t)\}_{t \geq 0}$: It will be more convenient to work in continuous time. For every quadruple of vertices $\vec{v} = (v_1, v_2, v_3, v_4) \in [n]^4$, let $\mathcal{P}_{\vec{v}}$ be a Poisson process with rate $\frac{1}{2n^3}$, independent between quadruples. The continuous time random graph process $\{\mathbf{BSR}^{(n)}(t)\}_{t \geq 0}$ is constructed recursively as follows. We denote the function $c(v)$ [resp. $c(\vec{v})$] associated with $\mathbf{BSR}^{(n)}(t-)$ as $c_{t-}(v)$ [resp. $c_{t-}(\vec{v})$]. Given $\mathbf{BSR}^{(n)}(t-)$, and that for some $\vec{v} \in [n]^4$, $\mathcal{P}_{\vec{v}}$ has a point at the time instant t , we define

$$\mathbf{BSR}^{(n)}(t) = \begin{cases} \mathbf{BSR}^{(n)}(t-) \cup (v_1, v_2) & \text{if } c_{t-}(\vec{v}) \in F \\ \mathbf{BSR}^{(n)}(t-) \cup (v_3, v_4) & \text{otherwise.} \end{cases} \quad (2.3)$$

The rationale behind this scaling for the rate of the Poisson point process is that the total rate of adding edges is

$$\frac{n^4}{2n^3} = \frac{n}{2}.$$

Thus with this scaling, for the F -BSR rule corresponding to the Erdős-Rényi evolution, the giant component emerges at time $t = 1$. To simplify notation, when there is no scope for confusion, we will suppress n in the notation. For example, we write $\mathbf{BSR}_t := \mathbf{BSR}^{(n)}(t)$.

Denote $\mathcal{C}_i^{(n)}(t)$ for the i -th largest component of \mathbf{BSR}_t . The work of Spencer and Wormald (see [27]) shows that for any given BSR, there exists a (model dependent) **critical time** $t_c > 0$ such that for $t < t_c$, $|\mathcal{C}_1^{(n)}(t)| = O(\log n)$ and for $t > t_c$, $|\mathcal{C}_1^{(n)}(t)| \sim f(t)n$ where $f(t) > 0$.

Along with the size of the components, another key quantity of interest is the complexity of components. Recall the definition of the surplus of a component from (1.1), and denote $\xi_i^{(n)}(t) := \mathbf{spls}(\mathcal{C}_i^{(n)}(t))$ for the surplus of the component $\mathcal{C}_i^{(n)}(t)$. We will be interested in the joint vector of ordered component sizes and corresponding surplus

$$((|\mathcal{C}_i(t)|, \xi_i(t)) : i \geq 1).$$

2.3. Augmented Multiplicative coalescent.

2.3.1. The multiplicative coalescent. Let $l^2 = \{x = (x_1, x_2, \dots) : \sum_i x_i^2 < \infty\}$. Then l^2 is a separable Hilbert space with the inner product $\langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i$, $x = (x_i), y = (y_i) \in l^2$. Let

$$l_{\downarrow}^2 = \{(x_1, x_2, \dots) : x_1 \geq x_2 \geq \dots \geq 0, \sum_i x_i^2 < \infty\}. \quad (2.4)$$

Then l_{\downarrow}^2 is a closed subset of l^2 which we equip with the metric inherited from l^2 . In [4] Aldous introduced a l_{\downarrow}^2 valued continuous time Markov process, called the *standard multiplicative coalescent*, that can be used to describe the asymptotic behavior of suitably scaled component size vector in Erdős-Rényi random graph evolution, near criticality. Subsequently, similar results have been shown to hold for other random graph models (see [6, 7] and references therein). We now give a brief description of this Markov process.

Fix $x = (x_i)_{i \in \mathbb{N}}$. Let $\{\xi_{i,j}, i, j \in \mathbb{N}\}$ be a collection of independent rate one Poisson processes. Given $t \geq 0$, consider the random graph with vertex set \mathbb{N} in which there exist $\xi_{i,j}([0, tx_i x_j / 2]) + \xi_{j,i}([0, tx_i x_j / 2])$ edges between (i, j) , $1 \leq i < j < \infty$, and there are $\xi_{i,i}([0, tx_i^2 / 2])$ self-loops at vertex $i \in \mathbb{N}$. The volume of a component \mathcal{C} of this graph is defined to be

$$\mathbf{vol}(\mathcal{C}) := \sum_{i \in \mathcal{C}} x_i.$$

Let $X_i(x, t)$ be the volume of the i -th largest (by volume) component. It can be shown that $X(x, t) = (X_i(x, t), i \geq 1) \in l_{\downarrow}^2$, a.s. (see Lemma 20 in [4]). Define

$$T_t : \mathbf{BM}(l_{\downarrow}^2) \rightarrow \mathbf{BM}(l_{\downarrow}^2),$$

as $T_t f(x) = \mathbb{E}(f(X(x, t)))$. It is easily checked that $(T_t)_{t \geq 0}$ satisfies the semigroup property $T_{t+s} = T_t T_s$, $s, t \geq 0$, and [4] shows that (T_t) is Feller in the sense that $T_t(C_b(l_{\downarrow}^2)) \subset C_b(l_{\downarrow}^2)$ for all $t \geq 0$. The paper [4] also shows that the semigroup (T_t) along with an initial distribution $\mu \in \mathcal{P}(l_{\downarrow}^2)$ determines a Markov process with values in l_{\downarrow}^2 and RCLL sample paths. Denoting by P^μ the probability distribution of this Markov process on $\mathcal{D}([0, \infty) : l_{\downarrow}^2)$, the Feller property says that $\mu \mapsto P^\mu$ is a continuous map. One special choice of initial distribution for this Markov process is particularly relevant for the study of asymptotics of random graph models.

We now describe this distribution. Let $\{W(t)\}_{t \geq 0}$ be a standard Brownian motion, and for a fixed $\lambda \in \mathbb{R}$, define

$$W_\lambda(t) = W(t) + \lambda t - \frac{t^2}{2}, \quad t \geq 0.$$

Let \hat{W}_λ denote the reflected version of W_λ , i.e.,

$$\hat{W}_\lambda(t) = W_\lambda(t) - \min_{0 \leq s \leq t} W_\lambda(s), \quad t \geq 0. \quad (2.5)$$

An excursion of \hat{W}_λ is an interval $(l, u) \subset [0, +\infty)$ such that $\hat{W}_\lambda(l) = \hat{W}_\lambda(u) = 0$ and $\hat{W}_\lambda(t) > 0$ for all $t \in (l, u)$. Define $u - l$ as the length of the excursion. Order the lengths of excursions of \hat{W}_λ as

$$X_1^*(\lambda) > X_2^*(\lambda) > X_3^*(\lambda) > \dots$$

and write $\mathbf{X}^*(\lambda) = (X_i^*(\lambda) : i \geq 1)$. Then $\mathbf{X}^*(\lambda)$ defines a l_\downarrow^2 valued random variable (see Lemma 25 in [4]) and let μ_λ be its probability distribution. Using the Feller property and connections with certain inhomogeneous random graph models, the paper [4] shows that $\mu_\lambda T_t = \mu_{\lambda+t}$, for all $\lambda \in \mathbb{R}$ and $t \geq 0$, where for $\mu \in \mathcal{P}(l_\downarrow^2)$, $\mu T_t \in \mathcal{P}(l_\downarrow^2)$ is defined in the usual way: $\mu T_t(A) = \int T_t(\mathbf{1}_A)(x) \mu(dx)$, $A \in \mathcal{B}(l_\downarrow^2)$. Using this consistency property one can determine a unique probability measure $\mu_{\text{MC}} \in \mathcal{P}(\mathcal{D}((-\infty, \infty) : l_\downarrow^2))$ such that, denoting the canonical coordinate process on $\mathcal{D}((-\infty, \infty) : l_\downarrow^2)$ by $\{\pi_t\}_{-\infty < t < \infty}$,

$$\mu_{\text{MC}} \circ (\pi_{t+\cdot})^{-1} = P^{\mu t}, \quad \text{for all } t \in \mathbb{R},$$

where $\pi_{t+\cdot}$ is the process $\{\pi_{t+s}\}_{s \geq 0}$. The measure μ_{MC} is known as the *standard multiplicative coalescent*. This measure plays a central role in characterizing asymptotic distribution of component size vectors in the critical window for random graph models [4, 6, 7].

2.3.2. The augmented multiplicative coalescent. We will now augment the above construction and introduce a measure on a larger space that can be used to describe the joint asymptotic behavior of the component size vector and the associated surplus vector, for a broad family of random graph models.

Let $\mathbb{N}^\infty = \{y = (y_1, \dots) : y_i \in \mathbb{N}, \text{ for all } i \geq 1\}$ and define

$$\mathbb{U}_\downarrow = \{(x_i, y_i)_{i \geq 1} \in l_\downarrow^2 \times \mathbb{N}^\infty : \sum_{i=1}^{\infty} x_i y_i < \infty \text{ and } y_m = 0 \text{ whenever } x_m = 0, m \geq 1\}.$$

We will view x_i as the volume of the i -th component and y_i the surplus of the i -th component of a graph with vertex set \mathbb{N} . Writing $x = (x_i)$ and $y = (y_i)$, we will sometimes denote (x_i, y_i) as $z = (x, y)$. We equip \mathbb{U}_\downarrow with the metric

$$\mathbf{d}_\mathbb{U}((x, y), (x', y')) = \left(\sum_{i=1}^{\infty} (x_i - x'_i)^2 \right)^{1/2} + \sum_{i=1}^{\infty} |x_i y_i - x'_i y'_i|. \quad (2.6)$$

The choice of this metric is discussed in Remark 4.15.

Let $\mathbb{U}_\downarrow^0 = \{(x_i, y_i)_{i \geq 1} \in \mathbb{U}_\downarrow : \text{if } x_k = x_m, k \leq m, \text{ then } y_k \geq y_m\}$. We now introduce the *augmented multiplicative coalescent* (AMC). This is a continuous time Markov process with values in $(\mathbb{U}_\downarrow^0, \mathbf{d}_\mathbb{U})$, whose dynamics can heuristically be described as follows: The process jumps at any given time instant from state $(x, y) \in \mathbb{U}_\downarrow^0$ to:

- (x^{ij}, y^{ij}) at rate $x_i x_j$, $i \neq j$, where (x^{ij}, y^{ij}) is obtained by merging components i and j into a component with volume $x_i + x_j$ and surplus $y_i + y_j$ and reordering the coordinates to obtain an element in \mathbb{U}_\downarrow^0 .

- (x, y^i) at rate $x_i^2/2$, $i \geq 1$, where (x, y^i) is the state obtained by increasing the surplus in the i -th component from y_i to $y_i + 1$ and reordering the coordinates (if needed) to obtain an element in \mathbb{U}_\downarrow^0 .

Whenever $z = (x, y) \in \mathbb{U}_\downarrow^0$ is such that $\sum_{i=1}^\infty x_i < \infty$, it is easy to construct a well defined Markov process $\{\mathbf{Z}(z, \lambda)\}_{\lambda \geq 0}$ that corresponds to the above transition mechanism, starting at time $\lambda = 0$ in the state z . However when $\sum_{i=1}^\infty x_i = \infty$, the existence of such a process requires more work. We show in Section 4 (see also Theorem 3.1) that in fact there is a well defined Markov process $\{\mathbf{Z}(z, \lambda)\}_{\lambda \geq 0}$ corresponding to the above dynamical description for *any* $z \in \mathbb{U}_\downarrow^0$. Define, for $\lambda \geq 0$, $\mathcal{T}_\lambda : \text{BM}(\mathbb{U}_\downarrow^0) \rightarrow \text{BM}(\mathbb{U}_\downarrow^0)$ as

$$(\mathcal{T}_\lambda f)(z) = \mathbb{E}f(\mathbf{Z}(z, \lambda)).$$

As for Aldous' multiplicative coalescent, there is one particular family of distributions that plays a special role. Recall the reflected parabolic Brownian motion $\hat{W}_\lambda(t)$ from (2.5). Let \mathcal{P} be a Poisson point process on $[0, \infty) \times [0, \infty)$ with intensity $\lambda_\infty^{\otimes 2}$ (where λ_∞ is the Lebesgue measure on $[0, \infty)$) independent of \hat{W}_λ . Let (l_i, r_i) be the i -th largest excursion of \hat{W}_λ . Define

$$X_i^*(\lambda) = r_i - l_i \text{ and } Y_i^*(\lambda) = |\mathcal{P} \cap \{(t, z) : 0 \leq z \leq \hat{W}_\lambda(t), l_i \leq t \leq r_i\}|.$$

Then $\mathbf{Z}^*(\lambda) = (\mathbf{X}^*(\lambda), \mathbf{Y}^*(\lambda))$ is a.s. a \mathbb{U}_\downarrow^0 valued random variable, where $\mathbf{X}^* = (X_i^*)_{i \geq 1}$ and $\mathbf{Y}^* = (Y_i^*)_{i \geq 1}$. Let ν_λ be its probability distribution. In Theorem 3.1 we will show that there exists a \mathbb{U}_\downarrow^0 valued stochastic process $(\mathbf{Z}(\lambda))_{-\infty < \lambda < \infty}$ such that $\mathbf{Z}(\lambda)$ has probability distribution ν_λ for every $\lambda \in (-\infty, \infty)$ and for all $f \in \text{BM}(\mathbb{U}_\downarrow^0)$, and $\lambda_1 < \lambda_2$, we have

$$\mathbb{E}[f(\mathbf{Z}(\lambda_2)) | \{\mathbf{Z}(\lambda)\}_{\lambda \leq \lambda_1}] = (\mathcal{T}_{\lambda_2 - \lambda_1} f)(\mathbf{Z}(\lambda_1)).$$

The process \mathbf{Z} will be referred to as the *standard augmented multiplicative coalescent*. We will also show that $\{\mathcal{T}_\lambda\}_{\lambda \geq 0}$ is a semigroup, which is *nearly Feller*, in the sense made precise in the statement of Theorem 3.1. It will be seen that this process plays a similar role in characterizing the asymptotic joint distributions of the component size and surplus vector in the critical window as Aldous' standard multiplicative coalescent does in the study of asymptotics of the component size vector.

3. RESULTS

Our first result establishes the existence of the standard augmented coalescent process. Let $\mathbb{U}_\downarrow^1 = \{z = (x, y) \in \mathbb{U}_\downarrow^0 : \sum_i x_i = \infty\}$.

Theorem 3.1. *There is a collection of maps $\{\mathcal{T}_t\}_{t \geq 0}$, $\mathcal{T}_t : \text{BM}(\mathbb{U}_\downarrow^0) \rightarrow \text{BM}(\mathbb{U}_\downarrow^0)$ and a \mathbb{U}_\downarrow^0 valued stochastic process $\{\mathbf{Z}(\lambda)\}_{-\infty < \lambda < \infty} = \{(\mathbf{X}(\lambda), \mathbf{Y}(\lambda))\}_{-\infty < \lambda < \infty}$ such that the following hold.*

- (i) $\{\mathcal{T}_t\}$ is a semigroup: $\mathcal{T}_t \circ \mathcal{T}_s = \mathcal{T}_{t+s}$, $s, t \geq 0$.
- (ii) $\{\mathcal{T}_t\}$ is nearly Feller: For all $t > 0$, $f \in \text{BM}(\mathbb{U}_\downarrow^0)$ and $\{z_n\} \subset \mathbb{U}_\downarrow^0$, such that f is continuous at all points in \mathbb{U}_\downarrow^1 and $z_n \rightarrow z$ for some $z \in \mathbb{U}_\downarrow^1$, we have $\mathcal{T}_t f(z_n) \rightarrow \mathcal{T}_t f(z)$.
- (iii) The marginal distribution of $\mathbf{Z}(\lambda)$ is characterized through the parabolic reflected Brownian motion \hat{W}_λ : For each $\lambda \in \mathcal{R}$, $\mathbf{Z}(\lambda)$ has the probability distribution ν_λ .
- (iv) The stochastic process $\{\mathbf{Z}(\lambda)\}$ satisfies the Markov property with semigroup $\{\mathcal{T}_t\}$: For all $f \in \text{BM}(\mathbb{U}_\downarrow^0)$, and $\lambda_1 < \lambda_2$, we have

$$\mathbb{E}[f(\mathbf{Z}(\lambda_2)) | \{\mathbf{Z}(\lambda)\}_{\lambda \leq \lambda_1}] = (\mathcal{T}_{\lambda_2 - \lambda_1} f)(\mathbf{Z}(\lambda_1)).$$

(v) If $f \in BM(\mathbb{U}_\downarrow^0)$ is such that $f(x, y) = g(x)$ for some $g \in BM(l_\downarrow^2)$, then

$$(\mathcal{T}_t f)(z) = (T_t g)(x), \quad \forall z = (x, y) \in \mathbb{U}_\downarrow^0.$$

Furthermore, $\{\mathbf{X}(\lambda)\}_{-\infty < \lambda < \infty}$ is Aldous's standard multiplicative coalescent.

A precise definition of \mathcal{T}_t can be found in Section 4. Theorem 3.1 will be proved in Section 5.

Throughout this work we fix $K \in \mathbb{N}_0$, $F \in \Omega_K^4$ and consider a F -BSR as introduced in Section 2.2.

The result below considers the asymptotics of the 'susceptibility functions'. For any given time t and fixed $k \geq 1$ define the k -susceptibility function

$$\mathcal{S}_k^{(n)}(t) \equiv \mathcal{S}_k(t) := \sum_{i \geq 1} |\mathcal{C}_i^{(n)}(t)|^k.$$

Define the scaled susceptibility functions by, for $k \geq 1$,

$$\bar{s}_k(t) := \frac{\mathcal{S}_k(t)}{n}. \quad (3.1)$$

Then Theorem 1.1 of [27] shows that for any bounded-size rule, there exists a monotonically increasing function $s_2 : [0, t_c) \rightarrow [0, \infty)$ satisfying $s_2(0) = 1$ and $\lim_{t \uparrow t_c} s_2(t) = \infty$, such that

$$\bar{s}_2(t) \xrightarrow{\mathbb{P}} s_2(t) \quad \forall t \in [0, t_c).$$

Part (iii) of the following result gives a similar result for \bar{s}_3 . Part (ii) in fact gives convergence of \bar{s}_2, \bar{s}_3 in a stronger sense. Part (i) of the theorem gives precise asymptotics of $s_2(t)$ and $s_3(t)$ as $t \uparrow t_c$.

Theorem 3.2. *There exist monotonically increasing functions $s_k : [0, t_c) \rightarrow [0, \infty)$, $k = 2, 3$, such that $s_2(0) = s_3(0) = 1$ and $\lim_{t \uparrow t_c} s_2(t) = \lim_{t \uparrow t_c} s_3(t) = \infty$, having the following properties.*

(i) *There exist $\alpha, \beta \in (0, \infty)$ such that*

$$s_2(t) = (1 + O(t_c - t)) \frac{\alpha}{t_c - t}, \quad s_3(t) = \beta [s_2(t)]^3 (1 + O(t_c - t)), \quad \text{as } t \uparrow t_c. \quad (3.2)$$

(ii) *For every $\gamma \in (1/6, 1/5)$,*

$$\sup_{t \in [0, t_n]} \left| \frac{n^{1/3}}{\bar{s}_2(t)} - \frac{n^{1/3}}{s_2(t)} \right| \xrightarrow{\mathbb{P}} 0 \quad (3.3)$$

$$\sup_{t \in [0, t_n]} \left| \frac{\bar{s}_3(t)}{(\bar{s}_2(t))^3} - \frac{s_3(t)}{(s_2(t))^3} \right| \xrightarrow{\mathbb{P}} 0, \quad (3.4)$$

where $t_n = t_c - n^{-\gamma}$.

(iii) *For all $t \in [0, t_c)$, $\bar{s}_2(t) \xrightarrow{\mathbb{P}} s_2(t)$, $\bar{s}_3(t) \xrightarrow{\mathbb{P}} s_3(t)$ as $n \rightarrow \infty$.*

We now state the main result which gives the asymptotic behavior in the critical scaling window as well as merging dynamics for all bounded-size rules.

Theorem 3.3 (Bounded-size rules: Convergence at criticality). *Let $\alpha, \beta \in (0, \infty)$ be as in Theorem 3.2. For $\lambda \in \mathbb{R}$ define*

$$\bar{\mathbf{C}}^{(n)}(\lambda) := \left(\frac{\beta^{1/3}}{n^{2/3}} \left| \mathcal{C}_i \left(t_c + \frac{\alpha \beta^{2/3}}{n^{1/3}} \lambda \right) \right| : i \geq 1 \right) \quad \text{and} \quad \bar{\mathbf{Y}}^{(n)}(\lambda) = \left(\xi_i \left(t_c + \frac{\alpha \beta^{2/3}}{n^{1/3}} \lambda \right) : i \geq 1 \right).$$

Then $\bar{\mathbf{Z}}^{(n)} = (\bar{\mathbf{C}}^{(n)}, \bar{\mathbf{Y}}^{(n)})$ is a stochastic process with sample paths in $\mathcal{D}((-\infty, \infty) : \mathbb{U}_\downarrow)$ and for any set of times $-\infty < \lambda_1 < \lambda_2 < \dots < \lambda_m < \infty$

$$\left(\bar{\mathbf{Z}}^{(n)}(\lambda_1), \dots, \bar{\mathbf{Z}}^{(n)}(\lambda_m) \right) \xrightarrow{d} (\mathbf{Z}(\lambda_1), \dots, \mathbf{Z}(\lambda_m)) \quad (3.5)$$

as $n \rightarrow \infty$, where \mathbf{Z} is as in Theorem 3.1.

3.1. Background. We now make some comments on the problem background and future directions.

- (a) **Critical random graphs:** Starting with the early work of Erdős-Rényi [15, 16], there is now a large literature on understanding phase transitions in random graph models, see e.g. [12, 13, 20] and the references therein. Proving and identifying phase transitions in dynamic random graph models such as the bounded-size rule requires a relatively new set of ideas and is much more recent [27]. The study of Erdős-Rényi random graph in the critical regime was carried out in [4, 18]. In particular the paper, [4] introduced the standard multiplicative coalescent to understand the merging dynamics of the Erdős-Rényi random graph at criticality. The barely subcritical and supercritical regimes of the Bohman-Frieze process were studied respectively in [22] and [19], with the latter identifying the scaling exponents for the susceptibility functions for the special case of the Bohman-Frieze (BF) process by using the special form of the differential equations for the BF process. The current work extends this result to all bounded-size rules (Theorem 3.2) by viewing such processes as random graph processes with immigration and attachment (see Section 6.5).
- (b) **Unbounded-size rules:** One of the reasons for renewed interest in such models is the recent study of the product rule ([1]), where as before one chooses two edges at random and then uses the edge that minimizes the product of the components at the end points of the chosen edges. This is an example of an unbounded-size rule and simulations in [1] suggest different behavior at criticality as compared to the usual Erdős-Rényi or BF random graph processes. There has been recent progress in rigorously understanding the continuity at the critical point [25] as well the subcritical regime [26]. Such unbounded rules can be regarded as formal limits of K -bounded-size rules analyzed in the current work, as $K \rightarrow \infty$. It would be of great interest to identify and understand the critical scaling window of such processes.
- (c) **Related open questions:** In the context of bounded-size rules our results suggest other related questions. In particular, there has been recent progress in understanding structural properties of the components of the Erdős-Rényi random graph at criticality, in particular see [2, 3] which use information about the surplus and component sizes in [4] to prove that the components viewed as metric spaces, converge to random fractals closely related to the continuum random tree [5] with shortcuts due to surplus edges. Our results strongly suggest the components in any bounded-size rule at criticality belong to the same universality class. Proving this will require substantially new ideas.

3.2. Organization of the paper. The two main results in this paper are Theorems 3.1 and 3.3. In Section 4 we introduce the semigroup $\{\mathcal{T}_t\}_{t \geq 0}$ and, as a first step towards Theorem 3.1, establish in Theorem 4.1 the existence of a \mathbb{U}_\downarrow^0 valued Markov process associated with this semigroup, starting from an arbitrary initial value. Then in Section 5 we complete the proof of Theorem 3.1. We then proceed to the analysis of bounded-size rules in Section 6 where we study the differential equation systems associated with the BSR process and prove Theorems 3.2. Finally in Section 7 we complete the proof of Theorem 3.3.

4. THE AUGMENTED MULTIPLICATIVE COALESCENT

We begin by making precise the formal dynamics of the augmented multiplicative coalescent process given in Section 2.3.2. Fix $(x, y) \in \mathbb{U}_\downarrow^0$. Let $\{\xi_{i,j}\}_{i,j \in \mathbb{N}}$ be a collection of i.i.d. rate one Poisson processes. Let $\mathbf{G}(z, t)$, where $z = (x, y)$, be the random graph on vertex set \mathbb{N} given as follows:

- (I) For $i \in \mathbb{N}$, put y_i initial self-loops to the vertex i .
- (II) For $i < j \in \mathbb{N}$, put $\xi_{i,j}([0, tx_i x_j / 2]) + \xi_{j,i}([0, tx_i x_j / 2])$ edges between vertices i and j . Also, for $i \in \mathbb{N}$, put additional $\xi_{i,i}([0, tx_i^2 / 2])$ self-loops to the vertex i .

Note that the total number of self-loops at a vertex i at time instant t is $y_i + \xi_{i,i}([0, tx_i^2 / 2])$. The self-loops coming from (I) and (II) will later be termed as ‘‘type I’’ and ‘‘type II’’ surplus.

Let $\mathcal{F}_t^x = \sigma\{\xi_{i,j}([0, sx_i x_j / 2]) : 0 \leq s \leq t, i, j \in \mathbb{N}\}$, $t \geq 0$.

Recall the volume of a component \mathcal{C} is defined to be $\mathbf{vol}(\mathcal{C}) = \sum_{i \in \mathcal{C}} x_i$. The surplus of a finite connected graph was defined in (1.1). For infinite graphs the definition requires some care. We define the surplus for a connected graph \mathbf{G} with vertex set a subset of \mathbb{N} as

$$\mathbf{spls}(\mathbf{G}) := \lim_{k \rightarrow \infty} \mathbf{spls}(\mathbf{G}^{[k]}),$$

where $\mathbf{G}^{[k]}$ is the **induced subgraph** that has the vertex set $[k]$ (the subgraph with vertex set $[k]$ and all edges between vertices in $[k]$ that are present in \mathbf{G}). It is easy to check that this definition of surplus does not depend on the labeling of the vertices. Further note that the surplus of a connected graph might be infinite with this definition.

Thus letting $\tilde{\mathcal{C}}_i(t)$ be the i -th largest component (in volume) in $\mathbf{G}(z, t)$, define $X_i(z, t) := \mathbf{vol}(\tilde{\mathcal{C}}_i(t))$ and $Y_i(z, t) := \mathbf{spls}(\tilde{\mathcal{C}}_i(t))$ to be the **volume** and the **surplus** of the i -th largest component at time t . In case two components have the same volume, the ordering of $(\tilde{\mathcal{C}}_i(t) : i \geq 1)$ is taken to be such that $Y_m(z, t) \geq Y_k(z, t)$ whenever $m \leq k$ and $X_m(z, t) = X_k(z, t)$.

Let $\mathbf{X}^z(t) := (X_i(z, t) : i \geq 1)$ and $\mathbf{Y}^z(t) := (Y_i(z, t) : i \geq 1)$. The paper [4] shows that $\mathbf{X}^z(t) \in l_\downarrow^2$ a.s. for all $t \geq 0$. The following result shows that $\mathbf{Z}^z(t) = (\mathbf{X}^z(t), \mathbf{Y}^z(t)) \in \mathbb{U}_\downarrow^0$ a.s., for all t .

Theorem 4.1. *Fix $z = (x, y) \in \mathbb{U}_\downarrow^0$ and let $(\mathbf{X}^z(t), \mathbf{Y}^z(t))_{t \geq 0}$ be the stochastic process described above, then for any fixed $t \geq 0$, $(\mathbf{X}^z(t), \mathbf{Y}^z(t)) \in \mathbb{U}_\downarrow^0$.*

The above theorem will be proved in Section 4.1. For $t \geq 0$, define $\mathcal{T}_t : \mathbf{BM}(\mathbb{U}_\downarrow^0) \rightarrow \mathbf{BM}(\mathbb{U}_\downarrow^0)$ as

$$\mathcal{T}_t f(z) = \mathbb{E} f(\mathbf{Z}^z(t)), \quad z \in \mathbb{U}_\downarrow^0, \quad f \in \mathbf{BM}(\mathbb{U}_\downarrow^0).$$

The following result shows that $\{\mathcal{T}_t\}$ is a semigroup that is (nearly) Feller.

Theorem 4.2. *For $t, s \geq 0$, $\mathcal{T}_t \circ \mathcal{T}_s = \mathcal{T}_{t+s}$. For all $t > 0$, $f \in \mathbf{BM}(\mathbb{U}_\downarrow^0)$ and $\{z_n\} \subset \mathbb{U}_\downarrow^0$, such that f is continuous at all points in \mathbb{U}_\downarrow^1 and $z_n \rightarrow z$ for some $z \in \mathbb{U}_\downarrow^1$, we have $\mathcal{T}_t f(z_n) \rightarrow \mathcal{T}_t f(z)$.*

The above theorem will be proved in Section 4.2. Throughout we will assume, without loss of generality, that for all $z \in \mathbb{U}_\downarrow^0$, \mathbf{Z}^z is constructed using the same set of Poisson processes $\{\xi_{i,j}\}$. This coupling of \mathbf{Z}^z for different values of z will not be noted explicitly in the statement of various results.

We begin with the following elementary lemma.

Lemma 4.3. *Let $\{\mathcal{F}_m\}_{m \in \mathbb{N}_0}$ be a filtration given on some probability space.*

(i) *Let $\{Z_m\}_{m \geq 0}$ be a $\{\mathcal{F}_m\}$ adapted sequence of nondecreasing random variables such that*

$Z_0 = 0$. Let $\lim_{m \rightarrow \infty} Z_m = Z_\infty$. Suppose there exists a nonnegative random variable U such that $U < \infty$ a.s. and $\sum_{m=1}^{\infty} \mathbb{E}[Z_m - Z_{m-1} | \mathcal{F}_{m-1}] \leq U$. Then for any $\epsilon \in (0, 1)$,

$$\mathbb{P}\{Z_\infty > \epsilon\} \leq \frac{1 + \epsilon}{\epsilon} \mathbb{E}[U \wedge 1].$$

(ii) Let $\{A_m\}$ be a sequence of events such that $A_m \in \mathcal{F}_m$. Suppose there exists a random variable $U < \infty$ a.s. such that $\sum_{m=1}^{\infty} \mathbb{E}[\mathbb{1}_{A_m} | \mathcal{F}_{m-1}] \leq U$. Then $\mathbb{P}\{A_m \text{ i.o.}\} = 0$. Furthermore,

$$\mathbb{P}\{\cup_{m=1}^{\infty} A_m\} \leq 2\mathbb{E}[U \wedge 1].$$

Proof: (i) Define $B_0 = 0$ and $B_m := \sum_{i=1}^m \mathbb{E}[Z_i - Z_{i-1} | \mathcal{F}_{i-1}]$ for $m = 1, 2, \dots$. Note that B_m is nondecreasing and \mathcal{F}_{m-1} -measurable. Define $\tau = \inf\{l : B_{l+1} > 1\}$ where the infimum over an empty set is taken to be ∞ . Since B_m is predictable, τ is a stopping time and, for all m , $B_{m \wedge \tau} \leq 1$. Let $B_\infty = \lim_{m \rightarrow \infty} B_m$. Since $Z_{m \wedge \tau} - B_{m \wedge \tau}$ is a martingale, by the optimal stopping theorem and monotone convergence,

$$\mathbb{E}[Z_\tau] = \lim_{m \rightarrow \infty} \mathbb{E}[Z_{m \wedge \tau}] = \lim_{m \rightarrow \infty} \mathbb{E}[B_{m \wedge \tau}] \leq \lim_{m \rightarrow \infty} \mathbb{E}[B_m \wedge 1] = \mathbb{E}[B_\infty \wedge 1].$$

Thus

$$\mathbb{P}\{Z_\infty > \epsilon\} \leq \mathbb{P}\{\tau < \infty\} + \frac{1}{\epsilon} \mathbb{E}[B_\infty \wedge 1] = \mathbb{P}\{B_\infty > 1\} + \frac{1}{\epsilon} \mathbb{E}[B_\infty \wedge 1] \leq \frac{1 + \epsilon}{\epsilon} \mathbb{E}[U \wedge 1].$$

(ii) The first statement is immediate from the Borel-Cantelli lemma (cf. [14, Theorem 5.3.2]). For the second statement note that for any $\epsilon \in (0, 1)$, we have $\cup_{m=1}^{\infty} A_m = \{\sum_{m=1}^{\infty} \mathbb{1}_{A_m} > \epsilon\}$. Now applying part (i) to $Z_m = \sum_{k=1}^m \mathbb{1}_{A_k}$ and taking $\epsilon \rightarrow 1$ yields the desired result. \square

Next, we present a result from [4] that will be used here. We begin with some notation. For $x \in l_{\downarrow}^2$, we write $x^{[k]} = (x_1, \dots, x_k, 0, 0, \dots)$ for the k -truncated version of x . Similarly, for a sequence $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \dots)$ of elements in l_{\downarrow}^2 , $x^{(n)[k]}$ is the k -truncation of $x^{(n)}$. For $z = (x, y)$, $z^{(n)} = (x^{(n)}, y^{(n)}) \in \cup_{\downarrow}^0 z^{[k]}, y^{[k]}, z^{(n)[k]}, y^{(n)[k]}$ are defined similarly.

Recall the construction of $\mathbf{G}(z, t)$ described in items (I) and (II) at the beginning of the section. We will distinguish the surplus created in $\tilde{C}_i(t)$ by the action in item (I) and that in item (II). The former will be referred to as the type I surplus and denoted by $\tilde{Y}_i(z, t)$ while the latter will be referred to as the type II surplus and denoted by $\hat{Y}_i(z, t) \equiv \hat{Y}_i(x, t)$. More precisely,

$$\tilde{Y}_i(z, t) = \sum_{j \in \tilde{C}_i(t)} y_j \quad \text{and} \quad \hat{Y}_i(z, t) = Y_i(z, t) - \tilde{Y}_i(z, t).$$

Also define

$$\tilde{R}(z, t) := \sum_{i=1}^{\infty} X_i(z, t) \tilde{Y}_i(z, t), \quad \hat{R}(x, t) \equiv \hat{R}(z, t) := \sum_{i=1}^{\infty} X_i(z, t) \hat{Y}_i(z, t)$$

and

$$R(z, t) := \sum_{i=1}^{\infty} X_i(z, t) Y_i(z, t), \quad S(x, t) \equiv S(z, t) := \sum_{i=1}^{\infty} (X_i(x, t))^2.$$

The following properties of S and \mathbf{X} have been established in [4, Proposition 5, Corollary 18, Lemma 22].

Theorem 4.4. (i) For every $x \in l_{\downarrow}^2$ and $t \geq 0$, we have $S(x, t) < \infty$ a.s. and $S(x^{[k]}, t) \uparrow S(x, t)$ as $k \rightarrow \infty$.

(ii) If $x^{(n)} \rightarrow x$ in l_{\downarrow}^2 , then $\mathbf{X}(x^{(n)}, t) \xrightarrow{\mathbb{P}} \mathbf{X}(x, t)$ in l_{\downarrow}^2 , as $n \rightarrow \infty$. In particular, $\{S(x^{(n)}, t)\}_{n \geq 1}$ is tight.

4.1. Existence of the augmented MC. This section proves Theorem 4.1. We begin by considering the type I surplus.

Proposition 4.5. *For any $t \geq 0$ and $z \in \mathbb{U}_\downarrow^0$, $\tilde{R}(z, t) = \sum_{i=1}^\infty X_i(z, t)\tilde{Y}_i(z, t) < \infty$ a.s.*

Proof of Proposition 4.5 is given below Lemma 4.7. The basic idea is to bound the truncated version $\tilde{R}^{[k]} = \tilde{R}(z^{[k]}, t)$ using a martingale argument, and then let $k \rightarrow \infty$. The truncation error is controlled using Lemma 4.6 below and a suitable supermartingale is constructed in Lemma 4.7.

Lemma 4.6. *For every $z \in \mathbb{U}_\downarrow^0$ and $t \geq 0$, as $k \rightarrow \infty$, $\tilde{R}(z^{[k]}, t) \rightarrow \tilde{R}(z, t) \leq \infty$ a.s.*

Proof: Fix $t \geq 0$. Denote by E_{ij} [resp. $E_{ij}^{[k]}$] the event that there exists a path from i to j in $\mathbf{G}(z, t)$ [resp. $\mathbf{G}(z^{[k]}, t)$], with the convention that $\mathbb{P}\{E_{ii}\} = \mathbb{P}\{E_{ii}^{[k]}\} = 1$. Let

$$f_i = \sum_{j=1}^\infty y_j \mathbb{1}_{E_{ij}}, \quad f_i^{[k]} = \sum_{j=1}^k y_j \mathbb{1}_{E_{ij}^{[k]}}.$$

Then

$$\tilde{R}(z, t) = \sum_{i=1}^\infty f_i x_i, \quad \tilde{R}(z^{[k]}, t) = \sum_{i=1}^\infty f_i^{[k]} x_i.$$

Since $E_{ij}^{[k]} \uparrow E_{ij}$, we have $f_i^{[k]} \uparrow f_i$. The result now follows from an application of monotone convergence theorem. \square

Lemma 4.7. *Suppose that $z = (x, y) = z^{[k]}$ for some $k \geq 1$ and that $\sum_j y_j \neq 0$. Then*

$$A_t \equiv A(z, t) := \log \tilde{R}(z, t) - \int_0^t S(z, u) du$$

is a supermartingale with respect to the filtration $\mathcal{F}_t^x = \sigma\{\xi_{i,j}([0, sx_i x_j/2]); 0 \leq s \leq t, i, j \in \mathbb{N}\}$.

Proof: From the construction of $\mathbf{Z}(z, \cdot)$ we see that $\tilde{R}(z, t)$ is a pure jump, nondecreasing process that at any time instant t , jumps at rate $X_i(z, t-)X_j(z, t-)$, $1 \leq i < j \leq k$, with jump sizes $B_{ij}(t-) = X_i(z, t-)\tilde{Y}_j(z, t-) + X_j(z, t-)\tilde{Y}_i(z, t-)$. Consequently $\log \tilde{R}(z, t)$ jumps at the same rate, with corresponding jump size $\log(1 + \frac{B_{ij}(t-)}{\tilde{R}(z, t-)})$. From this and elementary properties of Poisson processes it follows that

$$\log \tilde{R}(z, t) = \log \tilde{R}(z, 0) + \sum_{1 \leq i < j \leq k} \int_0^t \log \left(1 + \frac{B_{ij}(u)}{\tilde{R}(z, u)} \right) X_i(z, u) X_j(z, u) du + M(t),$$

where M is a \mathcal{F}_t^x martingale. Consequently, for $0 \leq s < t < \infty$

$$\log \tilde{R}(z, t) - \log \tilde{R}(z, s) = \sum_{1 \leq i < j \leq k} \int_s^t \log \left(1 + \frac{B_{ij}(u)}{\tilde{R}(z, u)} \right) X_i(z, u) X_j(z, u) du + M(t) - M(s). \quad (4.1)$$

Next note that, for $u \geq 0$

$$\begin{aligned}
& \sum_{1 \leq i < j \leq k} \log \left(1 + \frac{B_{ij}(u)}{\tilde{R}(z, u)} \right) X_i(z, u) X_j(z, u) \\
& \leq \sum_{1 \leq i < j \leq k} \frac{B_{ij}(u)}{\tilde{R}(z, u)} X_i(z, u) X_j(z, u) \\
& = \sum_{1 \leq i < j \leq k} \frac{X_i(z, u) \tilde{Y}_j(z, u) + X_j(z, u) \tilde{Y}_i(z, u)}{\tilde{R}(z, u)} X_i(z, u) X_j(z, u) \\
& \leq S(z, u).
\end{aligned}$$

Using this observation in (4.1) we now have

$$\mathbb{E} \left[\log \tilde{R}(z, t) - \log \tilde{R}(z, s) \mid \mathcal{F}_s^x \right] \leq \mathbb{E} \left[\int_s^t S(z, u) du \mid \mathcal{F}_s^x \right].$$

The result follows. \square

Proof of Proposition 4.5: Fix $z = (x, y) \in \mathbb{U}_\downarrow^0$. The result is trivially true if $\sum_i y_i = 0$. Assume now that $\sum_i y_i \neq 0$. For $k \geq 1$ and $a \in (0, \infty)$, define $T_a^{[k]} = \inf\{s \geq 0 : S(z^{[k]}, s) \geq a\}$. Fix $k \geq 1$ and assume without loss of generality that $\sum_{i=1}^k y_i > 0$. Write $R^{[k]}(t) = R(z^{[k]}, t)$, and $A^{[k]}(t) = A(z^{[k]}, t)$ where A is as in Lemma 4.7. From the supermartingale property $\mathbb{E}[A^{[k]}(T_a^{[k]} \wedge t)] \leq \mathbb{E}[A^{[k]}(0)] = \log \tilde{R}^{[k]}(0)$. Therefore

$$\mathbb{E} \left[\log \frac{\tilde{R}^{[k]}(T_a^{[k]} \wedge t)}{\tilde{R}^{[k]}(0)} \right] \leq \mathbb{E} \left[\int_0^{T_a^{[k]} \wedge t} S(z^{[k]}, u) du \right] \leq ta.$$

Thus

$$\mathbb{P}\{\tilde{R}^{[k]}(t) > m\} \leq \mathbb{P}\{\tilde{R}^{[k]}(t) > m, T_a^{[k]} > t\} + \mathbb{P}\{T_a^{[k]} \leq t\} \leq \frac{ta}{\log m - \log \tilde{R}^{[k]}(0)} + \mathbb{P}\{T_a^{[k]} \leq t\}.$$

By Lemma 4.6, $\tilde{R}^{[k]}(t) \rightarrow \tilde{R}(z, t)$, and by Theorem 4.4 (i), $S(z^{[k]}, t) \rightarrow S(z, t)$ when $k \rightarrow \infty$. Therefore letting $k \rightarrow \infty$ on both sides of the above inequality, we have

$$\mathbb{P}\{\tilde{R}(z, t) > m\} \leq \frac{ta}{\log m - \log \tilde{R}(z, 0)} + \mathbb{P}\{S(z, t) \geq a\}. \quad (4.2)$$

The result now follows on first letting $m \rightarrow \infty$ and then $a \rightarrow \infty$ in the above inequality. \square

The following result is an immediate consequence of the estimate in (4.2) and Theorem 4.4(ii).

Corollary 4.8. If $z^{(n)} \rightarrow z$ in \mathbb{U}_\downarrow^0 , then for every $t \geq 0$, $\{\tilde{R}(z^{(n)}, t)\}_{n \geq 1}$ is tight.

Next we consider the type II surplus. Let, for $x \in l_\downarrow^2$

$$\mathcal{G}_t^x := \sigma\{\{\xi_{i,j}([0, sx_i x_j / 2]) = 0\} : 0 \leq s \leq t, i, j \in \mathbb{N}\}.$$

The σ -field \mathcal{G}_t^x records whether or not i and j are in the same component at time s , for all i, j and for all $s \leq t$. In particular, components $\{\tilde{C}_i(s), i \geq 1, s \leq t\}$ can be determined from the information in \mathcal{G}_t^x and consequently, $\mathbf{X}(x, t)$ is \mathcal{G}_t^x measurable.

Lemma 4.9. (i) Fix $x \in l_{\downarrow}^2$ and $t \geq 0$. Then $\hat{R}(x, t) < \infty$ a.s.

(ii) Let $x^{(n)} \rightarrow x$ in l_{\downarrow}^2 . Then the sequence $\{\hat{R}(x^{(n)}, t)\}_{n \geq 1}$ is tight.

Proof: Note that (i) is an immediate consequence of (ii). Consider now (ii). For fixed $x \in l_{\downarrow}^2$ and $t \geq 0$, let $\hat{\mu}_i(x, t)$ denote the conditional law of $\hat{Y}_i(x, t)$, conditioned on \mathcal{G}_t^x . Then, for a.e. ω , $\hat{\mu}_i(x, t)$ is Poisson distribution with parameter

$$\int_0^t \sum_{j=1}^{\infty} \left(\sum_{k, k' \in \tilde{\mathcal{C}}_j(s)} \frac{1}{2} x_k x_{k'} \right) \mathbb{1}_{\{\tilde{\mathcal{C}}_j(s) \subset \tilde{\mathcal{C}}_i(t)\}} ds = \int_0^t \frac{1}{2} \sum_{j=1}^{\infty} (X_j(x, s))^2 \mathbb{1}_{\{\tilde{\mathcal{C}}_j(s) \subset \tilde{\mathcal{C}}_i(t)\}} ds \leq \frac{t}{2} (X_i(x, t))^2,$$

where the last inequality is a consequence of the inequality $\sum_{j: \tilde{\mathcal{C}}_j(s) \subset \tilde{\mathcal{C}}_i(t)} (X_j(x, s))^2 \leq (X_i(x, t))^2$. Therefore $\hat{\mu}_i(x, t) \leq_d \hat{\nu}_i(x, t)$, a.s., where $\hat{\nu}_i(x, t)$ is a random probability measure on \mathbb{N} such that for a.e. ω , $\hat{\nu}_i(x, t)$ is Poisson distribution with parameter $\frac{t}{2} (X_i(x, t, \omega))^2$.

A similar argument shows that the conditional distribution of $\sum_{i=1}^{\infty} \hat{Y}_i(x, t)$, given \mathcal{G}_t^x is a.s. stochastically dominated by a random measure on \mathbb{N} that, for a.e. ω has a Poisson distribution with parameter $\sum_{i=1}^{\infty} \frac{t}{2} (X_i(x, t, \omega))^2 = \frac{t}{2} S(x, t)$. Also, if $x^{(n)}$ is a sequence converging to x in l_{\downarrow}^2 , we have that for each n , the conditional distribution of $\sum_{i=1}^{\infty} \hat{Y}_i(x^{(n)}, t)$, given $\mathcal{G}_t^{x^{(n)}}$ is a.s. stochastically dominated by a Poisson random variable with parameter $\frac{t}{2} S(x^{(n)}, t)$. From Theorem 4.4(ii), $\{S(x^{(n)}, t)\}_{n \geq 1}$ is tight. Combining these facts we have that $\{\sum_{i=1}^{\infty} \hat{Y}_i(x^{(n)}, t)\}_{n \geq 1}$ is a tight family. Finally, note that $\hat{R}(x^{(n)}, t) \leq X_1(x^{(n)}, t) \left(\sum_{i=1}^{\infty} \hat{Y}_i(x^{(n)}, t) \right)$. The tightness of $\{\hat{R}(x^{(n)}, t)\}_{n \geq 1}$ now follows on combining the above established tightness of $\{\sum_{i=1}^{\infty} \hat{Y}_i(x^{(n)}, t)\}_{n \geq 1}$ and the tightness of $\{X_1(x^{(n)}, t)\}_{n \geq 1}$, where the latter is once again a consequence of Theorem 4.4(ii). \square

We now complete the proof of Theorem 4.1.

Proof of Theorem 4.1. Fix $z = (x, y) \in \mathbb{U}_{\downarrow}^0$ and $t \geq 0$. From Lemma 4.9 (i) $\hat{R}(x, t) < \infty$ a.s. Also, from Proposition 4.5, $\tilde{R}(z, t) < \infty$ a.s. The result now follows on recalling that $R(z, t) = \hat{R}(x, t) + \tilde{R}(z, t)$. \square

We also record the following consequence of Lemma 4.9 and Corollary 4.8 for future use.

Corollary 4.10. If $z^{(n)} \rightarrow z$ in $\mathbb{U}_{\downarrow}^0$, then $\{R(z^{(n)}, t)\}_{n \geq 1}$ is tight.

4.2. Feller property of the augmented MC. In this section, we will prove Theorem 4.2. In fact we will show that if $z^{(n)} = (x^{(n)}, y^{(n)})$ converges to $z = (x, y)$ in $\mathbb{U}_{\downarrow}^0$, and $z \in \mathbb{U}_{\downarrow}^1$, then

$$(\mathbf{X}(z^{(n)}, t), \mathbf{Y}(z^{(n)}, t)) \xrightarrow{\mathbb{P}} (\mathbf{X}(z, t), \mathbf{Y}(z, t)). \quad (4.3)$$

We start with the following elementary lemma.

Lemma 4.11. Suppose $(x, y), (x^{(n)}, y^{(n)}) \in \mathbb{U}_{\downarrow}$ for $n \geq 1$. Then

$$\lim_{n \rightarrow \infty} \mathbf{d}_{\mathbb{U}}((x, y), (x^{(n)}, y^{(n)})) = 0$$

if and only if the following three conditions hold:

- (i) $\lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} (x_i^{(n)} - x_i)^2 = 0$.
- (ii) $y_i^{(n)} = y_i$ for n sufficiently large, for all $i \geq 1$.
- (iii) $\lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} x_i^{(n)} y_i^{(n)} = \sum_{i=1}^{\infty} x_i y_i$.

Proof: The “only if” part is immediate. To see the “if” part, note that the first two conditions imply $\lim_{n \rightarrow \infty} x_i^{(n)} y_i^{(n)} = x_i y_i$ for all $i \geq 1$. By the third condition and Scheffe’s lemma, we now have $\lim_{n \rightarrow \infty} \sum_i |x_i^{(n)} y_i^{(n)} - x_i y_i| = 0$. The result follows. \square

The key ingredient in the proof is the following lemma the proof of which is given after Lemma 4.14.

Lemma 4.12. *Let $z^{(n)} = (x^{(n)}, y^{(n)})$ converge to $z = (x, y)$ in \mathbb{U}_\downarrow^0 . Suppose that $z \in \mathbb{U}_\downarrow^1$. Then*

- (i) $Y_i(z^{(n)}, t) \xrightarrow{\mathbb{P}} Y_i(z, t)$ for all $i \geq 1$.
- (ii) $\sum_{i=1}^{\infty} X_i(z^{(n)}, t) Y_i(z^{(n)}, t) \xrightarrow{\mathbb{P}} \sum_{i=1}^{\infty} X_i(z, t) Y_i(z, t)$.

Proof of Theorem 4.2 can now be completed as follows.

Proof of Theorem 4.2. The first part of the theorem is immediate from the construction given at the beginning of Section 4 and elementary properties of Poisson processes. For the second part, consider $z^{(n)} = (x^{(n)}, y^{(n)})$, $z = (x, y)$ as in the statement of the theorem. It suffices to prove (4.3). From Theorem 4.4(ii), $\mathbf{X}(z^{(n)}, t) \rightarrow \mathbf{X}(z, t)$ in probability, in l_\downarrow^2 . The result now follows on combining this convergence with the convergence in Lemma 4.12 (on noting that $(\mathbf{X}(z, t), \mathbf{Y}(z, t)) \in \mathbb{U}_\downarrow^1$ a.s.) and applying Lemma 4.11. \square

Rest of this section is devoted to the proof of Lemma 4.12. The key idea of the proof is as follows. Consider the induced subgraphs on the first k vertices $\mathbf{G}^{[k]} = \mathbf{G}(z^{[k]}, t)$ and $\mathbf{G}^{(n)[k]} = \mathbf{G}(z^{(n)[k]}, t)$. Since there are only finite number of vertices in $\mathbf{G}^{[k]}$, when $n \rightarrow \infty$, $\mathbf{G}^{(n)[k]}$ will eventually be identical to $\mathbf{G}^{[k]}$ almost surely. The main step in the proof is to control the difference between $\mathbf{G}^{(n)[k]}$ and $\mathbf{G}^{(n)}$ when k is large, uniformly for all n . For this we first analyze the difference between $\mathbf{G}^{(n)[k]}$ and $\mathbf{G}^{(n)[k+1]}$ in the lemma below.

Consider the set of vertices $[k+1] = \{1, 2, \dots, k, k+1\}$, and for every $i \in [k+1]$, let vertex i have label (x_i, y_i) representing its size and surplus, respectively. Suppose $x_1 \geq x_2 \geq \dots \geq x_{k+1}$. Fix $t > 0$. Define a random graph \mathbf{G}^* on the above vertex set as follows. For $i \leq k$, the number of edges, N_i , between i and $k+1$ is distributed as $\text{Poisson}(tx_i x_{k+1})$. In addition, there are $N_0 = \text{Poisson}(tx_{k+1}^2/2)$ self-loops to the vertex $k+1$. All the Poisson random variables are taken to be mutually independent.

Denote X_i and Y_i for the component volumes and surplus of the resulting star-like graph if i is the smallest labeled vertex in its component; otherwise let $X_i = Y_i = 0$. A precise definition of (X_i, Y_i) is as follows. Write $i \sim k+1$ if there is an edge between i and $k+1$ in \mathbf{G}^* . By convention $(k+1) \sim (k+1)$. Let $\mathcal{J}_k = \{i \in [k+1] : i \sim k+1\}$, and $i_0 = \min\{i : i \in \mathcal{J}_k\}$. Then

$$(X_i, Y_i) = \begin{cases} \left(\sum_{i \in \mathcal{J}_k} x_i, \sum_{i \in \mathcal{J}_k} y_i \right) & \text{if } i = i_0 \\ (0, 0) & \text{if } i \in \mathcal{J}_k \setminus \{i_0\} \\ (x_i, y_i) & \text{if } i \in [k+1] \setminus \mathcal{J}_k. \end{cases}$$

Define $R_k = \sum_{i=1}^k x_i y_i$, $S_k = \sum_{i=1}^k x_i^2$, $R_{k+1} = \sum_{i=1}^{k+1} X_i Y_i$. Then we have the following result.

Lemma 4.13. (i) $\mathbb{P}\{Y_i \neq y_i\} \leq tx_{k+1} y_{k+1} x_1 + tx_{k+1}^2 (1 + itx_1^2 + tS_k + tR_k x_1)$.
(ii) $\mathbb{E}[R_{k+1} - R_k] \leq x_{k+1} y_{k+1} (1 + tS_k) + x_{k+1}^2 (tR_k + t^2 S_k R_k + t^2 S_k x_1) + tx_{k+1}^3 (1 + 2tS_k + t^2 S_k^2)$.

Proof: (i) It is easy to see that, for $i = 1, \dots, k$,

$$\{Y_i \neq y_i\} \subset (\{y_{k+1} > 0\} \cap \{i \in \mathcal{J}_k\}) \cup \{N_0 \neq 0\} \cup_{j=1}^k \{N_j > 1\} \cup_{j < i} \{N_j N_i \neq 0\} \cup_{j: y_j > 0} \{N_j N_i \neq 0\}.$$

Using the observation that for a Poisson(λ) random variable Z , $\mathbb{P}\{Z \geq 1\} < \lambda$ and $\mathbb{P}\{Z \geq 2\} < \lambda^2$, we now have that

$$\begin{aligned} \mathbb{P}\{Y_i \neq y_i\} &\leq tx_i x_{k+1} \cdot y_{k+1} + \frac{tx_{k+1}^2}{2} + \sum_{j=1}^k (tx_j x_{k+1})^2 \\ &\quad + \sum_{j=1}^{i-1} tx_j x_{k+1} \cdot tx_i x_{k+1} + \sum_{j=1}^k tx_j x_{k+1} \cdot tx_i x_{k+1} \cdot y_j. \end{aligned}$$

The proof is now completed on collecting all the terms and using the fact that $x_i \leq x_1$ for every i .

(ii) Note that

$$X_0 = x_{k+1} + \sum_{j=1}^k x_j \mathbb{1}_{\{N_j \geq 1\}}, \quad Y_0 = y_{k+1} + \sum_{j=1}^k y_j \mathbb{1}_{\{N_j \geq 1\}} + N_0 + \sum_{j=1}^k (N_j - 1)^+.$$

Then

$$\begin{aligned} R_{k+1} - R_k &= X_0 Y_0 - \sum_{j \in \mathcal{J}_k} x_j y_j \\ &= x_{k+1} y_{k+1} + \sum_{j=1}^k (x_j y_{k+1} + x_{k+1} y_j) \mathbb{1}_{\{N_j \geq 1\}} + \sum_{1 \leq j < l \leq k} (x_j y_l + x_l y_j) \mathbb{1}_{\{N_j \geq 1\}} \mathbb{1}_{\{N_l \geq 1\}} \\ &\quad + N_0 X_0 + x_{k+1} \sum_{j=1}^k (N_j - 1)^+ + \sum_{j=1}^k x_j (N_j - 1)^+ \\ &\quad + \sum_{1 \leq j < l \leq k} (x_j \mathbb{1}_{\{N_j \geq 1\}} (N_l - 1)^+ + x_l \mathbb{1}_{\{N_l \geq 1\}} (N_j - 1)^+). \end{aligned}$$

The result now follows on taking expectations in the above equation and using the fact that $\mathbb{E}[(N_j - 1)^+] < (tx_j x_{k+1})^2$. \square

Recall that, by construction, $X_i(z, t) \geq X_{i+1}(z, t)$ for all $z \in \mathbb{U}_\downarrow$, $t \geq 0$ and $i \in \mathbb{N}$. The following lemma which is a key ingredient in the proof of Lemma 4.12 says that if $z \in \mathbb{U}_\downarrow^1$, ties do not occur, a.s.

Lemma 4.14. *Let $z \in \mathbb{U}_\downarrow^1$. Then for every $t > 0$ and $i \in \mathbb{N}$, $X_i(z, t) > X_{i+1}(z, t)$ a.s.*

Proof: Fix $t > 0$. Consider the graph $\mathbf{G}(z, t)$ and write $\mathcal{C}_{x_i} \equiv \mathcal{C}_{x_i}(t)$ for the component of vertex (x_i, y_i) at time t . It suffices to show for all $i \neq j$

$$\mathbb{P}\{|\mathcal{C}_{x_i}| = |\mathcal{C}_{x_j}|, \mathcal{C}_{x_i} \neq \mathcal{C}_{x_j}\} = 0. \quad (4.4)$$

The key property we shall use is that for $z = (x, y) \in \mathbb{U}_\downarrow^1$, $\sum_{i=1}^\infty x_i = \infty$. Now fix $i \geq 1$. It is enough to show that $|\mathcal{C}_{x_i}|$ has no atom i.e for all $(x, y) \in \mathbb{U}_\downarrow^1$

$$\mathbb{P}(|\mathcal{C}_{x_i}| = a) = 0, \quad \text{for any } a \geq 0. \quad (4.5)$$

To see this, first note that since $|\mathcal{C}_{x_i}| < \infty$ a.s., conditional on \mathcal{C}_{x_i} the vector $z^* = ((x_k, y_k) : x_k \notin \mathcal{C}_{x_i}) \in \mathbb{U}_\downarrow^1$ almost surely. Thus on the event $x_j \notin \mathcal{C}_{x_i}$, conditional on \mathcal{C}_{x_i} , using (4.5) with $a = |\mathcal{C}_{x_i}|$ implies that $\mathbb{P}(|\mathcal{C}_{x_j}| = |\mathcal{C}_{x_i}| \mid \mathcal{C}_{x_i}) = 0$ and this completes the proof. Thus it is enough to prove (4.5). For the rest of the argument, to ease notation let $i = 1$. Let us first show

the simpler assertion that the volume of direct neighbors of x_1 has a continuous distribution. More precisely, let $N_{i,j}(t) := \xi_{i,j}([0, tx_i x_j/2]) + \xi_{j,i}([0, tx_i x_j/2])$, $1 \leq i < j$, denote the number of edges between any two vertices x_i and x_j by time t . Then the volume of *direct* neighbors of the vertex x_1 is $L := \sum_{i=2}^{\infty} x_i \mathbf{1}_{\{N_{1,i}(t) \geq 1\}}$ and we will first show that L has no atom, namely

$$\mathbb{P}(L = a) = 0, \quad \text{for all } a \geq 0. \quad (4.6)$$

For any random variable X define the maximum atom size of X by

$$\mathbf{atom}(X) := \sup_{a \in \mathbb{R}} \mathbb{P}\{X = a\}.$$

For two independent random variables X_1 and X_2 we have $\mathbf{atom}(X_1 + X_2) \leq \min\{\mathbf{atom}(X_1), \mathbf{atom}(X_2)\}$. For $m \geq 2$, define $L_m = \sum_{i=m}^{\infty} x_i \mathbf{1}_{\{N_{1,i}(t) \geq 1\}}$. Since L_m and $L - L_m$ are independent, we have $\mathbf{atom}(L) \leq \mathbf{atom}(L_m)$. Define the event

$$E_m := \{N_{1,i}(t) \leq 1 \text{ for all } i \geq m\},$$

and write

$$L_m^*(t) := \sum_{i=m}^{\infty} x_i N_{1,i}(t).$$

Then $L_m^*(t)$ is a pure jump Levy process with Levy measure $\nu(du) = \sum_{i=m}^{\infty} x_i x_i \delta_{x_i}(du)$. By [17], such a Levy process has continuous marginal distribution since the Levy measure is infinite ($\nu(0, \infty) = (\sum_{i=m}^{\infty} x_i)x_1 = \infty$). Thus $L_m^*(t)$ has no atom. Next, for any $a \in \mathbb{R}$,

$$\begin{aligned} \mathbb{P}\{L_m = a\} &\leq \mathbb{P}\{E_m^c\} + \mathbb{P}\{E_m, L_m = a\} = \mathbb{P}\{E_m^c\} + \mathbb{P}\{E_m, L_m^*(t) = a\} \\ &\leq \sum_{i=m}^{\infty} \frac{(tx_1 x_i)^2}{2} + 0 = \frac{t^2 x_1^2}{2} \sum_{i=m}^{\infty} x_i^2. \end{aligned}$$

Thus $\mathbf{atom}(L) \leq \mathbf{atom}(L_m) \leq \frac{t^2 x_1^2}{2} \sum_{i=m}^{\infty} x_i^2$. Since m is arbitrary, we have $\mathbf{atom}(L) = 0$. Thus L is a continuous variable, and (4.6) is proved.

Let us now strengthen this to prove (4.5). Let $\tilde{\mathbf{G}}$ be the subgraph of $\mathbf{G}(z, t)$ obtained by deleting the vertex x_1 and all related edges. Let \tilde{X}_i be the volume of the i -th largest component of $\tilde{\mathbf{G}}$. Note that $\sum_{i=1}^{\infty} \tilde{X}_i = \sum_{i=2}^{\infty} x_i = \infty$ a.s. Conditional on $(\tilde{X}_i)_{i \geq 1}$, let $\tilde{N}_{1,i}$ have Poisson distribution with parameter $tx_1 \tilde{X}_i$. Then

$$\mathcal{C}_{x_1} \stackrel{d}{=} x_1 + \sum_{i=1}^{\infty} \tilde{X}_i \mathbf{1}_{\{\tilde{N}_{1,i} \geq 1\}},$$

where the second term has the same form as the random variable L . Using (4.6) completes the proof. \square

We now proceed to the proof of Lemma 4.12.

Proof of Lemma 4.12. Fix $t > 0$ and $z^{(n)}, z$ as in the statement of the lemma. Denote $Y^{[k]} = Y(z^{[k]}, t)$, $Y^{(n)[k]} = Y(z^{(n)[k]}, t)$. Similarly, denote $\mathcal{C}_i^{[k]}$ and $\mathcal{C}_i^{(n)[k]}$ for the corresponding i -th largest component; and $X_i^{[k]}$ and $X_i^{(n)[k]}$ for their respective sizes. Also, write $X^{(n)} = X(x^{(n)}, t)$ and define $Y^{(n)}, R^{(n)}, S^{(n)}$ similarly.

For $i \in \mathbb{N}$, define the event $E_i^{(n)[k]}$ as,

$$E_i^{(n)[k]} := \{\omega : X_j^{(n)[k]}(\omega) > X_{j+1}^{(n)}(\omega), \text{ for } j = 1, 2, \dots, i\},$$

and define $E_i^{[k]}$ similarly. Then

$$\begin{aligned} \mathbb{P}\{Y_i^{(n)} \neq Y_i(t)\} &\leq \mathbb{P}\{Y_i^{(n)} \neq Y_i^{(n)[k]}\} + \mathbb{P}\{Y_i^{(n)[k]} \neq Y_i^{[k]}\} + \mathbb{P}\{Y_i^{[k]} \neq Y_i(t)\} \\ &\leq \mathbb{P}\{Y_i^{(n)} \neq Y_i^{(n)[k]}, E_i^{(n)[k]}\} + \mathbb{P}\{(E_i^{(n)[k]})^c\} + \mathbb{P}\{Y_i^{(n)[k]} \neq Y_i^{[k]}\} + \mathbb{P}\{Y_i^{[k]} \neq Y_i(t)\}. \end{aligned} \quad (4.7)$$

Note that

$$E_i^{(n)[k]} \subset \{\omega : \mathcal{C}_j^{(n)[k]}(\omega) \subset \mathcal{C}_j^{(n)[m]}(\omega) \subset \mathcal{C}_j^{(n)}(\omega), \text{ for all } j = 1, 2, \dots, i \text{ and } m \geq k\}.$$

Thus the probability of the event $\{Y_i^{(n)[m+1]} \neq Y_i^{(n)[m]}, E_i^{(n)[k]}\}$, for $m \geq k$, can be estimated using Lemma 4.13 (i). More precisely, let $\mathcal{F}^{[m]} = \sigma\{\xi_{i,j}; i, j \leq m\}$ for $m \geq 1$. Then by Lemma 4.13 (i),

$$\begin{aligned} \mathbb{P}\{Y_i^{(n)[m+1]} \neq Y_i^{(n)[m]}, E_i^{(n)[k]} | \mathcal{F}^{[m]}\} &\leq t x_{m+1}^{(n)} y_{m+1}^{(n)} X_1^{(n)[m]} \\ &\quad + t (x_{m+1}^{(n)})^2 (1 + it (X_1^{(n)[m]})^2 + t S^{(n)[m]} + t R^{(n)[m]} X_1^{(n)[m]}), \end{aligned}$$

where $S^{(n)[m]} = \sum_i (X_i^{(n)[m]})^2$ and $R^{(n)[m]} = \sum_i (X_i^{(n)[m]} Y_i^{(n)[m]})$.

Note that $X_1^{(n)[k]} \leq X_1^{(n)}$, $R^{(n)[k]} \leq R^{(n)}$ and $S^{(n)[k]} \leq S^{(n)}$. Thus we have

$$\begin{aligned} \sum_{m=k}^{\infty} \mathbb{P}\{Y_i^{(n)[m+1]} \neq Y_i^{(n)[m]}, E_i^{(n)[k]} | \mathcal{F}^{[m]}\} &\leq t \left(\sum_{m=k+1}^{\infty} x_m^{(n)} y_m^{(n)} \right) X_1^{(n)} \\ &\quad + t \left(\sum_{m=k+1}^{\infty} (x_m^{(n)})^2 \right) (1 + it (X_1^{(n)})^2 + t S^{(n)} + t R^{(n)} X_1^{(n)}). \end{aligned}$$

Denote the right hand side of the above inequality as $U^{(n)[k]}$. Then by Lemma 4.3(ii), we have

$$\mathbb{P}\{Y_i^{(n)} \neq Y_i^{(n)[k]}, E_i^{(n)[k]}\} = \mathbb{P}\left(\bigcup_{m=k}^{\infty} \{Y_i^{(n)[m+1]} \neq Y_i^{(n)[m]}, E_i^{(n)[k]}\}\right) \leq 2\mathbb{E}[U^{(n)[k]} \wedge 1] \quad (4.8)$$

and therefore

$$\mathbb{P}\{Y_i^{(n)} \neq Y_i(t)\} \leq 2\mathbb{E}[U^{(n)[k]} \wedge 1] + \mathbb{P}\{(E_i^{(n)[k]})^c\} + \mathbb{P}\{Y_i^{(n)[k]} \neq Y_i^{[k]}\} + \mathbb{P}\{Y_i^{[k]} \neq Y_i(t)\}. \quad (4.9)$$

Next note that $X_1^{(n)}$, $S^{(n)}$ and $R^{(n)}$ are all tight sequences by Corollary 4.10 and Theorem 4.4(ii). Thus $(1 + it (X_1^{(n)})^2 + t S^{(n)} + t R^{(n)} X_1^{(n)})$ is also tight. Also, since $z^{(n)} \rightarrow z$,

$$\limsup_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{i=k+1}^{\infty} x_i^{(n)} y_i^{(n)} = 0 \text{ and } \limsup_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{i=k+1}^{\infty} (x_i^{(n)})^2 = 0.$$

Combining the above observations we have that $\limsup_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}\{U^{(n)[k]} > \epsilon\} = 0$ for all $\epsilon > 0$. From the inequality

$$\mathbb{E}[U^{(n)[k]} \wedge 1] \leq \mathbb{P}\{U^{(n)[k]} > \epsilon\} + \epsilon$$

we now see that

$$\limsup_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E}[U^{(n)[k]} \wedge 1] = 0. \quad (4.10)$$

Next, from a straightforward extension of Proposition 5 of Aldous [4] we have that $(\mathbf{X}^{(n)}, X_1^{(n)[k]}, \dots, X_i^{(n)[k]}) \xrightarrow{d} (\mathbf{X}(t), X_1^{[k]}, \dots, X_i^{[k]})$ in $\mathbb{I}_+^2 \times \mathbb{R}^i$ when $n \rightarrow \infty$, for each fixed i and k . Combining this with Lemma 4.14 we now see that for fixed i

$$\limsup_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}\{(E_i^{(n)[k]})^c\} = 0.$$

Also, for each fixed k

$$\limsup_{n \rightarrow \infty} \mathbb{P}\{Y_i^{(n)[k]} \neq Y_i^{[k]}\} = 0.$$

Observing that $\lim_{k \rightarrow \infty} Y_i^{[k]} = Y_i(t)$ and the last term in (4.9) does not depend on n , we have that

$$\limsup_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}\{Y_i^{[k]} \neq Y_i(t)\} = 0.$$

Part (i) of the lemma now follows on combining the above observations and taking limit as $n \rightarrow \infty$ and then $k \rightarrow \infty$ in (4.9).

We now prove part (ii) of the lemma. Note that

$$\liminf_{n \rightarrow \infty} R^{(n)} \geq \lim_{n \rightarrow \infty} R^{(n)[k]} = R^{[k]}.$$

With a similar argument as in Lemma 4.6, we have $R^{[k]} \rightarrow R(z, t)$ as $k \rightarrow \infty$. Thus sending $k \rightarrow \infty$ in the above display we have

$$\liminf_{n \rightarrow \infty} R^{(n)} \geq R(z, t). \quad (4.11)$$

To complete the proof, it suffices to show that

$$\text{For any } \epsilon > 0, \lim_{n \rightarrow \infty} \mathbb{P}\{R^{(n)} > R(z, t) + \epsilon\} = 0. \quad (4.12)$$

Note that

$$\begin{aligned} \mathbb{P}\{R^{(n)} - R(z, t) > \epsilon\} &\leq \mathbb{P}\{R^{(n)} - R^{(n)[k]} > \epsilon/2\} + \mathbb{P}\{R^{(n)[k]} - R(z, t) > \epsilon/2\} \\ &\leq \mathbb{P}\{R^{(n)} - R^{(n)[k]} > \epsilon/2\} + \mathbb{P}\{R^{(n)[k]} - R^{[k]} > \epsilon/2\}. \end{aligned} \quad (4.13)$$

The second term on the right side above goes to zero for each fixed k , as $n \rightarrow \infty$. For the first term, note that by Lemma 4.13(ii), for all $m \geq k$

$$\mathbb{E}[R^{(n)[m+1]} - R^{(n)[m]} | \mathcal{F}^{[m]}] \leq x_m^{(n)} y_m^{(n)} U_1^{(n)} + (x_m^{(n)})^2 U_2^{(n)} + (x_{m+1}^{(n)})^3 U_3^{(n)},$$

where $U_1^{(n)} = 1 + tS^{(n)}$, $U_2^{(n)} = tR^{(n)} + t^2S^{(n)}R^{(n)} + t^2S^{(n)}X_1^{(n)}$ and $U_3^{(n)} = t(1 + 2tS^{(n)} + t^2(S^{(n)})^2)$. Thus by Lemma 4.3 (i),

$$\mathbb{P}\{R^{(n)} - R^{(n)[k]} > \epsilon\} \leq (1 + 1/\epsilon)\mathbb{E}[U^{(n)[k]} \wedge 1],$$

where $U^{(n)[k]} = (\sum_{m=k+1}^{\infty} x_m^{(n)} y_m^{(n)})U_1^{(n)} + (\sum_{m=k+1}^{\infty} (x_m^{(n)})^2)U_2^{(n)} + (\sum_{m=k+1}^{\infty} (x_{m+1}^{(n)})^3)U_3^{(n)}$. Note that $U_1^{(n)}$, $U_2^{(n)}$ and $U_3^{(n)}$ are all tight sequences and $z^{(n)} \rightarrow z$. An argument similar to the one used to prove (4.10) now shows that, for all $\epsilon > 0$,

$$\limsup_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}\{R^{(n)} - R^{(n)[k]} > \epsilon\} \leq \left(1 + \frac{1}{\epsilon}\right) \limsup_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E}[U^{(n)[k]} \wedge 1] = 0.$$

The statement in (4.12) now follows on using the above convergence in (4.13) and combining it with the observation below (4.13). This completes the proof of part (ii). \square

Remark 4.15. Lemma 4.12 is at the heart of the (near) Feller property in Theorem 4.2 which is crucial for the proof of the joint convergence in (3.5). The proof of the lemma reveals the reason for considering the metric \mathbf{d}_\downarrow on \mathbb{U}_\downarrow .

One natural metric on \mathbb{U}_\downarrow , denoted by \mathbf{d}_1 , is the one obtained by replacing the second term in (2.6) with

$$\sum_{i=1}^{\infty} \frac{|y_i - y'_i|}{2^i} \wedge 1.$$

This metric corresponds to the topology on \mathbb{U}_\downarrow inherited from $\ell^2 \times \mathbb{N}^\infty$ taking the topology generated by the inner product $\langle \cdot, \cdot \rangle$ on ℓ^2 and the product topology on \mathbb{N}^∞ ; and then considering the product topology on $\ell^2 \times \mathbb{N}^\infty$.

Another metric (which we denote by \mathbf{d}_2) that can be considered on \mathbb{U}_\downarrow corresponds to replacing the second term in (2.6) with $\mathbf{d}_{vt}(\mu_z, \mu_{z'})$, where $\mu_z = \sum_{i=1}^\infty \delta_{z_i}$, $\mu_{z'} = \sum_{i=1}^\infty \delta_{z'_i}$ and \mathbf{d}_{vt} is the metric corresponding to the vague topology on the space of $\mathbb{N} \cup \{\infty\}$ valued locally finite measures on $(0, \infty) \times \mathbb{N}$.

The proof of Lemma 4.12 hinges upon the convergence of $\sum_{m=1}^\infty x_m^{(n)} y_m^{(n)}$ to $\sum_{m=1}^\infty x_m y_m$, as $n \rightarrow \infty$, even for the proof of convergence of $Y_i(z^{(n)}, t) \xrightarrow{\mathbb{P}} Y_i(z, t)$. Since \mathbf{d}_1 and \mathbf{d}_2 give no control over sums of the form $\sum_{m=1}^\infty x_m y_m$, this suggests that the convergence in \mathbf{d}_1 or \mathbf{d}_2 is “too weak” to yield the desired Feller property.

5. THE STANDARD AUGMENTED MULTIPLICATIVE COALESCENT.

In this section we prove Theorem 3.1. The Proposition 4 of [4] proves a very useful result on convergence of component size vectors of a general family of non-uniform random graph models to the ordered excursion lengths of \hat{W}_λ . We begin in this section by extending this result to the joint convergence of component size and component surplus vectors in \mathbb{U}_\downarrow , under a slight strengthening of the conditions assumed in [4].

Recall the excursion lengths and mark count process $\mathbf{Z}^*(\lambda) = (\mathbf{X}^*(\lambda), \mathbf{Y}^*(\lambda))$ defined in Section 2.3.2. Our first result below shows that, for fixed $\lambda \in \mathbb{R}$, $\mathbf{Z}^*(\lambda)$ arises as a limit of $\mathbf{Z}(z^{(n)}, q^{(n)})$ in \mathbb{U}_\downarrow for all sequences $\{z^{(n)}\} \subset \mathbb{U}_\downarrow$ and $q^{(n)} = q_\lambda^{(n)} \subset (0, \infty)$ that satisfy certain regularity conditions.

For $n \geq 1$, let $z^{(n)} = (x^{(n)}, y^{(n)}) \in \mathbb{U}_\downarrow^0$. Writing $z_i^{(n)} = (x_i^{(n)}, y_i^{(n)})$, $i \geq 1$, define

$$x^{*(n)} = \sup_{i \geq 1} x_i^{(n)}, \quad s_r^{(n)} = \sum_{i=1}^\infty (x_i^{(n)})^r, \quad r \geq 1.$$

Note that $x^{*(n)} = x_1^{(n)}$ since the sequence is ordered. Let $\{q^{(n)}\}$ be a nonnegative sequence. We will suppress (n) from the notation unless needed.

Theorem 5.1. *Let $z^{(n)} = (z_1^{(n)}, \dots) \in \mathbb{U}_\downarrow^0$ be such that $x_i^{(n)} = 0$ for all $i > n$ and $y_i^{(n)} = 0$ for all $i \geq 1$. Suppose that, as $n \rightarrow \infty$,*

$$\frac{s_3}{(s_2)^3} \rightarrow 1, \quad q - \frac{1}{s_2} \rightarrow \lambda, \quad \frac{x^*}{s_2} \rightarrow 0, \quad (5.1)$$

and, for some $\varsigma \in (0, \infty)$,

$$s_1 \cdot \left(\frac{x^*}{s_2} \right)^\varsigma \rightarrow 0. \quad (5.2)$$

Then $\mathbf{Z}^{(n)} = \mathbf{Z}(z^{(n)}, q^{(n)})$ converges in distribution in \mathbb{U}_\downarrow to $\mathbf{Z}^*(\lambda)$.

Remark: The convergence assumption in (5.1) is the same as that in Proposition 4 of [4]. The additional assumption in (5.2) is not very stringent as will be seen in Section 7 when this result is applied to a general family of bounded-size rules.

Given Theorem 5.1, the proof of Theorem 3.1 can now be completed as follows.

Proof of Theorem 3.1. The first two parts of the theorem were shown in Theorem 4.2. Also, part (v) of the theorem is immediate from the definition of $\{T_t\}$ in Section 2.3.1. Recall the definition of ν_λ from Section 2.3.2. In order to prove parts (iii)-(iv) it suffices to show that

$$\text{for any } \lambda_1, \lambda_2 \in \mathbb{R}, \lambda_1 \leq \lambda_2, \nu_{\lambda_1} \mathcal{T}_{\lambda_2 - \lambda_1} = \nu_{\lambda_2}. \quad (5.3)$$

Indeed, using the semigroup property of (\mathcal{T}_λ) and the above relation, it is straightforward to define a consistent family of finite dimensional distributions $\mu_{\lambda_1, \dots, \lambda_k}$ on $(\mathbb{U}_\downarrow)^{\otimes k}$, $-\infty < \lambda_1 < \lambda_2, \dots, \lambda_k < \infty$, $k \geq 1$, such that $\mu_\lambda = \nu_\lambda$ for every $\lambda \in \mathbb{R}$. The desired result then follows from Kolmogorov's consistency theorem.

We now prove (5.3). Let

$$z^{(n)} = (x^{(n)}, y^{(n)}), \quad x_i^{(n)} = n^{-2/3}, \quad y_i^{(n)} = 0, \quad i = 1, \dots, n, \quad q_{\lambda_j}^{(n)} = \lambda_j + n^{1/3}, \quad j = 1, 2.$$

We set $z_i^{(n)} = 0$ for $i > n$. Note that with this choice of $x^{(n)}$, $s_1 = n^{1/3}$, $s_2 = n^{-1/3}$, $s_3 = n^{-1}$ and so clearly (5.1) and (5.2) (with any $\varsigma > 1$) are satisfied with $q = q_{\lambda_j}$, $\lambda = \lambda_j$, $j = 1, 2$. Thus, denoting the distribution of $\mathbf{Z}(z^{(n)}, q_{\lambda_j}^{(n)})$ by $\nu_{\lambda_j}^{(n)}$, we have by Theorem 5.1 that

$$\nu_{\lambda_j}^{(n)} \rightarrow \nu_{\lambda_j}, \quad \text{as } n \rightarrow \infty. \quad (5.4)$$

Also, from the construction of $\mathbf{Z}(z, t)$ in Section 4, it is clear that $\nu_{\lambda_1}^{(n)} \mathcal{T}_{\lambda_2 - \lambda_1} = \nu_{\lambda_2}^{(n)}$. The result now follows on combining the convergence in (5.4) with Theorem 4.2 and observing that $\mathbf{Z}^*(\lambda) \in \mathbb{U}_\downarrow^1$ a.s. for every $\lambda \in \mathbb{R}$. \square

Rest of this section is devoted to the proof of Theorem 5.1 and is organized as follows. Recall the random graph process $\mathbf{G}(z, q)$, for $z \in \mathbb{U}_\downarrow$, $q \geq 0$, defined at the beginning of Section 4. In Section 5.1 we will give an equivalent in law construction of $\mathbf{G}(z, q)$, from [4], that defines the random graph simultaneously with a certain breadth-first-exploration random walk. The excursions of the reflected version of this walk encode the component sizes of the random graph while the area under the excursions gives the parameter of the Poisson distribution describing the (conditional) law of the surplus associated with the corresponding component. Using this construction, in Theorem 5.2, we will first prove a weaker result than Theorem 5.1 which proves the convergence in distribution of $\mathbf{Z}^{(n)}$ to $\mathbf{Z}^*(\lambda)$ in $l_\downarrow^2 \times \mathbb{N}^\infty$, where we consider the product topology on \mathbb{N}^∞ . This result is proved in Section 5.2. In Section 5.3 we will give the proof of Theorem 5.1 using Theorem 5.2 and an auxiliary tightness lemma (Lemma 5.4). Finally, proof of Lemma 5.4 is given in Section 5.4.

5.1. Breadth First Exploration Walk. In this section, following [4], we will give an equivalent in law construction of $\mathbf{G}(z, q)$ that defines the random graph simultaneously with a certain breadth-first-exploration random walk. Given $q \in (0, \infty)$ and $z \in \mathbb{U}_\downarrow^0$ such that $x_i = 0$ for all $i > n$ and $y_i = 0$ for all i , we will construct a random graph $\bar{\mathbf{G}}(z, q)$ that is equivalent in law to $\mathbf{G}(z, q)$, in two stages, as follows. We begin with a graph on $[n]$ with no edges. Let $\{\eta_{i,j}\}_{i,j \in \mathbb{N}}$ be independent Poisson point processes on $[0, \infty)$ such that η_{ij} for $i \neq j$ has intensity qx_j ; and for $i = j$ has intensity $qx_i/2$.

Stage I: The breadth-first-search forest and associated random walk: Choose a vertex $v(1) \in [n]$ with $\mathbb{P}(v(1) = i) \propto x_i$. Let

$$\mathbb{I}_1 = \{j \in [n] : j \neq v(1) \text{ and } \eta_{v(1),j} \cap [0, x_{v(1)}] \neq \emptyset\}.$$

Form an edge between $v(1)$ and each $j \in \mathbb{I}_1$. Let $c(1) = |\mathbb{I}_1|$. Let $m_{v(1),j}$ be the first point in $\eta_{v(1),j}$ for each $j \in \mathbb{I}_1$. Order the vertices in \mathbb{I}_1 according to increasing values of $m_{v(1),j}$ and label these as $v(2), \dots, v(c(1) + 1)$. Let

$$\mathcal{V}_1 = \{v(1)\}, \quad \mathcal{N}_1 = \{v(2), \dots, v(c(1) + 1)\}, \quad l_1 = x_{v(1)} \text{ and } d_1 = c(1).$$

Having defined $\mathcal{V}_{i'}$, $\mathcal{N}_{i'}$, $l_{i'}$, $d_{i'}$ and the edges up to step i' , with $\mathcal{V}_{i'} = \{v(1), \dots, v(i')\}$, $\mathcal{N}_{i'} = \{v(i'+1), v(i'+2), \dots, v(d_{i'} + 1)\}$ for $1 \leq i' \leq i - 1$, define, if $\mathcal{N}_{i-1} \neq \emptyset$

$$\mathbb{I}_i = \{j \in [n] : j \notin \mathcal{N}_{i-1} \cup \mathcal{V}_{i-1} \text{ and } \eta_{v(i),j} \cap [0, x_{v(i)}] \neq \emptyset\}$$

and form an edge between $v(i)$ and each $j \in \mathbb{I}_i$. Let $c(i) = |\mathbb{I}_i|$ and let $m_{v(i),j}$ be the first point in $\eta_{v(i),j}$ for each $j \in \mathbb{I}_i$. Order the vertices in \mathbb{I}_i according to increasing values of $m_{v(i),j}$ and label these as $v(d_{i-1} + 2), \dots, v(d_i + 1)$, where $d_i = d_{i-1} + c(i)$. Set

$$l_i = l_{i-1} + x_{v(i)}, \quad \mathcal{V}_i = \{v(1), \dots, v(i)\}, \quad \mathcal{N}_i = \{v(i+1), v(i+2), \dots, v(d_i + 1)\}.$$

In case $\mathcal{N}_{i-1} = \emptyset$, we choose $v(i) \in [n] \setminus \mathcal{V}_{i-1}$ with probability proportional to x_j , $j \in [n] \setminus \mathcal{V}_{i-1}$ and define $\mathbb{I}_i, c(i), d_i, l_i, \mathcal{V}_i, \mathcal{N}_i$ and the edges at step i exactly as above.

This procedure terminates after exactly n steps at which point we obtain a forest-like graph with no surplus edges. We will include surplus to this graph in stage II below.

Associate with the above construction an (interpolated) random walk process $H^{(n)}(\cdot)$ defined as follows. $H^{(n)}(0) = 0$ and

$$H^{(n)}(l_{i-1} + u) = H^{(n)}(l_{i-1}) - u + \sum_{j \notin \mathcal{V}_i \cup \mathcal{N}_{i-1}} x_j \mathbb{1}_{\{m_{v(i),j} < u\}} \quad \text{for } 0 < u < x_{v(i)}, \quad i = 1, \dots, n, \quad (5.5)$$

where by convention $l_0 = 0$ and $\mathcal{N}_0 = \emptyset$. This defines $H^{(n)}(t)$ for all $t \in [0, l_n)$. Define $H^{(n)}(t) = H^{(n)}(l_n -)$ for all $t \geq l_n$.

Stage II: Construction of surplus edges: For each $i = 1, \dots, n$, we construct surplus edges on the graph obtained in Stage I and a point process \mathcal{P}_x on $[0, l_n]$, simultaneously, as follows.

- (i) For each $v \in \mathbb{I}_i$ and $\tau \in \eta_{v(i),v} \cap [0, x_{v(i)}] \setminus \{m_{v(i),v}\}$, construct an edge between $v(i)$ and v . This corresponds to multi-edges between the two vertices $v(i)$ and v .
- (ii) For each $\tau \in \eta_{v(i),v(i)} \cap [0, x_{v(i)}]$, construct an edge between $v(i)$ and itself. This corresponds to self-loops at the vertex $v(i)$.
- (iii) For each $v(j) \in \mathcal{N}_{i-1} \setminus \{v(i)\}$ and $\tau \in \eta_{v(i),v(j)} \cap [0, x_{v(i)}]$, construct an edge between $v(i)$ and $v(j)$. This corresponds to additional edges between two vertices, $v(i)$ and $v(j)$, that were indirectly connected in stage I.

For each of the above cases, we also construct points for the point process \mathcal{P}_x at time $l_{i-1} + \tau \in [0, l_n]$.

This completes the construction of the graph $\bar{\mathbf{G}}(z, q)$ and the random walk $H^{(n)}(\cdot)$. This graph has the same law as $\mathbf{G}(z, q)$, so the associated component sizes and surplus vector denoted by $(\bar{\mathbf{X}}(z, q), \bar{\mathbf{Y}}(z, q))$ has the same law as that of $(\mathbf{X}(z, q), \mathbf{Y}(z, q))$. Furthermore, conditioned on $H^{(n)}$, \mathcal{P}_x is Poisson point process on $[0, l_n]$ whose intensity we denote by $r_x(t)$.

Using the above construction we will show in next section, as a first step, a weaker result than Theorem 5.1.

5.2. Convergence in $l_{\downarrow}^2 \times \mathbb{N}^{\infty}$. The following is the main result of this section.

Theorem 5.2. *Let $z^{(n)} \in \mathbb{U}_{\downarrow}^0$ and $q^{(n)} \in (0, \infty)$ be sequences that satisfy the conditions in Theorem 5.1. Then*

$$(\mathbf{X}^{(n)}, \mathbf{Y}^{(n)}) \xrightarrow{d} (\mathbf{X}^*(\lambda), \mathbf{Y}^*(\lambda)) \quad (5.6)$$

in the space $l_{\downarrow}^2 \times \mathbb{N}^{\infty}$ as $n \rightarrow \infty$, where we consider the product topology on \mathbb{N}^{∞} .

The key ingredient in the proof is the following result. With $z^{(n)}$ and $q^{(n)}$ as in the above theorem, define $\bar{\mathbf{X}}^{(n)} = \bar{\mathbf{X}}(z^{(n)}, q^{(n)})$, $\bar{\mathbf{Y}}^{(n)} = \bar{\mathbf{Y}}(z^{(n)}, q^{(n)})$ and $r^{(n)}(t) = r_{x^{(n)}}(t) \mathbb{1}_{[0, l_n]}(t)$, $t \geq 0$. Denote the random walk process from Section 5.1 constructed using $(x^{(n)}, q^{(n)})$ (rather than (x, q)), once more, by $H^{(n)}(\cdot)$.

Define the rescaled process $\bar{H}^{(n)}(\cdot)$ and its reflected version $\hat{H}^{(n)}(\cdot)$ as follows

$$\bar{H}^{(n)}(t) := \sqrt{\frac{s_2}{s_3}} H^{(n)}(t), \quad \hat{H}^{(n)}(t) := \bar{H}^{(n)}(t) - \min_{0 \leq u \leq t} \bar{H}^{(n)}(u). \quad (5.7)$$

Lemma 5.3. (i) As $n \rightarrow \infty$, the process $\bar{H}^{(n)} \xrightarrow{d} W_\lambda$ in $\mathcal{D}([0, \infty) : \mathbb{R})$.
(ii) For $n \geq 1$,

$$\sup_{t \geq 0} \left| r^{(n)}(t) - q \sqrt{\frac{s_3}{s_2}} \hat{H}^{(n)}(t) \right| \leq \frac{3}{2} qx_*. \quad (5.8)$$

Given Lemma 5.3, the proof of Theorem 5.2 can be completed as follows.

Proof of Theorem 5.2: The paper [4] shows that the vector $\bar{\mathbf{X}}^{(n)}$ can be represented as the ordered sequence of excursion lengths of the process $\hat{H}^{(n)}$. Also, weak convergence of $\bar{H}^{(n)}$ to W_λ in Lemma 5.3 (i) implies the convergence of $\hat{H}^{(n)}$ to \hat{W}_λ . Using these facts, Proposition 4 of [4] shows that $\bar{\mathbf{X}}^{(n)}$ converges in distribution to the ordered excursion length sequence of \hat{W}_λ , namely $\mathbf{X}^*(\lambda)$, in l_\downarrow^2 . Also, conditional on $\hat{H}^{(n)}$, \mathcal{P}_x is a Poisson point process on $[0, \infty)$ with rate $r^{(n)}(t)$ and for $i \geq 1$, $\bar{Y}_i^{(n)}$ has a Poisson distribution with parameter $\int_{[a_i^{(n)}, b_i^{(n)}]} r^{(n)}(s) ds$, where $a_i^{(n)}, b_i^{(n)}$ are the left and right endpoints of the i -th ordered excursion of $\hat{H}^{(n)}$. From conditions in (5.1) it follows that $x^* \rightarrow 0$ and $s_2 \rightarrow 0$, further more we have $qx^* \rightarrow 0$ and $q\sqrt{s_3/s_2} \rightarrow 1$. Lemma 5.3 (ii) then shows that $\int_{[a_i^{(n)}, b_i^{(n)}]} r^{(n)}(s) ds$ converges in distribution to $\int_{[a_i, b_i]} \hat{W}_\lambda(s) ds$, where a_i, b_i are the left and right endpoints of the i -th ordered excursion of \hat{W}_λ . In fact we have the joint convergence of $\left(\hat{H}^{(n)}, \left(\int_{[a_i^{(n)}, b_i^{(n)}]} r^{(n)}(s) ds \right)_{i \geq 1} \right)$ to $\left(\hat{W}_\lambda, \left(\int_{[a_i, b_i]} \hat{W}_\lambda(s) ds \right)_{i \geq 1} \right)$. This proves the convergence of $(\bar{\mathbf{X}}^{(n)}, \bar{\mathbf{Y}}^{(n)})$ to $(\mathbf{X}^*(\lambda), \mathbf{Y}^*(\lambda))$ in $l_\downarrow^2 \times \mathbb{N}^\infty$. The result follows since $(\bar{\mathbf{X}}^{(n)}, \bar{\mathbf{Y}}^{(n)})$ has the same law as $(\mathbf{X}^{(n)}, \mathbf{Y}^{(n)})$. \square

Proof of Lemma 5.3 Part (i) was proved in Proposition 4 of [4]. Consider now (ii). For $j = 1, 2, \dots$, define $\delta_{v(j)} = 1_{\{\mathcal{N}_{j-1} = \emptyset\}}$, i.e. $\delta_{v(j)}$ is 1 if $v(j)$ is the first vertex that is explored in its component during the breadth-first-search, and is 0 otherwise. It is easy to verify that $H^{(n)}$ satisfies

$$H^{(n)}(l_i) = - \sum_{j=1}^i \delta_{v(j)} x_{v(j)} + \sum_{v \in \mathcal{N}_i} x_v, \quad i = 1, \dots, n.$$

The above equation implies that for all $k \leq i$, $H^{(n)}(l_k) \geq - \sum_{j=1}^i \delta_{v(j)} x_{v(j)}$. In addition, taking $k_0 = \sup \{j \leq i : \delta_{v(j)} = 1\}$ we have $H^{(n)}(l_{k_0}) = - \sum_{j=1}^i \delta_{v(j)} x_{v(j)}$. In particular, this implies that $\inf_{j \leq i} H^{(n)}(l_j) = - \sum_{j=1}^i \delta_{v(j)} x_{v(j)}$. Also, from (5.5) we have that for $t \in (l_{i-1}, l_i]$, $H^{(n)}(t) \geq H^{(n)}(l_{i-1}) - x^*$. Consequently

$$\left| \inf_{0 \leq u \leq t} H^{(n)}(u) + \sum_{j=1}^{i-1} \delta_{v(j)} x_{v(j)} \right| = \left| \inf_{0 \leq u \leq t} H^{(n)}(u) - \inf_{\{j: l_j \leq t\}} H^{(n)}(l_j) \right| \leq x^*. \quad (5.9)$$

Let $\mathcal{N}_{i-1} = \{v(i), v(i+1), \dots, v(i+l)\}$. From the above expression for $H^{(n)}(l_i)$, we have that for $t \in (l_{i-1}, l_i]$

$$H^{(n)}(t) = \left(- \sum_{j=1}^{i-1} \delta_{v(j)} x_{v(j)} + \sum_{j=i}^{i+l} x_{v(j)} \right) - (t - l_{i-1}) + \sum_{j \notin \mathcal{V}_i \cup \mathcal{N}_{i-1}} x_j \mathbb{1}_{\{m_{v(i),j} < t - l_{i-1}\}}, \quad (5.10)$$

Also, accounting for the three sources of surplus described in Stage II of the construction, one has the following formula for $r^{(n)}(t)$ at time $t \in (l_{i-1}, l_i]$:

$$r^{(n)}(t) = q \cdot \left(\frac{x_{v(i)}}{2} + \sum_{j=i+1}^{i+l} x_{v(j)} + \sum_{j \notin \mathcal{V}_i \cup \mathcal{N}_{i-1}} x_j \mathbb{1}_{\{m_{v(i),j} < t - l_{i-1}\}} \right).$$

The three terms in the above expression correspond to self-loops; edges between vertices that in stage I were only connected indirectly; and additional edges between two vertices that were directly connected in stage I. Combining the above expression with (5.10) and (5.9), we have

$$\left| r^{(n)}(t) - q \cdot \left(H^{(n)}(t) - \min_{0 \leq s \leq t} H^{(n)}(s) \right) \right| \leq q \cdot \left(\left| \inf_{0 \leq s \leq t} H^{(n)}(s) + \sum_{j=1}^{i-1} \delta_{v(j)} x_{v(j)} \right| + \frac{x_{v(i)}}{2} \right) \leq \frac{3}{2} q x^*. \quad (5.11)$$

The result follows. \square

5.3. Proof of Theorem 5.1. In this section we complete the proof of Theorem 5.1. The key step in the proof is the following lemma whose proof is given in Section 5.4.

Lemma 5.4. *Let $z^{(n)} \in \mathbb{U}_{\downarrow}^0$ and $q^{(n)} \in (0, \infty)$ be as in Theorem 5.1. Let $\hat{H}^{(n)}$ be as introduced in (5.7). Then $\{\sup_{t \geq 0} \hat{H}^{(n)}(t)\}_{n \geq 1}$ is a tight family of \mathbb{R}_+ valued random variables.*

Remark 5.5. In fact one can establish a stronger statement, namely $\sup_{u \geq t} \sup_{n \geq 1} \hat{H}_u^{(n)} \rightarrow 0$ in probability as $t \rightarrow \infty$. Also, although not used in this work, using very similar techniques as in the proof of Lemma 5.4, it can be shown that $\sup_{u \geq t} \hat{W}_\lambda(u)$ converges a.s. to 0, as $t \rightarrow \infty$.

Proof of Theorem 5.1. Since $(\mathbf{X}^{(n)}, \mathbf{Y}^{(n)})$ has the same distributions as $(\bar{\mathbf{X}}^{(n)}, \bar{\mathbf{Y}}^{(n)})$, we can equivalently consider the convergence of the latter sequence. From Theorem 5.2 we have that $(\bar{\mathbf{X}}^{(n)}, \bar{\mathbf{Y}}^{(n)})$ converges to $(\mathbf{X}^*(\lambda), \mathbf{Y}^*(\lambda))$, in distribution, in $l_{\downarrow}^2 \times \mathbb{N}^\infty$ (with product topology on \mathbb{N}^∞). By appealing to Skorohod representation theorem, we can assume without loss of generality that the convergence is almost sure. By the definition of $\mathbf{d}_{\mathbb{U}_{\downarrow}}$ it now suffices to argue that

$$\sum_{i=1}^{\infty} |\bar{X}_i^{(n)} \bar{Y}_i^{(n)} - X_i^*(\lambda) Y_i^*(\lambda)| \xrightarrow{\mathbb{P}} 0.$$

Fix $\epsilon > 0$. Then, for any $k \in \mathbb{N}$,

$$\begin{aligned} \mathbb{P} \left\{ \sum_{i=1}^{\infty} |\bar{X}_i^{(n)} \bar{Y}_i^{(n)} - X_i^*(\lambda) Y_i^*(\lambda)| > \epsilon \right\} &\leq \mathbb{P} \left\{ \sum_{i=1}^k |\bar{X}_i^{(n)} \bar{Y}_i^{(n)} - X_i^*(\lambda) Y_i^*(\lambda)| > \frac{\epsilon}{3} \right\} \\ &\quad + \mathbb{P} \left\{ \sum_{i=k+1}^{\infty} \bar{X}_i^{(n)} \bar{Y}_i^{(n)} > \frac{\epsilon}{3} \right\} + \mathbb{P} \left\{ \sum_{i=k+1}^{\infty} X_i^*(\lambda) Y_i^*(\lambda) > \frac{\epsilon}{3} \right\}. \end{aligned} \quad (5.12)$$

From the convergence of $(\bar{\mathbf{X}}^{(n)}, \bar{\mathbf{Y}}^{(n)})$ to $(\mathbf{X}^*(\lambda), \mathbf{Y}^*(\lambda))$ in $l_1^2 \times \mathbb{N}^\infty$ we have that

$$\lim_{n \rightarrow \infty} \mathbb{P} \left\{ \sum_{i=1}^k |\bar{X}_i^{(n)} \bar{Y}_i^{(n)} - X_i^*(\lambda) Y_i^*(\lambda)| > \frac{\epsilon}{3} \right\} = 0.$$

Consider now the second term in (5.12). Let $E_L^{(n)} = \{\sup_{t \geq 0} r_t^{(n)} \leq L\}$. Then

$$\mathbb{P} \left\{ \sum_{i=k+1}^{\infty} \bar{X}_i^{(n)} \bar{Y}_i^{(n)} > \frac{\epsilon}{3} \right\} \leq \mathbb{P} \{(E_L^{(n)})^c\} + \frac{3}{\epsilon} \mathbb{E} \left[\mathbf{1}_{E_L^{(n)}} \left(\sum_{i=k+1}^{\infty} \bar{X}_i^{(n)} \bar{Y}_i^{(n)} \wedge 1 \right) \right].$$

Let $\mathcal{G} = \sigma\{\hat{H}^{(n)}(t) : t \geq 0\}$. Since $r_t^{(n)}$ is \mathcal{G} measurable for all $t \geq 0$, $E_L^{(n)} \in \mathcal{G}$. Then

$$\begin{aligned} \mathbb{E} \left[\mathbf{1}_{E_L^{(n)}} \left(\sum_{i=k+1}^{\infty} \bar{X}_i^{(n)} \bar{Y}_i^{(n)} \wedge 1 \right) \right] &= \mathbb{E} \left[\mathbf{1}_{E_L^{(n)}} \mathbb{E} \left[\sum_{i=k+1}^{\infty} \bar{X}_i^{(n)} \bar{Y}_i^{(n)} \wedge 1 \mid \mathcal{G} \right] \right] \\ &\leq \mathbb{E} \left[\mathbf{1}_{E_L^{(n)}} \left(\sum_{i=k+1}^{\infty} \mathbb{E} [\bar{X}_i^{(n)} \bar{Y}_i^{(n)} \mid \mathcal{G}] \wedge 1 \right) \right] \\ &\leq L \mathbb{E} \left[\left(\sum_{i=k+1}^{\infty} (\bar{X}_i^{(n)})^2 \right) \wedge 1 \right], \end{aligned}$$

where the last inequality follows on observing that, conditionally on \mathcal{G} , $\bar{Y}_i^{(n)}$ has a Poisson distribution with rate that is dominated by $\bar{X}_i^{(n)} \cdot (\sup_{t \geq 0} r_t^{(n)})$. Using the convergence of $\bar{\mathbf{X}}^{(n)}$ to \mathbf{X}^* , we now have

$$\limsup_{n \rightarrow \infty} \mathbb{E} \left[\mathbf{1}_{E_L^{(n)}} \left(\sum_{i=k+1}^{\infty} \bar{X}_i^{(n)} \bar{Y}_i^{(n)} \right) \wedge 1 \right] \leq L \mathbb{E} \left[\left(\sum_{i=k+1}^{\infty} (X_i^*(\lambda))^2 \right) \wedge 1 \right].$$

Let $\delta > 0$ be arbitrary. Using Lemma 5.4 and Lemma 5.3 (ii) we can choose $L \in (0, \infty)$ such that $\mathbb{P} \{(E_L^{(n)})^c\} \leq \delta$. Finally, taking limit as $n \rightarrow \infty$ in (5.12) we have that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mathbb{P} \left\{ \sum_{i=1}^{\infty} |\bar{X}_i^{(n)} \bar{Y}_i^{(n)} - X_i^*(\lambda) Y_i^*(\lambda)| > \epsilon \right\} &\leq \delta + L \mathbb{E} \left[\left(\sum_{i=k+1}^{\infty} (X_i^*(\lambda))^2 \right) \wedge 1 \right] \\ &\quad + \mathbb{P} \left\{ \sum_{i=k+1}^{\infty} X_i^*(\lambda) Y_i^*(\lambda) > \frac{\epsilon}{3} \right\}. \end{aligned} \quad (5.13)$$

The result now follows on sending $k \rightarrow \infty$ in the above display and recalling that $\sum_{i=1}^{\infty} (X_i^*(\lambda))^2 < \infty$ and $\sum_{i=1}^{\infty} X_i^*(\lambda) Y_i^*(\lambda) < \infty$ a.s. and $\delta > 0$ is arbitrary. \square

5.4. Proof of Lemma 5.4. In this section we prove Lemma 5.4. We will only treat the case $\lambda = 0$. The general case can be treated similarly. The key step in the proof is the following proposition whose proof is given at the end of the section.

Note that $\sup_{t \geq 0} |\bar{H}^{(n)}(t) - \bar{H}^{(n)}(t-)| \leq x^* \sqrt{s_2/s_3} \rightarrow 0$ as $n \rightarrow \infty$. Also, as $n \rightarrow \infty$, $qs_2 \rightarrow 1$. Thus, without loss of generality, we will assume that

$$\sup_{n \geq 1} \sup_{t \geq 0} |\bar{H}^{(n)}(t) - \bar{H}^{(n)}(t-)| \leq 1, \quad \sup_{n \geq 1} q^{(n)} s_2^{(n)} \leq 2. \quad (5.14)$$

Fix $\vartheta \in (0, 1/2)$ and define $t^{*(n)} = \left(\frac{s_2}{x^*}\right)^{\vartheta}$. Denote by $\{\mathcal{F}_t^{(n)}\}$ the filtration generated by $\{\bar{H}^{(n)}(t)\}_{t \geq 0}$. For ease of notation, we write $\sup_{t \in [a, b]} = \sup_{[a, b]}$. We will suppress (n) in the

notation, unless needed.

Proposition 5.6. *There exist $\Theta \in (0, \infty)$, events $G^{(n)}$, increasing $\mathcal{F}_t^{(n)}$ -stopping times $1 = \sigma_0^{(n)} < \sigma_1^{(n)} < \dots$, and a real positive sequence $\{\kappa_i\}$ with $\sum_{i=1}^{\infty} \kappa_i < \infty$, such that the following hold:*

(i) *For every $i \geq 1$, $\{\sigma_i^{(n)}\}_{n \geq 1}$ is tight.*

(ii) *For every $i \geq 1$,*

$$\mathbb{P} \left(\left\{ \sup_{[\sigma_{i-1}^{(n)}, \sigma_i^{(n)}]} \hat{H}^{(n)}(t) > 2\Theta + 1 \right\} \cap \{\sigma_{i-1}^{(n)} < t^{*(n)}\} \cap G^{(n)} \right) \leq \kappa_i.$$

(iii) *As $n \rightarrow \infty$, $\mathbb{P} \left\{ \sup_{[\sigma^{*(n)}, \infty)} \hat{H}^{(n)}(t) > \Theta; G^{(n)} \right\} \rightarrow 0$, where $\sigma^{*(n)} = \inf \{\sigma_i^{(n)} : \sigma_i^{(n)} \geq t^{*(n)}\}$.*

(iv) *As $n \rightarrow \infty$, $\mathbb{P}(G^{(n)}) \rightarrow 1$.*

Given Proposition 5.6, the proof of Lemma 5.4 can be completed as follows.

Proof of Lemma 5.4:

Fix $\epsilon \in (0, 1)$. Let $\Theta \in (0, \infty)$, $G^{(n)}$, $\sigma_i^{(n)}$, κ_i be as in Proposition 5.6. Choose $i_0 > 1$ such that $\sum_{i \geq i_0} \kappa_i \leq \epsilon$. Since $\{\sigma_{i_0-1}^{(n)}\}$ is tight, there exists $T \in (0, \infty)$ such that $\limsup_{n \rightarrow \infty} \mathbb{P} \{\sigma_{i_0-1}^{(n)} > T\} \leq \epsilon$. Thus for any $M' > 2\Theta + 1$, we have

$$\begin{aligned} \mathbb{P} \left\{ \sup_{[1, \infty)} \hat{H}^{(n)}(t) > M' \right\} &\leq \mathbb{P} \left\{ \sup_{[1, T]} \hat{H}^{(n)}(t) > M' \right\} + \mathbb{P} \{\sigma_{i_0-1}^{(n)} > T\} + \mathbb{P} \{(G^{(n)})^c\} \\ &\quad + \mathbb{P} \left\{ \sup_{[\sigma_{i_0-1}^{(n)}, \sigma^{*(n)}]} \hat{H}^{(n)}(t) > 2\Theta + 1; G^{(n)} \right\} + \mathbb{P} \left\{ \sup_{[\sigma^{*(n)}, \infty)} \hat{H}^{(n)}(t) > \Theta; G^{(n)} \right\}. \end{aligned}$$

Taking $\limsup_{n \rightarrow \infty}$ on both sides

$$\limsup_{n \rightarrow \infty} \mathbb{P} \left\{ \sup_{[1, \infty)} \hat{H}^{(n)}(t) > M' \right\} \leq \limsup_{n \rightarrow \infty} \mathbb{P} \left\{ \sup_{[1, T]} \hat{H}^{(n)}(t) > M' \right\} + \epsilon + 0 + \epsilon + 0.$$

Since $\left\{ \sup_{[1, T]} \hat{H}^{(n)}(t) \right\}_{n \geq 1}$ is tight, we have,

$$\limsup_{M' \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P} \left\{ \sup_{[1, \infty)} \hat{H}^{(n)}(t) > M' \right\} \leq 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary, the result follows. \square

We now proceed to the proof of Proposition 5.6. The following lemma is key.

Lemma 5.7. *There are $\{\mathcal{F}_t^{(n)}\}$ adapted processes $\{A^{(n)}(t)\}$, $\{B^{(n)}(t)\}$ and $\mathcal{F}_t^{(n)}$ -martingale $\{M^{(n)}(t)\}$ such that*

(i) *$A^{(n)}(\cdot)$ is a non-increasing function of t , a.s. For all $t \geq 0$, $\bar{H}^{(n)}(t) = \int_0^t A^{(n)}(u) du + M^{(n)}(t)$.*

(ii) *For $t \geq 0$, $\langle M^{(n)}, M^{(n)} \rangle_t = \int_0^t B^{(n)}(u) du$.*

(iii) *$\sup_{n \geq 1} \sup_{u \geq 0} B^{(n)}(u) \leq 2$.*

- (iv) With $G^{(n)} = \{A(t) < -t/2 \text{ for all } t \in [1, t^{*(n)}]\}$, $\mathbb{P}(G^{(n)}) \rightarrow 1$ as $n \rightarrow \infty$.
(v) For any $\alpha \in (0, \infty)$ and $t > 0$,

$$\mathbb{P} \left\{ \sup_{u \in [0, t]} |M^{(n)}(u)| > \alpha \right\} \leq 2 \exp \{ \alpha \} \cdot \exp \left\{ -\alpha \log \left(1 + \frac{\alpha}{2t} \right) \right\}. \quad (5.15)$$

Proof: Recall the notation from Section 5.1. Parts (i) and (ii) are proved in [4]. Furthermore, from Lemma 11 of [4] it follows that, for $t \in [l_{i-1}, l_i)$, writing $Q_2(t) = \sum_{j=1}^i (x_{v(j)})^2$, we have

$$A(t) \leq \sqrt{\frac{s_2}{s_3}} (-1 + qs_2 - qQ_2(t)), \quad B(t) \leq qs_2.$$

Part (iii) now follows on recalling from (5.14) that $qs_2 \leq 2$. To prove (iv) it suffices to show that

$$\sup_{t \leq t^*} \left| \frac{s_2}{s_3} Q_2(t) - t \right| \xrightarrow{\mathbb{P}} 0. \quad (5.16)$$

To prove this we will use the estimate on Page 832, Lemma 13 of [4], which says that for any fixed $\epsilon \in (0, 1)$, and $L \in (0, \infty)$

$$\mathbb{P} \left\{ \sup_{t \in [0, L]} \left| \frac{s_2}{s_3} Q_2(t) - t \right| > \epsilon \right\} = O \left(\frac{L^2 x^*}{s_2} + \sqrt{\frac{L(x^*)^2 s_2}{s_3}} + \frac{L^2 s_3}{s_2^2} + \sqrt{\frac{L s_3}{s_2}} + \frac{s_2^2}{(1 - 2L s_2)^+} \right).$$

Note that the first term on the right hand side determine its order when $L \rightarrow \infty$. Taking $L = t^*$ in the above estimate we see that, since $\vartheta \in (0, 1/2)$, the expression on the right side above goes to 0 as $n \rightarrow \infty$. This proves (5.16) and thus completes the proof of (iv). Finally, proof of (v) uses standard concentration inequalities for martingales. Indeed, recalling that the maximal jump size of \bar{H} , and consequently that of M , is bounded by 1 and $\langle M, M \rangle_t \leq 2t$, we have from Section 4.13, Theorem 5 of [23] that, for any fixed $\alpha > 0$ and $t > 0$,

$$\mathbb{P} \left\{ \sup_{u \in [0, t]} |M_u| > \alpha \right\} \leq 2 \exp \left\{ - \sup_{\lambda > 0} [\alpha \lambda - 2t \phi(\lambda)] \right\},$$

where $\phi(\lambda) = (e^\lambda - 1 - \lambda)$. A straightforward calculation shows

$$\sup_{\lambda > 0} [\alpha \lambda - 2t \phi(\lambda)] = \alpha \log \left(1 + \frac{\alpha}{2t} \right) - \left(\alpha - 2t \log \left(1 + \frac{\alpha}{2t} \right) \right) \geq \alpha \log \left(1 + \frac{\alpha}{2t} \right) - \alpha.$$

The result follows. \square

The bound (5.15) continues to hold if we replace $M(u)$ with $M(\tau + u) - M(\tau)$ for any finite stopping time τ . From this observation we immediately have the following corollary.

Corollary 5.8. Let M be as in Lemma 5.7. Then, for any finite stopping time τ :

- (i) $\mathbb{P} \left\{ \sup_{u \in [0, t]} |M(\tau + u) - M(\tau)| > \alpha \right\} \leq 2e^{-\alpha}$, whenever $\alpha > 2(e^2 - 1)t$.
(ii) $\mathbb{P} \left\{ \sup_{u \in [0, t]} |M(\tau + u) - M(\tau)| > \alpha \right\} \leq 2(2e/\alpha)^\alpha t^\alpha$, for all $t > 0$ and $\alpha > 0$.

Part (i) of the corollary is useful when α is large and part (ii) is useful when t is small. Finally we now give the proof of Proposition 5.6.

Proof of Proposition 5.6: From Lemma 5.3 (i) we have that $\hat{H}^{(n)}$ converges in distribution to \hat{W}_0 (recall we assume that $\lambda = 0$) as $n \rightarrow \infty$. Let $\{\epsilon_i\}_{i \geq 1}$ be a positive real sequence bounded by 1 and fix $\Theta \in (2, \infty)$. Choice of Θ and ϵ_i will be specified later in the proof. Let

$\sigma_0^{(n)} < \tau_1^{(n)} \leq \sigma_1^{(n)} < \tau_2^{(n)} \leq \sigma_2^{(n)} < \dots$ be a sequence of stopping times such that $\sigma_0^{(n)} = 1$, and for $i \geq 1$,

$$\tau_i^{(n)} = \inf\{t \geq \sigma_{i-1}^{(n)} + \epsilon_i : \hat{H}^{(n)}(t) \geq \Theta\} \wedge (\sigma_{i-1}^{(n)} + 1), \quad \sigma_i^{(n)} = \inf\{t \geq \tau_i^{(n)} : \hat{H}^{(n)}(t) \leq 1\}. \quad (5.17)$$

Similarly define stopping times $1 = \bar{\sigma}_0 < \bar{\tau}_1 \leq \bar{\sigma}_1 < \bar{\tau}_2 \leq \bar{\sigma}_2 < \dots$ by replacing $\hat{H}^{(n)}$ in (5.17) with \hat{W}_0 . Due to the negative quadratic drift in the definition of W_0 it follows that $\bar{\sigma}_i < \infty$ for every i and from the weak convergence of $\hat{H}^{(n)}$ to \hat{W}_0 it follows that $\sigma_i^{(n)} \rightarrow \bar{\sigma}_i$ and $\tau_i^{(n)} \rightarrow \bar{\tau}_i$, in distribution, as $n \rightarrow \infty$. Here we have used the fact that if ζ denotes the first time W_0 hits the level $\alpha \in (0, \infty)$ then, a.s., for any $\delta > 0$, there are infinitely many crossings of the level α in $(\zeta, \zeta + \delta)$. In particular we have that $\{\sigma_i^{(n)}\}_{n \geq 1}$ is a tight sequence, and this proves part (i) of Proposition 5.6.

For the rest of the proof we suppress (n) from the notation. Since the jump size of \hat{H} is bounded by 1, we have that $\sup_{[\sigma_{i-1}, \sigma_{i-1} + \epsilon_i]} \hat{H}(t) \leq \Theta$ implies $\sup_{[\sigma_{i-1}, \tau_i]} \hat{H}(t) \leq \Theta + 1$ and thus, in this case, when $t \in [\tau_i, \sigma_i]$, we have $\hat{H}(t) = \hat{H}(\tau_i) + (\bar{H}(t) - \bar{H}(\tau_i)) \leq \Theta + 1 + (\bar{H}(t) - \bar{H}(\tau_i))$. Let $G \equiv G^{(n)}$ be as in Lemma 5.7 (iv) and let $H_i = G \cap \{\sigma_{i-1} < t^*\}$, then writing $\mathbb{P}(\cdot \cap H_i)$ as $\mathbb{P}_i(\cdot)$,

$$\mathbb{P}_i \left\{ \sup_{[\sigma_{i-1}, \sigma_i]} \hat{H}(t) > 2\Theta + 1 \right\} \leq \mathbb{P}_i \left\{ \sup_{[\sigma_{i-1}, \sigma_{i-1} + \epsilon_i]} \hat{H}(t) > \Theta \right\} \quad (5.18)$$

$$+ \mathbb{P}_i \left\{ \sup_{[\tau_i, \sigma_i]} [\Theta + 1 + (\bar{H}(t) - \bar{H}(\tau_i))] > 2\Theta + 1 \right\}. \quad (5.19)$$

Denote the two terms on the right side by \mathbb{T}_1 and \mathbb{T}_2 respectively. Recalling that $\hat{H}(\sigma_{i-1}) \leq 2$, we have from the decomposition in Lemma 5.7 (i) and Corollary 5.8(ii) that

$$\mathbb{T}_1 \leq \mathbb{P} \left\{ \sup_{[\sigma_{i-1}, \sigma_{i-1} + \epsilon_i]} |M(t) - M(\sigma_{i-1})| > \frac{\Theta - 2}{2} \right\} \leq C_{\frac{\Theta-2}{2}} \epsilon_i^{(\Theta-2)/2}, \quad (5.20)$$

Here, for $\alpha > 0$, $C_\alpha = 2(2e/\alpha)^\alpha$ and we have used the fact that on H_i , $A(t) \leq -t/2 \leq 0$ for all $t \in [\sigma_{i-1}, \sigma_{i-1} + \epsilon_i]$.

Next, let $\{\delta_i\}_{i \geq 1}$ be a sequence of positive reals bounded by 1. Setting $d_i = \sum_{j=1}^{i-1} \epsilon_j$, we have

$$\begin{aligned} \mathbb{T}_2 &\leq \mathbb{P}_i \left\{ \sup_{[\tau_i, \tau_i + \delta_i]} (\bar{H}(t) - \bar{H}(\tau_i)) > \Theta \right\} + \mathbb{P}_i \left\{ \sup_{[\tau_i + \delta_i, \tau_i + 1]} (\bar{H}(t) - \bar{H}(\tau_i)) > \Theta \right\} + \mathbb{P} \{\sigma_i > \tau_i + 1\} \\ &\leq \mathbb{P} \left\{ \sup_{[\tau_i, \tau_i + \delta_i]} (M(t) - M(\tau_i)) > \Theta \right\} + \mathbb{P} \left\{ \sup_{[\tau_i + \delta_i, \tau_i + 1]} (M(t) - M(\tau_i)) > \Theta + \frac{\delta_i d_i}{2} \right\} \\ &\quad + \mathbb{P} \left\{ M(\tau_i + 1) - M(\tau_i) > -\Theta + \frac{d_i}{2} \right\} \\ &\leq C_\Theta \delta_i^\Theta + 2e^{-\delta_i d_i/2} + 2e^{\Theta - d_i/2}, \end{aligned} \quad (5.21)$$

whenever

$$\min\{\delta_i d_i/2, d_i/2 - \Theta\} > 2(e^2 - 1). \quad (5.22)$$

Fix $\Theta > 14$. Then $\max\{C_\Theta, C_{(\Theta-2)/2}\} \leq 2$. We will impose additional conditions on Θ later in the proof. Combining (5.20) and (5.21), we have

$$\mathbb{P}_i \left\{ \sup_{[\sigma_{i-1}, \sigma_i]} \hat{H}(t) > 2\Theta + 1 \right\} \leq 2(\epsilon_i^{(\Theta-2)/2} + \delta_i^\Theta + e^{-\delta_i d_i/2} + e^{\Theta-d_i/2}) \equiv \kappa_i. \quad (5.23)$$

Let

$$\epsilon_i = i^{-1/2}, \quad d_i = \sum_{j=1}^{i-1} \epsilon_j \sim i^{1/2}, \quad \delta_i = 1/\sqrt{d_i} \sim i^{-1/4}.$$

Then, (5.22) holds for i large enough, and

$$\kappa_i \sim 2(i^{-(\Theta-2)/4} + i^{-\Theta/4} + e^{-i^{1/4}/2} + e^{\Theta-i^{1/2}/2}),$$

which, since $\Theta > 14$, is summable. This proves part (ii) of the Proposition.

Now we consider part (iii). We will construct another sequence of stopping times with values in $[t^*, \infty)$, as follows. Define $\sigma_0^* := \inf\{\sigma_i : \sigma_i \geq t^*\} = \inf\{t \geq t^* : \hat{H}(t) \leq 1\}$, then define τ_i^*, σ_i^* for $i \geq 1$ similarly as in (5.17). Similar arguments as before give a bound as (5.23) with d_i replaced by t^* , δ_i replaced by $1/\sqrt{t^*}$, ϵ_i replaced with $1/t^*$ and Θ replaced by any $\Theta_0 > 14$. Namely,

$$\mathbb{P} \left\{ \sup_{[\sigma_{i-1}^*, \sigma_i^*]} \hat{H}(t) > 2\Theta_0 + 1; G^{(n)} \right\} \leq 2((1/t^*)^{(\Theta_0-2)/2} + (1/\sqrt{t^*})^{\Theta_0} + e^{-\sqrt{t^*}/2} + e^{\Theta_0-t^*/2}). \quad (5.24)$$

Here we have used the fact that since $A(t)$ is non-increasing, on $G^{(n)}$, $A(t) \leq -t^*/2$ for all $t \geq t^*$.

Recall that, by construction, $\hat{H}(t) = 0$ when $t \geq s_1$. So there exist i_0 such that $\tau_{i_0}^* = \infty$, in fact since $\sigma_i^* \geq \sigma_{i-1}^* + \epsilon$, we have that $i_0 \leq s_1/\epsilon$. Thus, we have from the above display that

$$\mathbb{P} \left\{ \sup_{[\sigma_0^*, \infty)} \hat{H}(t) > 2\Theta_0 + 1 \right\} \leq \frac{2s_1}{\epsilon} ((1/t^*)^{(\Theta_0-2)/2} + (1/\sqrt{t^*})^{\Theta_0} + e^{-\sqrt{t^*}/2} + e^{\Theta_0-t^*/2}).$$

Taking $\Theta > 29$, we have on setting $\Theta_0 = \frac{\Theta-1}{2}$ in the above display

$$\mathbb{P} \left\{ \sup_{[\sigma_0^*, \infty)} \hat{H}(t) > \Theta \right\} \leq 2s_1 \left(\left(\frac{1}{t^*} \right)^{(\Theta-1)/4-2} + \left(\frac{1}{t^*} \right)^{(\Theta-1)/4-1} + \frac{1}{t^*} e^{-\sqrt{t^*}/2} + \frac{1}{t^*} e^{(\Theta-1)/2-t^*/2} \right).$$

From (5.2) we have that $s_1 \cdot (\frac{1}{t^*})^{\varsigma/\vartheta} \rightarrow 0$. So if $\Theta \geq 4(\frac{\varsigma}{\vartheta} + 2) + 1$, the above expression approaches 0 as $n \rightarrow \infty$. The result now follows on taking $\Theta = \max\{29, 4(\frac{\varsigma}{\vartheta} + 2) + 1\}$. \square

6. BOUNDED-SIZE RULES AT TIME $t_c - n^{-\gamma}$

Throughout Sections 6 and 7 we take $T = 2t_c$ which is a convenient upper bound for the time parameters of interest. In this section we prove Theorem 3.2.

We begin with some notation associated with BSR processes, which closely follows [27]. Recall from Section 2.2 the set Ω_K and the random graph process $\mathbf{BSR}^{(n)}(t)$ associated with

a given K -BSR $F \subset \Omega_K^4$. Frequently we will suppress n in the notation. Also recall the definition of $c_t(v)$ from Section 2.2.

For $i \in \Omega_K$, define

$$X_i(t) = |\{v \in \mathbf{BSR}_t^{(n)} : c_t(v) = i\}| \text{ and } \bar{x}_i(t) = X_i(t)/n. \quad (6.1)$$

Denote by $\mathbf{BSR}^*(t)$ the subgraph of $\mathbf{BSR}(t)$ consisting of all components of size greater than K , and define, for $k = 1, 2, 3$

$$\mathcal{S}_{k,\varpi}(t) := \sum_{\{C \subset \mathbf{BSR}^*(t)\}} |C|^k \text{ and } \bar{s}_{k,\varpi}(t) = \mathcal{S}_{k,\varpi}(t)/n,$$

where $\{C \subset \mathbf{BSR}^*(t)\}$ denotes the collection of all components in $\mathbf{BSR}^*(t)$. For notational convenience in long formulae, we sometimes write $\mathbf{BSR}(t) = \mathbf{BSR}_t$ and similarly $\mathbf{BSR}^*(t) = \mathbf{BSR}_t^*$. Similar notation will be used throughout the paper.

Clearly

$$\mathcal{S}_k(t) = \mathcal{S}_{k,\varpi} + \sum_{i=1}^K i^{k-1} X_i(t), \quad \bar{s}_k(t) = \bar{s}_{k,\varpi} + \sum_{i=1}^K i^{k-1} \bar{x}_i(t). \quad (6.2)$$

Also note that $\mathcal{S}_1(t) = n$ and $\mathcal{S}_{1,\varpi}(t) = X_\varpi(t)$.

Recall the Poisson processes $\mathcal{P}_{\vec{v}}$ introduced in Section 2.2. Let $\mathcal{F}_t = \sigma\{\mathcal{P}_{\vec{v}}(s) : s \leq t, \vec{v} \in [n]^4\}$. For $T_0 \in [0, T]$ and a $\{\mathcal{F}_t\}_{0 \leq t < T_0}$ semi-martingale $\{J(t)\}_{0 \leq t < T_0}$ of the form

$$dJ(t) = \alpha(t)dt + dM(t), \quad \langle M, M \rangle_t = \int_0^t \gamma(s)ds, \quad (6.3)$$

where M is a $\{\mathcal{F}_t\}$ local martingale and γ is a progressively measurable process, we write $\alpha = \mathbf{d}(J)$, $M = \mathbf{M}(J)$ and $\gamma = \mathbf{v}(J)$.

Organization: The rest of this section is organized as follows. In Section 6.1, we state a recent result on BSR models and certain deterministic maps associated with the evolution of \mathbf{BSR}_t^* from [8] that will be used in this work. In Section 6.2, we will study the asymptotics of $\bar{s}_{2,\varpi}$ and $\bar{s}_{3,\varpi}$. In Section 6.3, we will complete the proof of Theorem 3.2(i). In Section 6.4, we will obtain some useful semi-martingale decompositions for certain functionals of \bar{s}_2 and \bar{s}_3 . In Section 6.5, we will prove parts (ii) and (iii) of Theorem 3.2.

6.1. Evolution of \mathbf{BSR}_t^* . We begin with the following lemma from [8] (see also [27]).

Lemma 6.1. (a) For each $i \in \Omega_K$, there exists a continuously differentiable function $x_i : [0, T] \rightarrow [0, 1]$ such that for any $\delta \in (0, 1/2)$, there exist $C_1, C_2 \in (0, \infty)$ such that for all n ,

$$\mathbb{P} \left(\sup_{i \in \Omega_K} \sup_{s \in [0, T]} |\bar{x}_i(t) - x_i(t)| > n^{-\delta} \right) < C_1 \exp \left(-C_2 n^{1-2\delta} \right).$$

(b) There exist polynomials $\{F_i^x(\mathbf{x})\}_{i \in \Omega_K}$, $\mathbf{x} = (x_i)_{i \in \Omega_K} \in \mathbb{R}^{K+1}$, such that $\mathbf{x}(t) = (x_i(t))_{i \in \Omega_K}$ is the unique solution to the differential equations:

$$x'_i(t) = F_i^x(\mathbf{x}(t)), \quad i \in \Omega_K, \quad t \in [0, T] \text{ with initial values } \mathbf{x}(0) = (1, 0, \dots, 0). \quad (6.4)$$

Furthermore, \bar{x}_i is a $\{\mathcal{F}_t\}_{0 \leq t < T}$ semi-martingale of the form (6.3) and

$$\sup_{0 \leq t < T} |\mathbf{d}(\bar{x}_i)(t) - F_i^x(\bar{\mathbf{x}}(t))| \leq \frac{K^2}{n}.$$

Also, for all $i \in \Omega_K$ and $t \in (0, T]$, we have $x_i(t) > 0$ and $\sum_{i \in \Omega_K} x_i(t) = 1$.

Recall that $\mathbf{BSR}^*(t)$ is the subgraph of $\mathbf{BSR}(t)$ consisting of all components of size greater than K . The evolution of this graph is governed by three type of events:

Type 1 (Immigrating vertices): This corresponds to the merger of two components of size bounded by K into a component of size larger than K . Such an event leads to the appearance of a new component in $\mathbf{BSR}^*(t)$ which we view as the immigration of a ‘vertex’ into $\mathbf{BSR}^*(t)$. Denote by $na_i^*(t)$ the rate at which a component of size $K+i$ immigrates into \mathbf{BSR}_t^* at time t . In [8] it is shown that there are polynomials $F_i^a(\mathbf{x})$ for $1 \leq i \leq K$ such that, with $\bar{\mathbf{x}}(t) = (\bar{x}_i(t))_{i \in \Omega_K}$

$$\sup_{t \in [0, \infty)} |a_i^*(t) - F_i^a(\bar{\mathbf{x}}(t))| \leq \frac{K}{n}. \quad (6.5)$$

We define, with $\mathbf{x}(t)$ as in Lemma 6.1,

$$a_i(t) := F_i^a(\mathbf{x}(t)), \quad i = 1, \dots, K. \quad (6.6)$$

Type 2 (Attachments): This event corresponds to a component of size at most K getting linked with some component of size larger than K . For $1 \leq i \leq K$, denote by $|C|c_i^*(t)$ the rate at which a component of size i attaches to a component C in \mathbf{BSR}_{t-}^* . Then (see [8]) there exist polynomials $F_i^c(\mathbf{x})$ for $1 \leq i \leq K$, such that $c_i^*(t) = F_i^c(\bar{\mathbf{x}}(t))$. Define

$$c_i(t) := F_i^c(\mathbf{x}(t)), \quad i = 1, \dots, K. \quad (6.7)$$

Type 3 (Edge formation): This event corresponds to the addition of an edge between components in \mathbf{BSR}_t^* . The occurrence of this event adds one edge between two vertices in \mathbf{BSR}_{t-}^* , the vertex set stays unchanged, whereas the edge set has one additional element. From [8], there is a polynomial $F^b(\mathbf{x})$ such that, defining $b^*(t) = F^b(\bar{\mathbf{x}}(t))$, the rate at which each pair of components $C_1 \neq C_2 \in \mathbf{BSR}_t^*$ merge at time t , equals $|C_1||C_2|b^*(t)/n$. Furthermore, define

$$b(t) := F^b(\mathbf{x}(t)). \quad (6.8)$$

From [8], $F_i^a(\mathbf{x})$, $F_i^c(\mathbf{x})$ and $F^b(\mathbf{x})$ are polynomials with positive coefficients, thus from the last statement in Lemma 6.1 we have that $b(t_c) \in (0, \infty)$.

6.2. Analysis of $\bar{s}_{2,\varpi}(t)$ and $\bar{s}_{3,\varpi}(t)$. Define functions $F_{2,\varpi}^s : [0, 1]^{K+1} \times \mathbb{R} \rightarrow \mathbb{R}$ and $F_{3,\varpi}^s : [0, 1]^{K+1} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ as

$$F_{2,\varpi}^s(\mathbf{x}, s_2) := \sum_{j=1}^K (K+j)^2 F_j^a(\mathbf{x}) + s_2 \sum_{j=1}^K 2j F_j^c(\mathbf{x}) + x_\varpi \sum_{j=1}^K j^2 F_j^c(\mathbf{x}) + (s_2)^2 F^b(\mathbf{x}), \quad (6.9)$$

for $(\mathbf{x}, s_2) \in [0, 1]^{K+1} \times \mathbb{R}$ and, for $(\mathbf{x}, s_2, s_3) \in [0, 1]^{K+1} \times \mathbb{R}^2$

$$\begin{aligned} F_{3,\varpi}^s(\mathbf{x}, s_2, s_3) &:= \sum_{j=1}^K (K+j)^3 F_j^a(\mathbf{x}) + s_3 \sum_{j=1}^K 3j F_j^c(\mathbf{x}) + 3s_2 \sum_{j=1}^K j^2 F_j^c(\mathbf{x}) \\ &\quad + x_\varpi \sum_{j=1}^K j^3 F_j^c(\mathbf{x}) + 3s_2 s_3 F^b(\mathbf{x}). \end{aligned} \quad (6.10)$$

The following lemma relates the evolution of $\bar{s}_{j,\varpi}(\cdot)$ to that of $F_{j,\varpi}^s(\bar{\mathbf{x}}(\cdot), \bar{s}_{2,\varpi}(\cdot))$. By definition, $\bar{s}_{j,\varpi}(t)$ is a non-decreasing process with RCLL paths, and therefore a semi-martingale. For $T_0 \in [0, T]$, a stochastic process $\{\xi(t)\}_{0 \leq t < T_0}$, and a nonnegative sequence $\alpha(n)$, the quantity $O_{T_0}(\xi(t)\alpha(n))$ will represent a stochastic process $\{\eta(t)\}_{0 \leq t < T_0}$ such that for some $d_1 \in (0, \infty)$, $\eta(t) \leq d_1 \xi(t)\alpha(n)$, for all $0 \leq t < T_0$ and $n \geq 1$.

Lemma 6.2. *The processes $\bar{s}_{j,\varpi}$, $j = 2, 3$, are $\{\mathcal{F}_t\}_{0 \leq t < t_c}$ semi-martingales of the form (6.3) and*

$$\begin{aligned} |\mathbf{d}(\bar{s}_{2,\varpi})(t) - F_{2,\varpi}^s(\bar{\mathbf{x}}(t), \bar{s}_{2,\varpi}(t))| &= O_{t_c}(S_4(t)/n^2) \\ |\mathbf{d}(\bar{s}_{3,\varpi})(t) - F_{3,\varpi}^s(\bar{\mathbf{x}}(t), \bar{s}_{2,\varpi}(t), \bar{s}_{3,\varpi}(t))| &= O_{t_c}(S_5(t)/n^2). \end{aligned}$$

Proof: Note that $\mathcal{S}_{2,\varpi}$ and $\mathcal{S}_{3,\varpi}$ have jumps at time instant t with rates and values $\Delta\mathcal{S}_{2,\varpi}(t)$, $\Delta\mathcal{S}_{3,\varpi}(t)$, respectively, given as follows.

- for each $1 \leq i \leq K$, with rate $na_i^*(t)$,

$$\Delta\mathcal{S}_{2,\varpi}(t) = (K + i)^2, \quad \Delta\mathcal{S}_{3,\varpi}(t) = (K + i)^3.$$

- for each $1 \leq i \leq K$ and $\mathcal{C} \subset \mathbf{BSR}_{t-}^*$, at rate $|\mathcal{C}|c_i^*(t)$,

$$\Delta\mathcal{S}_{2,\varpi}(t) = 2|\mathcal{C}|i + i^2, \quad \Delta\mathcal{S}_{3,\varpi}(t) = 3|\mathcal{C}|^2i + 3|\mathcal{C}|i^2 + i^3.$$

- for all unordered pair $\mathcal{C}, \tilde{\mathcal{C}} \subset \mathbf{BSR}_{t-}^*$, such that $\mathcal{C} \neq \tilde{\mathcal{C}}$, at rate $|\mathcal{C}||\tilde{\mathcal{C}}|b^*(t)/n$,

$$\Delta\mathcal{S}_{2,\varpi}(t) = 2|\mathcal{C}||\tilde{\mathcal{C}}|, \quad \Delta\mathcal{S}_{3,\varpi}(t) = 3|\mathcal{C}|^2|\tilde{\mathcal{C}}| + 3|\mathcal{C}||\tilde{\mathcal{C}}|^2.$$

Thus

$$\begin{aligned} \mathbf{d}(\mathcal{S}_{2,\varpi})(t) &= \sum_{j=1}^K (K + j)^2 na_j^*(t) + \sum_{j=1}^K \sum_{\mathcal{C} \subset \mathbf{BSR}_t^*} (2j|\mathcal{C}| + j^2)|\mathcal{C}|c_j^*(t) + \sum_{\mathcal{C} \neq \tilde{\mathcal{C}} \subset \mathbf{BSR}_t^*} 2|\mathcal{C}||\tilde{\mathcal{C}}| \frac{b^*(t)|\mathcal{C}||\tilde{\mathcal{C}}|}{n} \\ &= \sum_{j=1}^K (K + j)^2 na_j^*(t) + \sum_{j=1}^K 2jc_j^*(t)\mathcal{S}_{2,\varpi}(t) + \sum_{j=1}^K j^2c_j^*(t)X_\varpi(t) \\ &\quad + \frac{b^*(t)}{n}(\mathcal{S}_{2,\varpi}^2(t) - \mathcal{S}_{4,\varpi}(t)) \\ &= n(F_{2,\varpi}^s(\bar{\mathbf{x}}, \bar{s}_{2,\varpi}) + O(1/n) + O_{t_c}(S_{4,\varpi}(t)/n^2)) \end{aligned} \tag{6.11}$$

and

$$\begin{aligned} \mathbf{d}(\mathcal{S}_{3,\varpi})(t) &= \sum_{j=1}^K (K + j)^3 na_j^*(t) + \sum_{j=1}^K \sum_{\mathcal{C} \subset \mathbf{BSR}_t^*} (3j|\mathcal{C}|^2 + 3j^2|\mathcal{C}| + j^3)|\mathcal{C}|c_j^*(t) \\ &\quad + \sum_{\mathcal{C} \neq \tilde{\mathcal{C}} \subset \mathbf{BSR}_t^*} (3|\mathcal{C}|^2|\tilde{\mathcal{C}}| + 3|\mathcal{C}||\tilde{\mathcal{C}}|^2) \frac{b^*(t)|\mathcal{C}||\tilde{\mathcal{C}}|}{n} \\ &= \sum_{j=1}^K (K + j)^3 na_j^*(t) + \sum_{j=1}^K 3jc_j^*(t)\mathcal{S}_{3,\varpi}(t) + \sum_{j=1}^K 3j^2c_j^*(t)\mathcal{S}_{2,\varpi}(t) \\ &\quad + \sum_{j=1}^K j^3c_j^*(t)X_\varpi(t) + \frac{3b^*(t)}{n}(\mathcal{S}_{3,\varpi}(t)\mathcal{S}_{2,\varpi}(t) - \mathcal{S}_{5,\varpi}(t)) \\ &= n(F_{3,\varpi}^s(\bar{\mathbf{x}}(t), \bar{s}_{2,\varpi}(t), \bar{s}_{3,\varpi}(t)) + O(1/n) + O_{t_c}(S_{4,\varpi}(t)/n^2)). \end{aligned}$$

The result follows. \square

6.3. Proof of Theorem 3.2(i). In this section we prove Theorem 3.2(i). We begin with the following differential equations, whose solutions play an important role in defining $s_2(\cdot)$ and $s_3(\cdot)$ that appear in Theorem 3.2.

Consider the equations

$$s'_{2,\varpi}(t) = F_{2,\varpi}^s(\mathbf{x}(t), s_{2,\varpi}(t)), \quad s_{2,\varpi}(0) = 0, \quad t \geq 0 \quad (6.12)$$

$$s'_{3,\varpi}(t) = F_{3,\varpi}^s(\mathbf{x}(t), s_{2,\varpi}(t), s_{3,\varpi}(t)), \quad s_{3,\varpi}(0) = 0, \quad t \geq 0. \quad (6.13)$$

The following lemma describes some properties of solutions to the above differential equations.

Lemma 6.3. *Equation (6.12) and (6.13) have unique solutions $s_{2,\varpi}(t)$ and $s_{3,\varpi}(t)$ for $t \in [0, t_c)$. Furthermore $s_{2,\varpi}(t)$ and $s_{3,\varpi}(t)$ are non-negative, increasing and $\lim_{t \uparrow t_c} s_{2,\varpi}(t) = \lim_{t \uparrow t_c} s_{3,\varpi}(t) = \infty$.*

Proof: The existence and uniqueness of solution to (6.12) follows from Lemma 6.1(b) and Theorem 2.2 in [27]. Furthermore, Theorem 2.2 in [27] also implies $\lim_{t \uparrow t_c} s_{2,\varpi}(t) = \infty$.

Note that $F_{3,\varpi}^s$ is a polynomial and the right hand side of (6.13) is linear in $s_{3,\varpi}$, thus (6.13) has a unique solution on $[0, t_c)$ as well. Since $s_{2,\varpi}(t)$ explodes at t_c , we have $\lim_{t \uparrow t_c} s_{3,\varpi}(t) = \infty$. The monotonicity of $s_{2,\varpi}(t)$ and $s_{3,\varpi}(t)$ follow from the positivity of the functions $\{F_j^a : 1 \leq j \leq K\}$, F^b , $\{F_j^c : 1 \leq j \leq K\}$ that appear in the definition of the functions $F_{2,\varpi}^s$ and $F_{3,\varpi}^s$ in (6.9) and (6.10) respectively. \square

Define $s_k : [0, t_c) \rightarrow [0, \infty)$, $k = 2, 3$, as follows.

$$s_k(t) := s_{k,\varpi}(t) + \sum_{i=1}^K i^{k-1} x_i(t), \quad \text{for } k = 2, 3. \quad (6.14)$$

Then using (6.12) - (6.13) we get the following differential equations for s_2 and s_3 .

Lemma 6.4. *The functions s_2, s_3 are continuously differentiable on $[0, t_c)$, $\lim_{t \uparrow t_c} s_2(t) = \lim_{t \uparrow t_c} s_3(t) = \infty$, and can be characterized as the unique solutions of the following differential equations*

$$\begin{aligned} s'_2(t) &= F_2^s(\mathbf{x}(t), s_2(t)), \quad s_2(0) = 1, \\ s'_3(t) &= F_3^s(\mathbf{x}(t), s_2(t), s_3(t)), \quad s_3(0) = 1. \end{aligned}$$

where the function $F_2^s(\cdot)$ and $F_3^s(\cdot)$ are defined as

$$\begin{aligned} F_2^s(\mathbf{x}, s_2) &:= F_{2,\varpi}^s \left(\mathbf{x}, s_2 - \sum_{i=1}^K i x_i \right) + \sum_{i=1}^K i F_i^x(\mathbf{x}), \\ F_3^s(\mathbf{x}, s_2, s_3) &:= F_{3,\varpi}^s \left(\mathbf{x}, s_2 - \sum_{i=1}^K i x_i, s_3 - \sum_{i=1}^K i^2 x_i \right) + \sum_{i=1}^K i^2 F_i^x(\mathbf{x}). \end{aligned}$$

Proof: The differentiability of $s_2(\cdot)$, $s_3(\cdot)$ on $[0, t_c)$ and the form of F_2^s and F_3^s follow from (6.14) and the differential equations (6.12), (6.13) and (6.4). The uniqueness of the solution follows from the fact that F_2^s and F_3^s are polynomials. The limit behavior as $t \uparrow t_c$ follows from the definition of s_k . \square

The following lemma defines the two parameters α and β that appear in Theorems 3.2 and 3.3. Recall from Section 6.1 that $b(t_c) \in (0, \infty)$.

Lemma 6.5. *The following two limits exist,*

$$\alpha := \lim_{t \rightarrow t_c^-} (t_c - t)s_2(t), \quad \beta := \lim_{t \rightarrow t_c^-} \frac{s_3(t)}{(s_2(t))^3}.$$

Furthermore, $\alpha, \beta \in (0, \infty)$ and $\alpha = 1/b(t_c)$.

Proof: By (6.14), for $k = 2, 3$, $|s_k(t) - s_{k,\varpi}(t)| \leq K^k$. Since $\lim_{t \rightarrow t_c^-} s_{k,\varpi}(t) = \infty$, we thus have that $\lim_{t \rightarrow t_c^-} s_k(t)/s_{k,\varpi}(t) = 1$. Write $y_\varpi(t) = 1/s_{2,\varpi}(t)$ and $z_\varpi(t) = y_\varpi^3(t)s_{3,\varpi}(t)$, it suffices to show that:

$$\lim_{t \rightarrow t_c^-} \frac{t_c - t}{y_\varpi(t)} = \lim_{t \rightarrow t_c^-} -\frac{1}{y'_\varpi(t)} = \frac{1}{b(t_c)}, \quad \text{and} \quad \lim_{t \rightarrow t_c^-} z_\varpi(t) \in (0, \infty). \quad (6.15)$$

Define $A_l(t) = \sum_{i=1}^K (K+i)^l a_i(t)$ and $C_l(t) = \sum_{i=1}^K i^l c_i(t)$ for $l = 1, 2, 3$. Then by (6.12), (6.13), (6.9) and (6.10), the derivative of $y_\varpi(t)$ and $z_\varpi(t)$ can be written as follows (we omit t from the notation):

$$y'_\varpi = -(A_2 + C_2 x_\varpi) y_\varpi^2 - 2C_1 y_\varpi - b, \quad (6.16)$$

$$\begin{aligned} z'_\varpi &= y_\varpi^3 [A_3 + 3C_1 s_{3,\varpi} + 3C_2 s_{2,\varpi} + C_3 x_\varpi + 3b s_{2,\varpi} s_{3,\varpi}] \\ &\quad - 3y_\varpi^2 s_{3,\varpi} [(A_2 + C_2 x_\varpi) y_\varpi^2 + 2C_1 y_\varpi + b] \\ &= -(3y_\varpi A_2 + 3y_\varpi C_2 x_\varpi + 3C_1) z_\varpi + (y_\varpi^3 A_3 + 3y_\varpi^2 C_2 + y_\varpi^3 C_3 x_\varpi) \\ &= -B_1 z_\varpi + B_2, \end{aligned} \quad (6.17)$$

where $B_1(t) = (3y_\varpi(t)A_2(t) + 3y_\varpi(t)C_2(t)x_\varpi(t) + 3C_1(t))$ and $B_2(t) = (y_\varpi^3(t)A_3(t) + 3y_\varpi^2(t)C_2(t) + y_\varpi^3(t)C_3(t)x_\varpi(t))$. Since $\lim_{t \rightarrow t_c^-} y_\varpi(t) = 0$, we have $\lim_{t \rightarrow t_c^-} y'_\varpi(t) = -b(t_c)$ which proves the first statement in (6.15).

Choose $t_1 \in (0, t_c)$ such that $y_\varpi(t), z_\varpi(t) \in (0, \infty)$ for all $t \in (t_1, t_c)$. Then from (6.17), for all such t

$$z_\varpi(t) = \int_{t_1}^t e^{-\int_s^t B_1(u)du} B_2(s)ds + z_\varpi(t_1) e^{-\int_{t_1}^t B_1(u)du}.$$

Since B_1, B_2 are nonnegative and $\sup_{t \in [t_1, t_c]} \{B_1(t) + B_2(t)\} < \infty$, we have $\lim_{t \rightarrow t_c^-} z_\varpi(t) \in (0, \infty)$. This completes the proof of (6.15). The result follows. \square

We now complete the proof of Theorem 3.2(i).

Proof of Theorem 3.2(i): Let α, β be as introduced in Lemma 6.5. From Lemma 6.4 it follows that $y(t) = 1/s_2(t)$ and $z(t) = y^3(t)s_3(t)$, for $0 \leq t < t_c$, solve the differential equations

$$y'(t) = F^y(\mathbf{x}(t), y(t)), \quad z'(t) = F^z(\mathbf{x}(t), y(t), z(t)), \quad y(0) = z(0) = 1, \quad (6.18)$$

where $F^y : [0, 1]^{K+2} \rightarrow \mathbb{R}$ and $F^z : [0, 1]^{K+2} \times \mathbb{R} \rightarrow \mathbb{R}$ are defined as

$$F^y(\mathbf{x}, y) := -y^2 F_2^s(\mathbf{x}, 1/y), \quad F^z(\mathbf{x}, y, z) := 3z F^y(\mathbf{x}, y)/y + y^3 F_3^s(\mathbf{x}, 1/y, z/y^3), \quad (6.19)$$

$(\mathbf{x}, y, z) \in [0, 1]^{K+2} \times \mathbb{R} \rightarrow \mathbb{R}$. We claim that $F^y(\mathbf{x}, y)$ and $F^z(\mathbf{x}, y, z)$ are polynomials. To see this note that, by (6.9), for any $u \in \mathbb{R}$

$$y^2 F_{2,\varpi}^s(\mathbf{x}, u/y) = y^2 \sum_{j=1}^K (K+j)^2 F_j^a(\mathbf{x}) + uy \sum_{j=1}^K 2j F_j^c(\mathbf{x}) + y^2 x_\varpi \sum_{j=1}^K j^2 F_j^c(\mathbf{x}) + u^2 F^b(\mathbf{x}).$$

From the definition of F_2^s in Lemma 6.4 it is now clear that $F^y(\mathbf{x}, y)$ is a polynomial. To check that $F^z(\mathbf{x}, y, z)$ is a polynomial, one only needs to examine the expression

$$-3zyF_{2,\varpi}^s(\mathbf{x}, \frac{1}{y} - \beta_1) + y^3F_{3,\varpi}^s(\mathbf{x}, \frac{1}{y} - \beta_1, \frac{z}{y^3} - \beta_2), \text{ where } \beta_k = \sum_{i=1}^K i^k x_i, i = 1, 2.$$

Note that the non-polynomial part in the first term is $-3yz \cdot \frac{1}{y^2} F^b(\mathbf{x})$, which gets cancelled with the non-polynomial part of the second term, namely $y^3 \cdot \frac{3}{y} \cdot \frac{z}{y^3} F^b(\mathbf{x})$. This proves the claim. Also, from (6.10) it is clear that the order of z in $F^z(\mathbf{x}, y, z)$ is at most 1.

Thus (6.18) has a unique solution. Also, defining $y(t_c) = \lim_{t \rightarrow t_c^-} y(t) = \lim_{t \rightarrow t_c^-} y_\varpi(t)$ and $z(t_c) = \lim_{t \rightarrow t_c^-} z(t) = \lim_{t \rightarrow t_c^-} z_\varpi(t)$, we see that y, z are twice continuously differentiable (from the left) at t_c . Furthermore, $y'(t_c^-) = -\alpha^{-1}$ and $z(t_c^-) = \beta$. Thus we have

$$y(t) = \frac{1}{\alpha}(t_c - t)(1 + O(t_c - t)), \quad z(t) = \beta(1 + O(t_c - t)), \text{ as } t \uparrow t_c.$$

The result follows. \square

6.4. Asymptotic analysis of $\bar{s}_2(t)$ and $\bar{s}_3(t)$. In preparation for the proof of Theorem 3.2(ii), in this section we will obtain some useful semi-martingale decompositions for $Y(t) := \frac{1}{\bar{s}_2(t)}$ and $Z(t) := \frac{\bar{s}_3(t)}{(\bar{s}_2(t))^3}$. Throughout this section and next we will denote $|\mathcal{C}_1^{(n)}(t)|$ as $I(t)$. Recall the functions F_2^s, F_3^s introduced in Lemma 6.4.

Lemma 6.6. *The processes \bar{s}_2 and \bar{s}_3 are $\{\mathcal{F}_t\}_{0 \leq t < t_c}$ semi-martingales of the form (6.3) and the following equations hold.*

$$(a) \quad \mathbf{d}(\bar{s}_2)(t) = F_2^s(\bar{\mathbf{x}}(t), \bar{s}_2(t)) + O_{t_c}(I^2(t)\bar{s}_2(t)/n).$$

$$(b) \quad \mathbf{d}(\bar{s}_3)(t) = F_3^s(\bar{\mathbf{x}}(t), \bar{s}_2(t), \bar{s}_3(t)) + O_{t_c}(I^3(t)\bar{s}_2(t)/n).$$

$$(c) \quad \mathbf{v}(\bar{s}_2)(t) = O_{t_c}(I^2(t)\bar{s}_2^2(t)/n).$$

Proof: Parts (a) and (b) are immediate from (6.2), Lemma 6.1(b) and Lemma 6.2. For part (c), recall the three types of events described in Section 6.1. For type 1, $\Delta\bar{s}_2(t)$ is bounded by $2K^2/n$ and the total rate of such events is bounded by $n/2$. For type 2, the attachment of a size j component, $1 \leq j \leq K$, to a component \mathcal{C} in \mathbf{BSR}_t^* occurs at rate $|\mathcal{C}|c_j^*(t)$ and produces a jump $\Delta\bar{s}_2(t) = 2j|\mathcal{C}|/n$. For type 3, components \mathcal{C} and $\tilde{\mathcal{C}}$ merge at rate $|\mathcal{C}||\tilde{\mathcal{C}}|b^*(t)/n$ and produce a jump $\Delta\bar{s}_2(t) = 2|\mathcal{C}||\tilde{\mathcal{C}}|/n$. Thus for $t \in [0, t_c)$, $\mathbf{v}(\bar{s}_2)(t)$ can be estimated as

$$\begin{aligned} \mathbf{v}(\bar{s}_2)(t) &\leq \frac{n}{2} \left(\frac{2K^2}{n} \right)^2 + \sum_{j=1}^K \sum_{\mathcal{C} \in \mathbf{BSR}_t^*} \left(\frac{2j|\mathcal{C}|}{n} \right)^2 |\mathcal{C}|c_j^*(t) + \sum_{\mathcal{C} \neq \tilde{\mathcal{C}} \in \mathbf{BSR}_t^*} \left(\frac{2|\mathcal{C}||\tilde{\mathcal{C}}|}{n} \right)^2 \frac{b^*(t)|\mathcal{C}||\tilde{\mathcal{C}}|}{n} \\ &\leq \frac{2K^4}{n} + \frac{4K^2\mathcal{S}_3}{n^2} + \frac{4(\mathcal{S}_3)^2}{n^3} = O_{t_c} \left(\frac{I^2(t)\bar{s}_2^2(t)}{n} \right). \end{aligned}$$

This proves (c). \square

In the next lemma, we obtain a semi-martingale decomposition for Y .

Lemma 6.7. *The process $Y(t) = 1/\bar{s}_2(t)$ is a $\{\mathcal{F}_t\}_{0 \leq t < t_c}$ semi-martingale of the form (6.3) and*

(i) *With $F^y(\cdot)$ as defined in (6.19),*

$$\mathbf{d}(Y)(t) = F^y(\bar{\mathbf{x}}(t), Y(t)) + O_{t_c} \left(\frac{I^2(t)Y(t)}{n} \right). \quad (6.20)$$

(ii)

$$\mathbf{v}(Y)(t) = O_{t_c} \left(\frac{I^2(t)Y^2(t)}{n} \right).$$

Proof: Note that

$$\Delta Y(t) = \frac{1}{\bar{s}_2 + \Delta \bar{s}_2} - \frac{1}{\bar{s}_2} = -\frac{\Delta \bar{s}_2}{\bar{s}_2^2} + \frac{(\Delta \bar{s}_2)^2}{\bar{s}_2^2(\bar{s}_2 + \Delta \bar{s}_2)} = -\frac{\Delta \bar{s}_2}{\bar{s}_2^2} + O_{t_c} \left(\frac{(\Delta \bar{s}_2)^2}{\bar{s}_2^3} \right). \quad (6.21)$$

Thus by Lemma 6.6(a), we have,

$$\begin{aligned} \mathbf{d}(Y)(t) &= -\frac{1}{(\bar{s}_2(t))^2} \mathbf{d}(\bar{s}_2)(t) + O_{t_c} \left(\frac{1}{(\bar{s}_2(t))^3} \mathbf{v}(\bar{s}_2)(t) \right) \\ &= \left(-\frac{1}{(\bar{s}_2(t))^2} \right) \left(F_2^s(\bar{\mathbf{x}}(t), \bar{s}_2(t)) + O_{t_c} \left(\frac{I^2(t)\bar{s}_2(t)}{n} \right) \right) + O_{t_c} \left(\frac{1}{(\bar{s}_2(t))^3} \cdot \frac{I^2(t)\bar{s}_2^2(t)}{n} \right) \\ &= F^y(\bar{\mathbf{x}}(t), Y(t)) + O_{t_c} \left(\frac{I^2(t)Y(t)}{n} \right). \end{aligned}$$

This proves (i). For (ii), note that (6.21) also implies $(\Delta Y(t))^2 \leq \frac{(\Delta \bar{s}_2)^2}{\bar{s}_2^4}$. We then have

$$\mathbf{v}(Y)(t) \leq \frac{2\mathbf{v}(\bar{s}_2)(t)}{\bar{s}_2^4} = O_{t_c} \left(\frac{I^2(t)Y^2(t)}{n} \right).$$

The result follows. \square

We now give a semi-martingale decomposition for $Z(t) = \bar{s}_3(t)/(\bar{s}_2(t))^3$.

Lemma 6.8. *The process $Z(t) = \bar{s}_3(t)/(\bar{s}_2(t))^3$ is a $\{\mathcal{F}_t\}_{0 \leq t < t_c}$ semi-martingale of the form (6.3) and*

(i) *With $F^z(\cdot)$ as defined in (6.19),*

$$\mathbf{d}(Z)(t) = F^z(\bar{\mathbf{x}}(t), Y(t), Z(t)) + O_{t_c} \left(\frac{I^3(t)Y^2(t)}{n} \right).$$

(ii)

$$\mathbf{v}(Z)(t) = O_{t_c} \left(\frac{I^4(t)Y^4(t)}{n} + \frac{I^6(t)Y^6(t)}{n} \right).$$

Proof: Note that

$$\Delta Z = Y^3 \Delta \bar{s}_3 + 3Y^2 \bar{s}_3 \Delta Y + R(\Delta Y, \Delta \bar{s}_3),$$

where $R(\Delta Y, \Delta \bar{s}_3)$ is the error term which, using the observations that $\bar{s}_3 \leq I\bar{s}_2$, $\Delta \bar{s}_3 \leq 3I\Delta \bar{s}_2$ and $|\Delta Y| \leq Y^2 \Delta \bar{s}_2$, can be bounded as follows.

$$\begin{aligned} |R(\Delta Y, \Delta \bar{s}_3)| &\leq 3Y^2 |\Delta Y| |\Delta \bar{s}_3| + 3Y \bar{s}_3 |\Delta Y|^2 \\ &\leq 3Y^2 \cdot Y^2 \Delta \bar{s}_2 \cdot 3I\Delta \bar{s}_2 + 3I \cdot (Y^2 \Delta \bar{s}_2)^2 = 12IY^4 \cdot (\Delta \bar{s}_2)^2. \end{aligned}$$

From Lemma 6.6(b), Lemma 6.7(i) and Lemma 6.6(c), we have

$$\begin{aligned}
\mathbf{d}(Z)(t) &= Y^3(t)\mathbf{d}(\bar{s}_3)(t) + 3Y^2(t)\bar{s}_3(t)\mathbf{d}(Y)(t) + O_{t_c}(I(t)Y^4(t)\mathbf{v}(\bar{s}_2)(t)) \\
&= Y^3(t) \left(F_3^s(\bar{\mathbf{x}}(t), \bar{s}_2(t), \bar{s}_3(t)) + O_{t_c}\left(\frac{I^3(t)\bar{s}_2(t)}{n}\right) \right) \\
&\quad + 3Y^2(t)\bar{s}_3(t) \left(F^y(\bar{\mathbf{x}}(t), Y(t)) + O_{t_c}\left(\frac{I^2(t)Y(t)}{n}\right) \right) + O_{t_c}\left(\frac{I^3(t)Y^2(t)}{n}\right) \\
&= F^z(\bar{\mathbf{x}}(t), Y(t), Z(t)) + O_{t_c}\left(\frac{I^3(t)Y^2(t)}{n}\right).
\end{aligned}$$

This proves (i). For (ii), note that

$$Y^3|\Delta\bar{s}_3| + 3Y^2\bar{s}_3|\Delta Y| \leq Y^3 \cdot 3I|\Delta\bar{s}_2| + 3Y^2 \cdot I\bar{s}_2 \cdot Y^2|\Delta\bar{s}_2| = 6Y^3I|\Delta\bar{s}_2|.$$

Thus,

$$|\Delta Z| \leq 6Y^3I|\Delta\bar{s}_2| + 12IY^4 \cdot (\Delta\bar{s}_2)^2.$$

Applying Lemma 6.6(c) we now have,

$$\mathbf{v}(Z)(t) = O_{t_c}(Y^6I^2\mathbf{v}(\bar{s}_2)(t)) + O_{t_c}\left(\frac{I^6Y^6}{n}\right) = O_{t_c}\left(\frac{I^4Y^4}{n} + \frac{I^6Y^6}{n}\right).$$

The result follows. \square

6.5. Proof of Theorem 3.2(ii). We begin with an upper bound on the size of the largest component at time $t \leq t_n = t_c - n^{-\gamma}$ for $\gamma \in (0, 1/4)$, which has been proved in [8], and will play an important role in the proof of Theorem 3.2(ii).

Theorem 6.9 ([8, Theorem 1.2] **Barely subcritical regime**). *Fix $\gamma \in (0, 1/4)$. Then there exists $C_3 \in (0, \infty)$ such that, as $n \rightarrow \infty$,*

$$\mathbb{P}\left\{I^{(n)}(t) \leq C_3 \frac{(\log n)^4}{(t_c - t)^2}, \forall t < t_c - n^{-\gamma}\right\} \rightarrow 1.$$

The next lemma is an elementary consequence of Gronwall's inequality.

Lemma 6.10. *Let $\{t_n\}$ be a sequence of positive reals such that $t_n \in [0, t_c)$ for all n . Suppose that $U^{(n)}$ is a semi-martingale of the form (6.3) with values in $\mathbb{D} \subset \mathbb{R}$. Let $g : [0, t_c) \times \mathbb{D} \rightarrow \mathbb{R}$ be such that, for some $C_4(g) \in (0, \infty)$,*

$$\sup_{t \in [0, t_c)} |g(t, u_1) - g(t, u_2)| \leq C_4(g)|u_1 - u_2|, \quad u_1, u_2 \in \mathbb{D}. \quad (6.22)$$

Let $\{u(t)\}_{t \in [0, t_c)}$ be the unique solution of the differential equation

$$u'(t) = g(t, u(t)), \quad u(0) = u_0.$$

Further suppose that there exist positive sequences:

- (i) $\{\theta_1(n)\}$ such that, whp, $|U^{(n)}(0) - u_0| \leq \theta_1(n)$.
- (ii) $\{\theta_2(n)\}$ such that, whp,

$$\int_0^{t_n} |\mathbf{d}(U^{(n)})(t) - g(t, U^{(n)}(t))| dt \leq \theta_2(n).$$

- (iii) $\{\theta_3(n)\}$ such that, whp, $\langle \mathbf{M}(U^{(n)}), \mathbf{M}(U^{(n)}) \rangle_{t_n} \leq \theta_3(n)$.

Then, whp,

$$\sup_{0 \leq t \leq t_n} |U^{(n)}(t) - u(t)| \leq e^{C_4(g)t_c} (\theta_1(n) + \theta_2(n) + \theta_4(n)),$$

where $\theta_4 = \theta_4(n)$ is any sequence satisfying $\sqrt{\theta_3(n)} = o(\theta_4(n))$.

Proof: We suppress n from the notation unless needed. Using the Lipschitz property of g , we have, for all $t \in [0, t_n]$,

$$\begin{aligned} |U(t) - u(t)| &\leq |U(0) - u_0| + \int_0^t |\mathbf{d}(U)(s) - g(s, U(s))| ds + \int_0^t |g(s, U(s)) - g(s, u(s))| ds + |\mathbf{M}(U)(t)| \\ &\leq |U(0) - u_0| + \int_0^t |\mathbf{d}(U)(s) - g(s, U(s))| ds + |\mathbf{M}(U)(t)| + C_4 \int_0^t |U(s) - u(s)| ds. \end{aligned}$$

Then by Gronwall's lemma

$$\sup_{0 \leq t \leq t_n} |U(t) - u(t)| \leq \left(|U(0) - u_0| + \int_0^{t_n} |\mathbf{d}(U)(s) - g(s, U(s))| ds + \sup_{0 \leq t \leq t_n} |\mathbf{M}(U)(t)| \right) e^{C_4 t_c}. \quad (6.23)$$

Let $\tau^{(n)} = \inf\{t \geq 0 : \langle \mathbf{M}(U), \mathbf{M}(U) \rangle_t > \theta_3(n)\}$. By Doob's inequality

$$\mathbb{E} \left[\sup_{0 \leq t \leq t_n} |\mathbf{M}(U)(t \wedge \tau)|^2 \right] \leq 4\mathbb{E} [|\mathbf{M}(U)(t_n \wedge \tau)|^2] = 4\mathbb{E} [\langle \mathbf{M}(U), \mathbf{M}(U) \rangle_{t_n \wedge \tau}] \leq 4\theta_3(n).$$

Then for any $\theta_4(n)$ such that $\theta_3 = o((\theta_4)^2)$, we have

$$\begin{aligned} \mathbb{P} \left\{ \sup_{0 \leq t \leq t_n} |\mathbf{M}(U)(t)| > \theta_4(n) \right\} &\leq \mathbb{P} \{ \tau^{(n)} < t_n \} + \mathbb{P} \left\{ \sup_{0 \leq t \leq t_n} |\mathbf{M}(U)(t \wedge \tau)| > \theta_4(n) \right\} \\ &\leq \mathbb{P} \{ \langle \mathbf{M}(U), \mathbf{M}(U) \rangle_{t_n} > \theta_3(n) \} + 4\theta_3(n)/\theta_4^2(n) \rightarrow 0. \end{aligned}$$

The result now follows on using the above observation in (6.23). \square

Proof of Theorem 3.2(ii): Let y and z be as in the proof of Theorem 3.2(i). It suffices to show

$$\sup_{0 \leq t \leq t_n} |Y(t) - y(t)| n^{1/3} \xrightarrow{\mathbb{P}} 0 \quad (6.24)$$

$$\sup_{0 \leq t \leq t_n} |Z(t) - z(t)| \xrightarrow{\mathbb{P}} 0. \quad (6.25)$$

We begin by proving the following weaker result than (6.24):

$$\sup_{0 \leq t \leq t_n} |Y(t) - y(t)| = O(n^{-1/5}), \text{ whp.} \quad (6.26)$$

Recalling from the proof of Theorem 3.2(i) that $\mathbf{x} \mapsto F^y(\mathbf{x}, y)$ is Lipschitz for $\mathbf{x} \in [0, 1]^{K+1}$, uniformly for all $y \in [0, 1]$, we get for some $d_1 \in (0, \infty)$

$$\sup_{0 \leq t \leq t_c} |F^y(\bar{\mathbf{x}}(t), Y(t)) - F^y(\mathbf{x}(t), Y(t))| \leq d_1 \sup_{i \in \Omega_K} \sup_{0 \leq t \leq t_c} |\bar{x}_i(t) - x_i(t)|.$$

From Lemma 6.7(i) and Lemma 6.1(a) we now get for some $d_2 \in (0, \infty)$, whp,

$$|\mathbf{d}(Y)(t) - F^y(\mathbf{x}(t), Y(t))| \leq d_2 \left(\frac{I^2(t)Y(t)}{n} + n^{-2/5} \right), \text{ for all } t \in [0, t_n].$$

Thus, from Theorem 6.9 and recalling that $\gamma < 1/5$, we have whp,

$$\begin{aligned} \int_0^{t_n} |\mathbf{d}(Y)(t) - F^y(\mathbf{x}(t), Y(t))| dt &= O\left(\int_0^{t_n} \frac{(\log n)^8}{n(t_c - t)^4} dt + n^{-2/5}\right) \\ &= O((\log n)^8 n^{3\gamma-1}) + O(n^{-2/5}) = O(n^{-2/5}). \end{aligned}$$

Next, by Lemma 6.7(ii) and using the fact $Y(t) \leq 1$ for all $t \in [0, t_c]$,

$$\begin{aligned} \langle \mathbf{M}(Y), \mathbf{M}(Y) \rangle_{t_n} &= O\left(\int_0^{t_n} \frac{I^2(t)Y^2(t)}{n} dt\right) = O\left(\int_0^{t_n} \frac{I^2(t)}{n} dt\right) \\ &= O\left(\int_0^{t_n} \frac{(\log n)^8}{n(t_c - t)^4} dt\right) = O((\log n)^8 n^{3\gamma-1}). \end{aligned} \quad (6.27)$$

The statement in (6.26) now follows on observing that $((\log n)^8 n^{3\gamma-1})^{1/2} = o(n^{-1/5})$ and applying Lemma 6.10 with $\mathbb{D} := [0, 1]$, $g(t, y) := F^y(\mathbf{x}(t), y)$, $\theta_1 = 0$, $\theta_2 = n^{-2/5}$ and $\theta_3 = (\log n)^8 n^{3\gamma-1}$.

We now strengthen the estimate in (6.26) to prove (6.24). From Theorem 3.2(i) it follows that $y(t_n) = 1/s_2(t_n) = \Theta(n^{-\gamma})$. Since $\gamma < 1/5$, from (6.26) we have, whp, $Y(t) \leq 2y(t)$ for all $t \leq t_n$. Thus from the first equality in (6.27) and Theorem 3.2(i) we get, whp,

$$\langle \mathbf{M}(Y), \mathbf{M}(Y) \rangle_{t_n} = O\left(\int_0^{t_n} \frac{I^2(t)y^2(t)}{n} dt\right) = O\left(\int_0^{t_n} \frac{(\log n)^8}{n(t_c - t)^2} dt\right) = O((\log n)^8 n^{\gamma-1}).$$

Since $((\log n)^8 n^{\gamma-1})^{1/2} = o(n^{-2/5})$, applying Lemma 6.10 again gives

$$\sup_{0 \leq t \leq t_n} |Y(t) - y(t)| = O(n^{-2/5}), \text{ whp.} \quad (6.28)$$

This proves (6.24).

We now prove (6.25). We will apply Lemma 6.10 to $\mathbb{D} := \mathbb{R}$ and $g(t, z) := F^z(\mathbf{x}(t), y(t), z)$. As noted in the proof of Theorem 3.2(i), g defined as above satisfies (6.22).

We now verify the three assumptions in Lemma 6.10. Note that (i) is satisfied with $\theta_1 = 0$, since $Z(0) = z(0) = 1$. Next, by Lemma 6.8(ii) and the fact $Y(t) \leq 2y(t)$ for $t \leq t_n$, whp, we have

$$\begin{aligned} \langle \mathbf{M}(Z), \mathbf{M}(Z) \rangle_{t_n} &= O\left(\int_0^{t_n} \left(\frac{I^4(t)Y^4(t)}{n} + \frac{I^6(t)Y^6(t)}{n}\right) dt\right) \\ &= O\left(\int_0^{t_n} \left(\frac{(\log n)^{16}}{n(t_c - t)^4} + \frac{(\log n)^{24}}{n(t_c - t)^6}\right) dt\right) = O((\log n)^{24} n^{5\gamma-1}). \end{aligned}$$

Since $\gamma < 1/5$, we can find $\theta_4(n) \rightarrow 0$ such that $\sqrt{(\log n)^{24} n^{5\gamma-1}} = o(\theta_4(n))$. Thus (iii) in Lemma 6.10 is satisfied. Next recall from the proof of Theorem 3.2(i) that $g(t, z)$ is linear in z . Also, $Z(t) \leq I(t)$. Thus from Lemma 6.1 and (6.28), for some $d_3 \in (0, \infty)$ whp, for all $t \leq t_n$

$$\begin{aligned} &|F^z(\bar{\mathbf{x}}(t), Y(t), Z(t)) - g(t, Z(t))| \\ &\leq d_3(1 + Z(t)) \left(\sup_{1 \leq i \leq K} \sup_{0 \leq t \leq t_n} |\bar{x}_i(t) - x_i(t)| + \sup_{0 \leq t \leq t_n} |Y(t) - y(t)| \right) = I(t)O(n^{-2/5}). \end{aligned}$$

By Lemma 6.8(i) and the above bound,

$$\begin{aligned}
 \int_0^{t_n} |\mathbf{d}(Z)(t) - g(t, Z(t))| dt &= O\left(\int_0^{t_n} n^{-2/5} I(t) dt\right) + O\left(\int_0^{t_n} \frac{y^2(t) I^3(t)}{n} dt\right) \\
 &= O\left(\int_0^{t_n} \frac{(\log n)^4 n^{-2/5}}{(t_c - t)^2} dt\right) + O\left(\int_0^{t_n} \frac{(\log n)^{12}}{(t_c - t)^4} dt\right) \\
 &= O((\log n)^4 n^{\gamma-2/5}) + O((\log n)^{12} n^{3\gamma-1}). \tag{6.29}
 \end{aligned}$$

This verifies (ii) in Lemma 6.10 with $\theta_2(n) = O((\log n)^{12} n^{3\gamma-1})$. From Lemma 6.10 we now have

$$\sup_{0 \leq t \leq t_n} |Z(t) - z(t)| = O(\theta_1(n) + \theta_2(n) + \theta_4(n)) = o(1).$$

The result follows. \square

Part (iii) of Theorem 3.2 is a simple consequence of part (ii).

Proof of Theorem 3.2(iii): The convergence when $t = 0$ is trivial. Consider now $t > 0$. Since $t_n \rightarrow t_c$ as $n \rightarrow \infty$, we have from part (ii) of the theorem that, for fixed $t \in (0, t_c)$,

$$\frac{1}{\bar{s}_2(t)} \xrightarrow{\mathbb{P}} \frac{1}{s_2(t)}, \quad \frac{\bar{s}_3(t)}{(\bar{s}_2(t))^3} \xrightarrow{\mathbb{P}} \frac{s_3(t)}{(s_2(t))^3}.$$

Also, since $t > 0$, we have that $x_i(t) > 0$ for all $i \in \Omega_K$, thus $s_2(t) > 0$. Theorem 3.2(iii) is now immediate. \square

7. COUPLING WITH THE MULTIPLICATIVE COALESCENT

In this section we prove Theorem 3.3. Throughout this section we fix $\gamma \in (1/6, 1/5)$. The basic idea of the proof is as follows. Recall $\alpha, \beta \in (0, \infty)$ from Theorem 3.2 (see also Lemma 6.5). We begin by approximating the BSR random graph process by a process which until time $t_n := t_c - n^{-\gamma}$ is identical to the BSR process and in the time interval $[t_n, t_c + \alpha\beta^{2/3} \frac{\lambda}{n^{1/3}}]$ evolves as an Erdős-Rényi process, namely over this interval edges between any pair of vertices appear at rate $1/\alpha n$, and self loops at any given vertex appear at rate $1/2\alpha n$. Asymptotic behavior of this random graph is analyzed using Theorem 5.1. Theorems 3.2 and 6.9 help in verifying the conditions (5.1) and (5.2) in the statement of Theorem 5.1. We then complete the proof of Theorem 3.3 by arguing that the ‘difference’ between the BSR process and the modified random graph process is asymptotically negligible.

Let

$$t_n = t_c - \alpha\beta^{2/3} \frac{\lambda_n}{n^{1/3}} \text{ where } \lambda_n = \frac{n^{1/3-\gamma}}{\alpha\beta^{2/3}}. \tag{7.1}$$

Throughout this section, for $\lambda \in \mathbb{R}$, we denote $t^\lambda = t_c + \alpha\beta^{2/3} \lambda/n^{1/3}$. Recall the random graph process $\mathbf{BSR}^*(t)$ introduced in Section 6. Denote by $(|\mathcal{C}_i^*(t)|, \xi_i^*(t))_{i \geq 1}$ the vector of ordered component size and corresponding surplus in $\mathbf{BSR}^*(t)$ (the components are denoted by $\mathcal{C}_i^*(t)$). Let, for $\lambda \in \mathbb{R}$,

$$\bar{\mathbf{C}}^{(n),*}(\lambda) = \left(\frac{\beta^{1/3}}{n^{2/3}} |\mathcal{C}_i^*(t^\lambda)| : i \geq 1 \right), \quad \bar{\mathbf{Y}}^{(n),*}(\lambda) = \left(\xi_i^*(t^\lambda) : i \geq 1 \right).$$

For $i \geq 1$, denote $\bar{C}_i^{(n),*}(\lambda)$ and $\bar{Y}_i^{(n),*}(\lambda)$ for the i -th coordinate of $\bar{C}^{(n),*}(\lambda)$ and $\bar{Y}^{(n),*}(\lambda)$ respectively. Write $\bar{Y}_i^{(n),*} = \tilde{\xi}_i^{(n)} + \hat{\xi}_i^{(n)}$ where $\tilde{\xi}_i^{(n)}(\lambda)$ represents the surplus in $\mathbf{BSR}^*(t^\lambda)$ that is created before time t_n , namely

$$\tilde{\xi}_i^{(n)}(\lambda) = \sum_{j: \mathcal{C}_j^*(t_n) \subset \mathcal{C}_i^*(t^\lambda)} \bar{Y}_j^{(n),*}(-\lambda_n).$$

In Section 7.2 we will show that the contribution from $\tilde{\xi}^{(n)}(\lambda) := (\tilde{\xi}_i^{(n)}(\lambda) : i \geq 1)$ is asymptotically negligible. First, in Section 7.1 below we analyze the contribution from the ‘new surplus’, i.e. $\hat{\xi}^{(n)} := (\hat{\xi}_i^{(n)} : i \geq 1)$.

7.1. Surplus created after time t_n . The main result of this section is as follows. Recall the process $\mathbf{Z}(\lambda) = (\mathbf{X}(\lambda), \mathbf{Y}(\lambda))$ introduced in Theorem 3.1.

Theorem 7.1. *For every $\lambda \in \mathbb{R}$, as $n \rightarrow \infty$, $(\bar{C}^{(n),*}(\lambda), \hat{\xi}^{(n)}(\lambda))$ converges in distribution, in \mathbb{U}_\downarrow , to $(\mathbf{X}(\lambda), \mathbf{Y}(\lambda))$.*

The basic idea in the proof of the above theorem is to argue that $\mathbf{BSR}^*(t^\lambda)$ ‘lies between’ two Erdős-Rényi random graph processes $\mathbf{G}^{(n),-}(t^\lambda)$ and $\mathbf{G}^{(n),+}(t^\lambda)$, whp, and then apply Theorem 5.1 to each of these processes. For a graph \mathbf{G} , denote by $|\mathcal{C}_i(\mathbf{G})|$ and $\xi_i(\mathbf{G})$ the size and surplus, respectively, of the i -th largest component, $\mathcal{C}_i(\mathbf{G})$ of graph \mathbf{G} . We begin with the following lemma. Recall λ_n from (7.1).

Lemma 7.2. *There exists a construction of $\{\mathbf{BSR}^*(t)\}_{t \geq 0}$ along with two other random graph processes $\{\mathbf{G}^{(n),-}(t)\}_{t \geq 0}$ and $\{\mathbf{G}^{(n),+}(t)\}_{t \geq 0}$ such that:*

(i) *With high probability,*

$$\mathbf{G}^{(n),-}(t^\lambda) \subset \mathbf{BSR}^*(t^\lambda) \subset \mathbf{G}^{(n),+}(t^\lambda) \quad \text{for all } \lambda \in [-\lambda_n, \lambda_n]. \quad (7.2)$$

(ii) *Let for $i \geq 1$, $\bar{C}_i^{(n),\mp}(\lambda) = \frac{\beta^{1/3}}{n^{2/3}} |\mathcal{C}_i(\mathbf{G}^{(n),\mp}(t^\lambda))|$ and*

$$\bar{Y}_i^{(n),\mp}(\lambda) = \xi_i(\mathbf{G}^{(n),\mp}(t^\lambda)) - \sum_{j: \mathcal{C}_j(\mathbf{G}^{(n),\mp}(t_n)) \subset \mathcal{C}_i(\mathbf{G}^{(n),\mp}(t^\lambda))} \xi_j(\mathbf{G}^{(n),\mp}(t_n)).$$

Then, for all $\lambda \in \mathbb{R}$

$$(\bar{C}^{(n),\bullet}(\lambda), \bar{Y}^{(n),\bullet}(\lambda)) \xrightarrow{d} (\mathbf{X}(\lambda), \mathbf{Y}(\lambda)), \quad \bullet = -, +,$$

where \xrightarrow{d} denotes weak convergence in \mathbb{U}_\downarrow .

We remark that $\bar{Y}^{(n),\mp}(\lambda)$ represents the surplus in $\mathbf{G}^{(n),\mp}(t^\lambda)$ created after time instant t_n . Proof of the lemma relies on the following proposition which is an immediate consequence of Theorem 3.2 and Theorem 6.9.

Proposition 7.3. *There exists a $\kappa \in (0, \frac{1}{3} - \gamma)$ such that*

$$\frac{\bar{s}_3(t_n)}{(\bar{s}_2(t_n))^3} \xrightarrow{\mathbb{P}} \beta, \quad \frac{n^{1/3}}{\bar{s}_2(t_n)} - \frac{n^{1/3-\gamma}}{\alpha} \xrightarrow{\mathbb{P}} 0, \quad \frac{I^{(n)}(t_n)}{n^{2\gamma+\kappa}} \xrightarrow{\mathbb{P}} 0.$$

We now prove Lemma 7.2.

Proof of Lemma 7.2:

We suppress n in the notation for the random graph processes. Write $t_n^+ := t_c + n^{-\gamma}$. Let $\mathbf{BSR}(t)$ for $t \in [0, t_n^+]$ be constructed as in Section 2.2 and define $\mathbf{BSR}^*(t)$ for $t \in [0, t_n]$ as in Section 6. Set

$$\mathbf{G}^{(n),-}(t) = \mathbf{G}^{(n),+}(t) = \mathbf{BSR}^*(t), \quad \text{for } t \in [0, t_n].$$

We now give the construction of these processes for $t \in [t_n, t_n^+]$.

The construction is done in two rounds. In the first round, we construct processes $\mathbf{G}^{I,-}(t)$, $\mathbf{BSR}^{I,*}(t)$ and $\mathbf{G}^{I,+}(t)$ for $t \in [t_n, t_n^+]$ by using only the information about immigrations and attachments in $\mathbf{BSR}(t)$, while the edge formation between large components is ignored. We first construct the process $\{\mathbf{BSR}^I(t)\}_{t \in [t_n, t_n^+]}$ as follows. Let $\mathbf{BSR}^I(t_n) := \mathbf{BSR}(t_n)$. For $t > t_n$, $\mathbf{BSR}^I(t)$ is constructed along with and same as $\mathbf{BSR}(t)$, except for when

$$c_{t-}(\vec{v}) \in \{\vec{j} \in \Omega_K^4 : \vec{j} \in F, j_1 = j_2 = \varpi \quad \text{or} \quad \vec{j} \notin F, j_3 = j_4 = \varpi\},$$

in which case no edge is added to $\mathbf{BSR}^I(t)$.

Let $\bar{x}_i(t), a_i^*(t), b^*(t), c_i^*(t)$, $1 \leq i \leq K$, $t \in [t_n, t_n^+]$, be the processes determined from $\{\mathbf{BSR}(t)\}_{t \in [t_n, t_n^+]}$ as in Section 6. These processes will be used in the second round of the construction.

Now define $\mathbf{BSR}^{I,*}(t)$ to be the subgraph that consists of all large components (components of size greater than K) in $\mathbf{BSR}^I(t)$, and then define $\mathbf{G}^{I,-}(t)$ and $\mathbf{G}^{I,+}(t)$ for $t \in [t_n, t_n^+]$ as follows:

$$\mathbf{G}^{I,-}(t) \equiv \mathbf{BSR}^{I,*}(t_n), \quad \text{and} \quad \mathbf{G}^{I,+}(t) \equiv \mathbf{BSR}^{I,*}(t_n^+).$$

Then

$$\mathbf{G}^{I,-}(t) \subset \mathbf{BSR}^{I,*}(t) \subset \mathbf{G}^{I,+}(t) \quad \text{for all } t \in [t_n, t_n^+].$$

We now proceed to the second round of the construction. Let

$$E_n = \left\{ b(t_c) - n^{-1/6} < b^*(t) < b(t_c) + n^{-1/6}, \text{ for all } t \in [t_n, t_n^+] \right\}.$$

Note that Lemma 6.1 and (6.8) implies that with probability at least $1 - C_1 e^{-C_2 n^{1/5}}$,

$$\begin{aligned} \sup_{t \in (t_n, t_n^+)} |b^*(t) - b(t_c)| &\leq \sup_{t \in (t_n, t_n^+)} |b^*(t) - b(t)| + \sup_{t \in (t_n, t_n^+)} |b(t) - b(t_c)| \\ &\leq d_1 n^{-2/5} + d_2 n^{-\gamma} = o(n^{-1/6}). \end{aligned}$$

Thus $\mathbb{P}\{E_n^c\} \rightarrow 0$ as $n \rightarrow \infty$. Since we only need the coupling to be good with high probability, it suffices to construct the coupling of the three processes until the first time $t \in [t_n, t_n^+]$ when $b^*(t) \notin [b(t_c) - n^{-1/6}, b(t_c) + n^{-1/6}]$. Equivalently, we can assume without loss of generality that $b^*(t) \in [b(t_c) - n^{-1/6}, b(t_c) + n^{-1/6}]$, for all $t \in [t_n, t_n^+]$, a.s.

We will construct $\mathbf{G}^+(t)$, $\mathbf{BSR}^*(t)$ and $\mathbf{G}^-(t)$ by adding new edges between components in the three random graph processes $\mathbf{G}^{I,-}(t)$, $\mathbf{BSR}^{I,*}(t)$ and $\mathbf{G}^{I,+}(t)$ such that, at time $t \in [t_n, t_n^+]$ edges are added between each pair of vertices in $\mathbf{G}^{I,-}(t)$, $\mathbf{BSR}^{I,*}(t)$ and $\mathbf{G}^{I,+}(t)$, at rates $\frac{1}{n}(b(t_c) - n^{-1/6})$, $\frac{1}{n}b^*(t)$ and $\frac{1}{n}(b(t_c) + n^{-1/6})$, respectively. The precise mechanism is as follows.

We first construct $\mathbf{G}^+(t)$ for $t \in (t_n, t_n^+]$ by adding edges between every pair of vertices in $\mathbf{G}^{I,+}(t)$ at the rate $\frac{1}{n}(b(t_c) + n^{-1/6})$ and creating self-loops at the rate $\frac{1}{2n}(b(t_c) + n^{-1/6})$ for each vertex in $\mathbf{G}^{I,+}(t)$.

Next, we construct $\mathbf{BSR}^*(t)$ and $\mathbf{G}^-(t)$ through successive thinning of $\mathbf{G}^+(t)$, thus obtaining the desired coupling. Let (e_1, e_2, \dots) be the sequence of edges that are added to $\mathbf{G}^{I,+}(t)$ to obtain $\mathbf{G}^+(t)$. Let (u_1, u_2, \dots) be i.i.d Uniform $[0, 1]$ random variables that are also independent of the random variables used to construct $\mathbf{G}^{I,-}, \mathbf{BSR}^{I,*}, \mathbf{G}^{I,+}, \mathbf{G}^+$. Suppose at time t_k , we have $\mathbf{G}^+(t_k) = \mathbf{G}^+(t_k-) \cup \{e_k\}$, where $e_k = \{v_1, v_2\}$. We set $\mathbf{BSR}^*(t_k) = \mathbf{BSR}^*(t_k-) \cup \{e_k\}$

if and only if

$$v_1, v_2 \in \overline{\mathbf{BSR}}^{I,*}(t_k-) \text{ and } u_k \leq \frac{b^*(t_k)}{b(t_c) + n^{-1/6}},$$

otherwise let $\mathbf{BSR}^*(t_k) = \mathbf{BSR}^*(t_k-)$. This defines the process $\mathbf{BSR}^*(t)$ (with the correct probability law) such that the second inclusion in (7.2) is satisfied. Finally, construct $\mathbf{G}^-(t)$ by a thinning of $\mathbf{BSR}^*(t)$ exactly as above by replacing $\frac{b^*(t_k)}{b(t_c) + n^{-1/6}}$ with $\frac{b(t_c) - n^{-1/6}}{b^*(t_k)}$. Then $\mathbf{G}^-(t)$, for $t \in [t_n, t_n^+]$ is an Erdős-Rényi type processes and the first inclusion in (7.2) is satisfied. This completes the proof of the first part of the lemma.

We now prove (ii). Consider first the case $\bullet = -$. We will apply Theorem 5.1. With notation as in that theorem, it follows from the Erdős-Rényi dynamics of $\mathbf{G}^{(n),-}(t)$ that, the distribution of $(\bar{\mathbf{C}}^{(n),-}(\lambda), \bar{\mathbf{Y}}^{(n),-}(\lambda))$, conditioned on $\{\mathcal{P}_{\vec{v}}(t), t \leq t_n, \vec{v} \in [n]^4\}$, for each $\lambda \in [-\lambda_n, \lambda_n]$, is same as the distribution of $\mathbf{Z}(z^{(n)}, q^{(n)})$, where $z^{(n)} = (\bar{\mathbf{C}}^{(n),-}(-\lambda_n), \mathbf{0})$, $\mathbf{0}$ denotes the vector $(0, 0, \dots)$ and $q^{(n)}$ is determined by the equality

$$q^{(n)} \bar{\mathbf{C}}_i^{(n),-}(-\lambda_n) \bar{\mathbf{C}}_j^{(n),-}(-\lambda_n) = \frac{\alpha \beta^{2/3}}{n^{1/3}} (\lambda + \lambda_n) \frac{(b(t_c) - n^{-1/6})}{n} |\mathcal{C}_i(\mathbf{G}^{(n),-}(t_n))| |\mathcal{C}_j(\mathbf{G}^{(n),-}(t_n))|,$$

for $i \neq j$. Recalling that $\alpha b(t_c) = 1$ it then follows that $q^{(n)} = \lambda + \frac{n^{1/3-\gamma}}{\alpha \beta^{2/3}} + O(n^{1/6-\gamma})$. We now verify the conditions of Theorem 5.1. Taking $x^{(n)} = \bar{\mathbf{C}}^{(n),-}(-\lambda_n)$ we see with, x^*, s_k , $k = 1, 2, 3$ as in Theorem 5.1,

$$s_1^{(n)} \leq \beta^{1/3} n^{1/3}, \quad s_2^{(n)} = \frac{\beta^{2/3}}{n^{4/3}} \sum_{\mathcal{C} \subset \mathbf{BSR}^*(t_n)} |\mathcal{C}|^2, \quad s_3^{(n)} = \frac{\beta}{n^2} \sum_{\mathcal{C} \subset \mathbf{BSR}^*(t_n)} |\mathcal{C}|^3.$$

Recall the definition of \bar{s}_k and $\bar{s}_{k,\varpi}$ from (3.1) and Section 6. Then

$$s_2^{(n)} = \frac{\beta^{2/3} \bar{s}_{2,\varpi}(t_n)}{n^{1/3}}, \quad s_3^{(n)} = \frac{\beta \bar{s}_{3,\varpi}(t_n)}{n}, \quad x^{*(n)} = \beta^{1/3} \frac{I(t_n)}{n^{2/3}}.$$

From the first two convergences in Proposition 7.3 and recalling that, for $k = 1, 2$, $|\bar{s}_{k,\varpi} - \bar{s}_k| \leq K^k$, we immediately get that the first two convergences in (5.1) hold. Also,

$$\frac{x^*}{s_2} = \frac{I(t_n)}{\beta^{1/3} n^{1/3} \bar{s}_{2,\varpi}(t_n)} = \frac{I(t_n)}{\beta^{2/3} n^{\gamma+1/3}} O(1) \rightarrow 0, \text{ in probability,}$$

where the second equality is consequence of the second convergence in Proposition 7.3, and the convergence of the last term follows from the third convergence in Proposition 7.3. This proves the third convergence in (5.1).

Finally we note that the convergence in (5.2) holds with $\varsigma = \frac{1}{1-3(\gamma+\kappa)}$, where κ is as in Proposition 7.3, since

$$s_1 \left(\frac{x^*}{s_2} \right)^\varsigma \leq O(1) n^{1/3} \left(\frac{I(t_n)}{n^{\gamma+1/3}} \right)^\varsigma = O(1) \left(\frac{I(t_n)}{n^{2\gamma+\kappa}} \right)^\varsigma \rightarrow 0,$$

where the last equality follows from our choice of ς and the convergence is a consequence of Proposition 7.3. Thus we have verified all the conditions in Theorem 5.1 and therefore we have from this result that $(\bar{\mathbf{C}}^{(n),-}(\lambda), \bar{\mathbf{Y}}^{(n),-}(\lambda))$ converges in distribution, in \mathbb{U}_\downarrow , to $(\mathbf{X}^*(\lambda), \mathbf{Y}^*(\lambda))$ proving part (ii) of the lemma for $\bullet = -$.

To prove part (ii) of the lemma for $\bullet = +$, one needs slightly more work. Once more we will apply Theorem 5.1. As before, conditioned on $\{\bar{\mathbf{C}}^{(n),+}(\lambda_0) : \lambda_0 \leq -\lambda_n\}$, for each $\lambda \in [-\lambda_n, \lambda_n]$, the distribution of $(\bar{\mathbf{C}}^{(n),+}(\lambda), \bar{\mathbf{Y}}^{(n),+}(\lambda))$ is same as the distribution of $\mathbf{Z}(\bar{z}^{(n)}, \bar{q}^{(n)})$, where

$\bar{z}^{(n)} = (\bar{\mathbf{C}}^{(n),+}(-\lambda_n), \mathbf{0})$ and $\bar{q}^{(n)} = \lambda + \frac{n^{1/3-\gamma}}{\alpha\beta^{2/3}} + O(n^{1/6-\gamma})$. Taking $x^{(n)} = \bar{\mathbf{C}}^{(n),+}(-\lambda_n)$ we see with, $x^*, s_k, k = 1, 2, 3$ as in Theorem 5.1,

$$s_1^{(n)} \leq \beta^{1/3} n^{1/3}, \quad s_2^{(n)} = \frac{\beta^{2/3}}{n^{4/3}} \sum_{\mathcal{C} \subset \mathbf{BSR}^{I,*}(t_n^+)} |\mathcal{C}|^2, \quad s_3^{(n)} = \frac{\beta}{n^2} \sum_{\mathcal{C} \subset \mathbf{BSR}^{I,*}(t_n^+)} |\mathcal{C}|^3.$$

Next note that for any component $\mathcal{C} \subset \mathbf{G}^-(t_n) = \mathbf{BSR}^{I,*}(t_n)$ there is a unique component $\mathcal{C}^+ \subset \mathbf{G}^+(t_n) = \mathbf{BSR}^{I,*}(t_n^+)$, such that $\mathcal{C} \subset \mathcal{C}^+$. Denote by \mathcal{C}_i the i -th largest component in $\mathbf{BSR}^{I,*}(t_n)$, and let \mathcal{C}_i^+ be the corresponding component in $\mathbf{BSR}^{I,*}(t_n^+)$ such that $\mathcal{C}_i \subset \mathcal{C}_i^+$. Denote by N the number of immigrations that occur during $[t_n, t_n^+]$ in $\mathbf{BSR}^{I,*}$, and denote by $\{\tilde{\mathcal{C}}_i^+\}_{i=1}^N$ the components in $\mathbf{BSR}^{I,*}(t_n^+)$ resulting from these immigrations. Then

$$s_2^{(n)} = \frac{\beta^{2/3} \bar{s}_2^+}{n^{1/3}}, \quad s_3^{(n)} = \frac{\beta \bar{s}_3^+}{n}, \quad x^{*(n)} = \beta^{1/3} \frac{I^+}{n^{2/3}},$$

where

$$\begin{aligned} \bar{s}_2^+ &:= \frac{1}{n} \left(\sum_{i=1}^{\infty} |\mathcal{C}_i^+|^2 + \sum_{i=1}^N |\tilde{\mathcal{C}}_i^+|^2 \right), \\ \bar{s}_3^+ &:= \frac{1}{n} \left(\sum_{i=1}^{\infty} |\mathcal{C}_i^+|^3 + \sum_{i=1}^N |\tilde{\mathcal{C}}_i^+|^3 \right), \\ I^+ &:= \max \left\{ \max_i |\mathcal{C}_i^+|, \max_i |\tilde{\mathcal{C}}_i^+| \right\}. \end{aligned}$$

To complete the proof it suffices to show that the statement in Proposition 7.3 holds with $(\bar{s}_2(t_n), \bar{s}_3(t_n), I^{(n)}(t_n))$ replaced with $(\bar{s}_2^+, \bar{s}_3^+, I^+)$. This follows from Proposition 7.4 given below and hence completes the proof of the lemma. \square

Proposition 7.4. *With notation as in the proof of Lemma 7.2, as $n \rightarrow \infty$, we have*

$$I^+ = O(I), \quad \frac{\bar{s}_2^+}{\bar{s}_2(t_n)} \xrightarrow{\mathbb{P}} 1, \quad \frac{\bar{s}_3^+}{\bar{s}_3(t_n)} \xrightarrow{\mathbb{P}} 1, \quad \frac{n^{1/3}}{\bar{s}_2(t_n)} - \frac{n^{1/3}}{\bar{s}_2^+} \xrightarrow{\mathbb{P}} 0.$$

Proof. The proof is similar to that of Proposition 8.1 in [7] thus we only give a sketch.

Observe that the total rate of attachments is $\sum_{i=1}^K c_i^*(t) \leq 1$ and each attachment has size no bigger than K . Recall that \mathcal{C}_i denotes the i -th largest component in $\mathbf{BSR}^{I,*}(t_n)$. Denote by $V_i(t), t \in [t_n, t_n^+]$, the stochastic process defining the size of the component containing \mathcal{C}_i in $\mathbf{BSR}^{I,*}(t)$. Note that $V_i(t_n) = |\mathcal{C}_i|$ and $V_i(t_n^+) = |\mathcal{C}_i^+|$. Then $V_i(t)/K$ can be stochastically dominated by a Yule process starting with $\lceil |\mathcal{C}_i|/K \rceil$ particles and birth rate K . Using this and an argument similar to [7], it follows that,

$$|\mathcal{C}_i^+| - |\mathcal{C}_i| \leq_d K \cdot \text{Negative-Binomial}(\lceil |\mathcal{C}_i|/K \rceil, e^{-2Kn^{-\gamma}}).$$

Next, note that the immigrations are of size no bigger than $2K$, and thus for the same reason, we have the bound,

$$|\tilde{\mathcal{C}}_i^+| \leq_d 2K + K \cdot \text{Negative-Binomial}(2, e^{-2Kn^{-\gamma}}).$$

Since the total number of vertices is n , the number of immigrations N can be bounded by n/K .

With the above three bounds the proof of the proposition follows exactly as the proof of Proposition 8.1 in [7] with obvious changes needed due to the constant K that appears in the above bounds. Details are omitted. \square

The next proposition says that the inclusion in (7.2) can be strengthened to component-wise inclusion.

Proposition 7.5. *Fix $\lambda \in \mathbb{R}$ and $i_0 \geq 1$. Then, as $n \rightarrow \infty$,*

$$\mathbb{P} \left\{ \mathcal{C}_i(\mathbf{G}^{(n),-}(t^\lambda)) \subset \mathcal{C}_i(\mathbf{BSR}^*(t^\lambda)) \subset \mathcal{C}_i(\mathbf{G}^{(n),+}(t^\lambda)) \forall 1 \leq i \leq i_0 \right\} \rightarrow 1.$$

Proof: From Lemma 7.2 and Lemma 15 in [6] (see also Section 8.2 of [7] for a similar argument), we have, as $n \rightarrow \infty$,

$$(\bar{\mathbf{C}}^{(n),-}(\lambda), \bar{\mathbf{C}}^{(n),*}(\lambda), \bar{\mathbf{C}}^{(n),+}(\lambda)) \xrightarrow{d} (\mathbf{X}(\lambda), \mathbf{X}(\lambda), \mathbf{X}(\lambda)), \quad (7.3)$$

in $l_{\downarrow}^2 \times l_{\downarrow}^2 \times l_{\downarrow}^2$, where \mathbf{X} is as in Theorem 3.1. Define events E_n, F_n as

$$E_n = \left\{ \bar{\mathbf{C}}_i^{(n),-}(\lambda) > \bar{\mathbf{C}}_{i+1}^{(n),+}(\lambda) : 1 \leq i \leq i_0 \right\}, F_n = \left\{ \mathbf{G}^{(n),-}(\lambda) \subset \mathbf{BSR}^*(\lambda) \subset \mathbf{G}^{(n),+}(\lambda) \right\}.$$

Then on the set $E_n \cap F_n$

$$\mathcal{C}_i(\mathbf{G}^{(n),-}(\lambda)) \subset \mathcal{C}_i(\mathbf{BSR}^*(\lambda)) \subset \mathcal{C}_i(\mathbf{G}^{(n),+}(\lambda)), \forall 1 \leq i \leq i_0.$$

From Lemma 7.2 (i) $\mathbb{P}\{F_n^c\} \rightarrow 1$. Also

$$\begin{aligned} \limsup_n \mathbb{P}(E_n^c) &\leq \limsup_n \mathbb{P} \left\{ \bar{\mathbf{C}}_i^{(n),-}(\lambda) \leq \bar{\mathbf{C}}_{i+1}^{(n),+}(\lambda) \text{ for some } 1 \leq i \leq i_0 \right\} \\ &\leq \mathbb{P} \left\{ \mathbf{X}_i(\lambda) \leq \mathbf{X}_{i+1}(\lambda) \text{ for some } 1 \leq i \leq i_0 \right\} = 0. \end{aligned}$$

This shows that $\mathbb{P}(E_n \cap F_n) \rightarrow 1$ as $n \rightarrow \infty$. The result follows. \square

We will also need the following elementary lemma. Proof is omitted.

Lemma 7.6. *Let $\eta^{(n),-}, \eta^{(n),+}, \eta^*$ be real random variables such that $\eta^{(n),-} \leq \eta^{(n),+}$ with high probability. Further assume $\eta^{(n),-} \xrightarrow{d} \eta^*$ and $\eta^{(n),+} \xrightarrow{d} \eta^*$. Then $\eta^{(n),+} - \eta^{(n),-} \xrightarrow{\mathbb{P}} 0$. Furthermore, if $\eta^{(n)}$ are random variables such that $\eta^{(n),-} \leq \eta^{(n)} \leq \eta^{(n),+}$ with high probability, then $\eta^{(n)} \xrightarrow{d} \eta^*$ and $\eta^{(n)} - \eta^{(n),-} \xrightarrow{\mathbb{P}} 0$.*

We now complete the proof of Theorem 7.1.

Proof of Theorem 7.1: From Lemma 7.2 (ii) we have that

$$\left(\bar{\mathbf{C}}^{(n),-}(\lambda), \bar{\mathbf{Y}}^{(n),-}(\lambda), \sum_{i=1}^{\infty} \bar{\mathbf{C}}_i^{(n),-}(\lambda) \mathbf{Y}_i^{(n),-}(\lambda) \right) \xrightarrow{d} \left(\mathbf{X}(\lambda), \mathbf{Y}(\lambda), \sum_{i=1}^{\infty} \mathbf{X}_i(\lambda) \mathbf{Y}_i(\lambda) \right), \quad (7.4)$$

in $l_{\downarrow}^2 \times \mathbb{N}^{\infty} \times \mathbb{R}$, where on \mathbb{N}^{∞} we consider the product topology.

In order to prove the theorem it suffices, in view of Lemma 4.11, to show that

$$\left(\bar{\mathbf{C}}^{(n),*}(\lambda), \hat{\xi}^{(n)}(\lambda), \sum_{i=1}^{\infty} \bar{\mathbf{C}}_i^{(n),*}(\lambda) \hat{\xi}_i^{(n)}(\lambda) \right) \xrightarrow{d} \left(\mathbf{X}(\lambda), \mathbf{Y}(\lambda), \sum_{i=1}^{\infty} \mathbf{X}_i(\lambda) \mathbf{Y}_i(\lambda) \right), \quad (7.5)$$

in $l_{\downarrow}^2 \times \mathbb{N}^{\infty} \times \mathbb{R}$.

From Proposition 7.5, we have for any $i_0 \in \mathbb{N}$, with high probability

$$\bar{Y}_i^{(n),-}(\lambda) \leq \hat{\xi}_i^{(n)}(\lambda) \leq \bar{Y}_i^{(n),+} \text{ for } 1 \leq i \leq i_0.$$

Also, from Lemma 7.2 (i), whp,

$$\sum_{i=1}^{\infty} \bar{C}_i^{(n),-}(\lambda) \bar{Y}_i^{(n),-}(\lambda) \leq \sum_{i=1}^{\infty} \bar{C}_i^{(n)}(\lambda) \bar{Y}_i^{(n)}(\lambda) \leq \sum_{i=1}^{\infty} \bar{C}_i^{(n),+}(\lambda) \bar{Y}_i^{(n),+}(\lambda).$$

From Lemma 7.6 and Lemma 7.2 (ii), we then have

$$\left(\left| \hat{\xi}^{(n)}(\lambda) - \bar{Y}^{(n),-}(\lambda) \right|, \sum_{i=1}^{\infty} \bar{C}_i^{(n),*}(\lambda) \hat{\xi}_i^{(n)}(\lambda) - \sum_{i=1}^{\infty} \bar{C}_i^{(n),-}(\lambda) \bar{Y}_i^{(n),-}(\lambda) \right) \xrightarrow{\mathbb{P}} 0,$$

in $\mathbb{N}^{\infty} \times \mathbb{R}$, where for $y = (y_1, y_2, \dots) \in \mathbb{Z}^{\infty}$, $|y| = (|y_1|, |y_2|, \dots)$. The convergence in (7.5) now follows on combining (7.4) and (7.3). The result follows. \square

7.2. Proof of Theorem 3.3. As a first step towards the proof we show the following convergence result for one dimensional distributions.

Theorem 7.7. *For every $\lambda \in \mathbb{R}$, as $n \rightarrow \infty$, $(\bar{C}^{(n)}(\lambda), \bar{Y}^{(n)}(\lambda))$ converges in distribution, in \mathbb{U}_{\downarrow} , to $(\mathbf{X}(\lambda), \mathbf{Y}(\lambda))$.*

Proof. Fix $\lambda \in \mathbb{R}$. We first argue that

$$(\bar{C}^{(n),*}(\lambda), \bar{Y}^{(n),*}(\lambda)) \xrightarrow{d} (\mathbf{X}(\lambda), \mathbf{Y}(\lambda)), \text{ in } \mathbb{U}_{\downarrow}. \quad (7.6)$$

For this, it suffices to show that

$$\sum_{i=1}^{\infty} \tilde{\xi}_i^{(n)}(\lambda) \bar{C}_i^{(n),*}(\lambda) \xrightarrow{\mathbb{P}} 0. \quad (7.7)$$

Define

$$E_n = \left\{ I^{(n)}(s) \leq C_3 \frac{(\log n)^4}{(t_c - s)^2} \text{ for } s \leq t_c - n^{-\gamma} \right\}.$$

By Theorem 6.9, $\mathbb{P}\{E_n^c\} \rightarrow 0$ and $E_n \in \tilde{\mathcal{F}}(\lambda) := \sigma\{|\mathcal{C}_i(s)| : i \geq 1, s \leq t^\lambda\}$ for all $\lambda \geq -\lambda_n$. We begin by showing that there exists $d_1 \in (0, \infty)$ such that, for all $i \in \mathbb{N}$,

$$\mathbb{E} \left[\tilde{\xi}_i^{(n)}(\lambda) \mid \tilde{\mathcal{F}}_\lambda \right] \mathbf{1}_{E_n} \leq d_1 \bar{C}_i^{(n),*}(\lambda) n^{\gamma-1/3} (\log n)^4. \quad (7.8)$$

Note that at any time $s < t^\lambda$, for a component $\mathcal{C} \subset \mathbf{BSR}^{(n)}(s)$, there are at most $2|\mathcal{C}|^2 n^2$ quadruples of vertices which may provide a surplus edge within \mathcal{C} . Since edges are formed at rate $2/n^3$, we have that

$$\begin{aligned} \mathbb{E} \left[\tilde{\xi}_i^{(n)}(\lambda) \mid \tilde{\mathcal{F}}_\lambda \right] &\leq \int_0^{t_n} \left[\sum_{j: \mathcal{C}_j(\mathbf{BSR}^{(n)}(s)) \subset \mathcal{C}_i(\mathbf{BSR}^*(t^\lambda))} \frac{1}{2n^3} 2n^2 |\mathcal{C}_j(\mathbf{BSR}^{(n)}(s))|^2 \right] ds \\ &\leq \frac{1}{n} |\mathcal{C}_i(\mathbf{BSR}^*(t^\lambda))| \int_0^{t_n} I(s) ds. \end{aligned}$$

Thus, for some $d_0, d_1 \in (0, \infty)$

$$\mathbb{E} \left[\tilde{\xi}_i^{(n)}(\lambda) \mid \tilde{\mathcal{F}}(\lambda) \right] \mathbf{1}_{E_n} \leq d_0 \frac{\bar{C}_i^{(n),*}(\lambda)}{n^{1/3}} \int_0^{t_c - n^{-\gamma}} \frac{(\log n)^4}{(t_c - s)^2} ds \leq d_1 \bar{C}_i^{(n),*}(\lambda) n^{\gamma-1/3} (\log n)^4.$$

This proves (7.8). As an immediate consequence of this inequality we have that

$$\begin{aligned} \mathbb{E} \left[\sum_i \tilde{\xi}_i^{(n)}(\lambda) \bar{C}_i^{(n),*}(\lambda) \mid \tilde{\mathcal{F}}(\lambda) \right] \mathbf{1}_{E_n} &= \sum_i \bar{C}_i^{(n),*}(\lambda) \mathbf{1}_{E_n} \mathbb{E} \left[\tilde{\xi}_i^{(n)}(\lambda) \mid \tilde{\mathcal{F}}(\lambda) \right] \\ &\leq d_1 n^{\gamma-1/3} (\log n)^4 \sum_i \left(\bar{C}_i^{(n),*}(\lambda) \right)^2. \end{aligned}$$

Observing that $\gamma - 1/3 < 0$ and, from Theorem 7.1, that $\sum_i \left(\bar{C}_i^{(n),*}(\lambda) \right)^2$ converges in distribution, we have that

$$\mathbb{E} \left[\sum_i \tilde{\xi}_i^{(n)}(\lambda) \bar{C}_i^{(n),*}(\lambda) \mid \tilde{\mathcal{F}}(\lambda) \right] \mathbf{1}_{E_n} \xrightarrow{\mathbb{P}} 0.$$

Since $\mathbb{P}\{E_n\} \rightarrow 1$, letting $\eta^{(n)} = \sum_i \tilde{\xi}_i^{(n)}(\lambda) \bar{C}_i^{(n),*}(\lambda)$, we have that $\mathbb{E}(\eta^{(n)} \mid \tilde{\mathcal{F}}(\lambda)) \rightarrow 0$ in probability. Convergence in (7.7) now follows from Markov's inequality on noting that, as $n \rightarrow \infty$,

$$\mathbb{E}[\eta^{(n)} \wedge 1] = E \left[E[\eta^{(n)} \wedge 1 \mid \tilde{\mathcal{F}}(\lambda)] \right] \leq \mathbb{E} \left[\mathbb{E}[\eta^{(n)} \mid \tilde{\mathcal{F}}(\lambda)] \wedge 1 \right] \rightarrow 0.$$

This proves (7.6). Next note that

$$\sum_{i=1}^{\infty} |\bar{C}_i^{(n)}(\lambda) - \bar{C}_i^{(n),*}(\lambda)|^2 \leq \frac{n}{n^{4/3}} O(1) \rightarrow 0, \text{ as } n \rightarrow \infty. \quad (7.9)$$

Also,

$$\begin{aligned} \mathbb{E} \left[\bar{Y}_i^{(n)}(\lambda) \mid \tilde{\mathcal{F}}(\lambda) \right] \mathbf{1}_{\{|C_i(t^\lambda)| \leq K\}} &\leq \left[\int_0^{t^\lambda} \sum_{j: C_j(s) \subset C_i(t^\lambda)} \frac{1}{2n^3} 2n^2 |C_j(s)|^2 ds \right] \mathbf{1}_{\{|C_i(t^\lambda)| \leq K\}} \\ &\leq \frac{K^2}{n} \cdot t^\lambda = O(n^{-1}). \end{aligned}$$

Thus, as $n \rightarrow \infty$,

$$\begin{aligned} &\mathbb{E} \sum_{i=1}^{\infty} |\bar{C}_i^{(n)}(\lambda) \bar{Y}_i^{(n)}(\lambda) - \bar{C}_i^{(n),*}(\lambda) \bar{Y}_i^{(n),*}(\lambda)| \\ &= \sum_{i=1}^{\infty} \mathbb{E} \left[\bar{C}_i^{(n)}(\lambda) \bar{Y}_i^{(n)}(\lambda) \mathbf{1}_{\{|C_i(t^\lambda)| \leq K\}} \right] \leq O(n^{-1}) \mathbb{E} \left[\sum_{i=1}^{\infty} \bar{C}_i^{(n)}(\lambda) \right] = O(n^{-2/3}) \rightarrow 0. \end{aligned}$$

The result now follows on combining the above convergence with (7.9) and (7.6). \square

Remark 7.8. The proofs of Theorems 7.1 and 7.7 in fact establish the following stronger statement: For all $\lambda \in \mathbb{R}$,

$$\begin{aligned} &\left(\left| \bar{Y}^{(n),-}(\lambda) - \bar{Y}^{(n)}(\lambda) \right|, \sum_{i=1}^{\infty} |\bar{C}_i^{(n),-}(\lambda) - \bar{C}_i^{(n)}(\lambda)|^2, \right. \\ &\quad \left. \sum_{i=1}^{\infty} |\bar{C}_i^{(n),-}(\lambda) \bar{Y}_i^{(n),-}(\lambda) - \bar{C}_i^{(n)}(\lambda) \bar{Y}_i^{(n)}(\lambda)| \right) \rightarrow (0, 0, 0), \end{aligned}$$

in probability, in $\mathbb{N}^\infty \times \mathbb{R} \times \mathbb{R}$.

Proof of Theorem 3.3: For simplicity we present the proof for the case $m = 2$. The general case can be treated similarly. Fix $-\infty < \lambda_1 < \lambda_2 < \infty$. Denote, for $\lambda \in \mathbb{R}$, $\bar{\mathbf{Z}}^{(n),-}(\lambda) = (\bar{\mathbf{C}}^{(n),-}(\lambda), \bar{\mathbf{Y}}^{(n),-}(\lambda))$. In view of Remark 7.8 it suffices to show that, as $n \rightarrow \infty$,

$$(\bar{\mathbf{Z}}^{(n),-}(\lambda_1), \bar{\mathbf{Z}}^{(n),-}(\lambda_2)) \xrightarrow{d} (\mathbf{Z}(\lambda_1), \mathbf{Z}(\lambda_2)),$$

for which it is enough to show that for all $f_1, f_2 \in C_b(\mathbb{U}_\downarrow^0)$

$$\mathbb{E} \left[f_1(\bar{\mathbf{Z}}^{(n),-}(\lambda_1)) f_2(\bar{\mathbf{Z}}^{(n),-}(\lambda_2)) \right] \rightarrow \mathbb{E} [f_1(\mathbf{Z}(\lambda_1)) f_2(\mathbf{Z}(\lambda_2))]. \quad (7.10)$$

Note that the left side of (7.10) equals

$$\mathbb{E} \left[f_1(\bar{\mathbf{Z}}^{(n),-}(\lambda_1)) \mathcal{T}_{\lambda_2 - \lambda_1} f_2(\bar{\mathbf{Z}}^{(n),-}(\lambda_1)) \right],$$

which using Theorem 3.1 (2), Lemma 7.2 (ii) and the fact that $\mathbf{Z}(\lambda) \in \mathbb{U}_\downarrow^1$ a.s., converges to

$$\mathbb{E} [f_1(\mathbf{Z}(\lambda_1)) \mathcal{T}_{\lambda_2 - \lambda_1} f_2(\mathbf{Z}(\lambda_1))] = \mathbb{E} [f_1(\mathbf{Z}(\lambda_1)) f_2(\mathbf{Z}(\lambda_2))],$$

where the last equality follows from Theorem 3.1 (3). This proves (7.10) and the result follows. \square

ACKNOWLEDGEMENTS

We thank the referee for a very careful reading of the manuscript and for the many suggestions that significantly improved the presentation. AB and XW has been supported in part by the National Science Foundation (DMS-1004418, DMS-1016441, DMS-13051120), the Army Research Office (W911NF-0-1-0080, W911NF-10-1-0158) and the US-Israel Binational Science Foundation (2008466). SB and XW have been supported in part by the National Science Foundation (DMS-1105581, DMS-1310002).

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