Stability of Constrained Markov Modulated Diffusions

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Abstract: A family of constrained diffusions in random environment is considered. Constraint set is a polyhedral cone and coefficients of the diffusion are governed by, in addition to the system state, a finite state Markov process that is independent of the driving noise. Such models arise as limit objects in the heavy traffic analysis of stochastic processing networks (SPN) with Markov modulated arrival and processing rates. We give sufficient conditions (which in particular includes a requirement on the regularity of the underlying Skorohod map) for positive recurrence and geometric ergodicity. When the coefficients only depend on the modulating Markov process (i.e. they are independent of the system state), a complete characterization for stability and transience is provided. The case, where the pathwise Skorohod problem is not well-posed but the underlying reflection matrix is completely-S, is treated as well. As consequences of geometric ergodicity various results, such as exponential integrability of invariant measures and CLT for fluctuations of long time averages of process functionals about their stationary values, are obtained. Conditions for stability are formulated in terms of the averaged drift, where the average is taken with respect to the stationary distribution of the modulating Markov process. Finally, steady state distributions of the underlying SPN are considered and it is shown that, under suitable conditions, such distributions converge to the unique stationary distribution of the constrained random environment diffusion.

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1. Introduction

We study a family of constrained diffusions in random environment. Such models arise as diffusion approximations of stochastic processing networks (SPN) in heavy traffic. One such setting has been considered in [7] where it is shown that state processes for certain critically

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loaded generalized Jackson networks for which the arrival and service rates depend on the system state and values of an independent finite state Markov process, under a suitable scaling, converge to constrained random environment diffusions of the form studied in the current work. Our objective in this work is to study stability properties and invariant measures of such diffusions and argue that the latter serve as good approximations for the steady states of the corresponding SPN model.

Let $G \subset \mathbb{R}^K$ be a convex polyhedral cone with vertex at the origin given as the intersection of half spaces $G_i, i = 1, \cdots, N$. Denote by $n_i$ and $d_i$ the inward normal and constraint direction associated with $G_i$. We assume that the Skorohod map $\Gamma$ defined by the data $\{(d_i, n_i) : i = 1, 2, \ldots, N\}$ is well posed and Lipschitz continuous (Assumption 2.1). The map $\Gamma$ takes a RCLL trajectory $\psi : [0, \infty) \rightarrow \mathbb{R}^K$ to another RCLL trajectory $\phi$, which stays within $G$ at all times, in a manner that is uniquely determined by the set of reflection directions $\{d_i : i = 1, 2, \ldots, N\}$. A precise description and definition of the Skorohod map is given in Section 2.1. The Markov modulated diffusion process we study is constrained to take values in $G$ and is defined through the equation

$$X(t) = \Gamma \left( z + \int_0^t b(X(s), Y(s))ds + \int_0^t \sigma(X(s), Y(s))dW(s) \right)(t), \hspace{1em} t \geq 0. \hspace{1em} (1.1)$$

The process $(X, Y)$ is defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$ such that $Y$ is a $\{\mathcal{F}_t\}$ Markov process with state space $L \doteq \{1, 2, \ldots, L\}$, infinitesimal generator $\mathbb{Q}$, and a unique stationary distribution $\{q_j^* : j \in L\}$ (see Assumption 2.4), and $W$ is a $\{\mathcal{F}_t\}$ standard Brownian motion. We also assume Lipschitz continuity on $\sigma$ and $b$ (Assumption 2.2) and boundedness and uniform nondegeneracy of $\sigma$ (Assumption 2.3). We now introduce the main stability assumption on the drift coefficient $b$. Let

$$\mathcal{C} \doteq \left\{-\sum_{i=1}^N \zeta_i d_i : \zeta_i \geq 0, i \in \{1, \cdots, N\}\right\} \hspace{1em} (1.2)$$

and, for $\delta > 0$,

$$\mathcal{C}(\delta) \doteq \{v \in \mathcal{C} : \text{dist}(v, \partial \mathcal{C}) \geq \delta\}, \hspace{1em} \delta \in (0, \infty). \hspace{1em} (1.3)$$

The cone in (1.2) plays a key role in the stability analysis of constrained diffusions (see [16], [3], [1], [6]). For example, it follows from results in [1] that if the drift and diffusion coefficients do not depend on the process $Y$ (i.e., for all $(x, y) \in G \times L$, $b(x, y) \equiv b(x)$ and $\sigma(x, y) \equiv \sigma(x)$), and for some $\delta_0 > 0$, $b(x) \in \mathcal{C}(\delta_0)$ for all $x \in G$, then the Markov process $X$ is positive recurrent and consequently has a unique invariant probability measure. In the Markov modulated setting considered here the cone $\mathcal{C}$ once again plays a key role, however instead of requiring that $b(x, y) \in \mathcal{C}(\delta_0)$ for all $x, y$, we impose a substantially weaker condition that the drift averaged over $y$ according to the stationary distribution $q^*$, is in $\mathcal{C}(\delta_0)$, for all $x$. For technical reasons (see Remark 3.1) we are only able to treat the case where the drift can be decomposed as $b(x, y) = b_1(x) + b_2(y)$. For a general drift field $b(x, y)$ our condition for stability is somewhat stronger, as described below. Write $b$ as

$$b(x, y) = b_1(x, y) + b_2(y), \hspace{1em} (x, y) \in G \times L, \hspace{1em} (1.4)$$
where \( b_1 : G \times \mathbb{L} \to \mathbb{R}^K \) and \( b_2 : \mathbb{L} \to \mathbb{R}^K \) are measurable maps. Define \( b^*_2 = \sum_{j \in \mathbb{L}} q^*_j b_2(j) \) and \( b^*(x,y) = b_1(x,y) + b^*_2 \). Then our stability assumption (Assumption 2.5) requires that \( b^*(x,y) \in C(\delta_0) \) for some \( \delta_0 > 0 \). In Theorem 2.2 we show that, under Assumptions 2.1 - 2.5, \((X,Y)\) is positive recurrent and has a unique invariant probability measure. For the case when \( b_1 = 0 \), we obtain a sharper result (Theorem 2.3) which says that if \( b^*(\equiv b^*_2) \) is in the interior of \( \mathcal{C} \), then \((X,Y)\) is positive recurrent, and if \( b^* \) is not in \( \mathcal{C} \), \((X,Y)\) is transient. Under the same stability condition, we identify an appropriate exponentially growing Lyapunov function \( V \) and establish the \( V \)-uniform ergodicity of \((X,Y)\). As consequences of this geometric ergodicity property we obtain exponential integrability of the invariant probability measure, uniform time estimates for polynomial moments of all orders and certain functional central limit results (Theorem 2.4).

We note that our stability condition (Assumption 2.5) on the drift vector field allows for the drift to be “transient” in some states of the Markov process \( Y \). For example, consider vectors \( v_1, v_2 \in \mathbb{R}^K \) such that \( v_1 \in \mathcal{C}^c \) and \( v_2 \in \mathcal{C}^c \). Then it is well known that if \( b(x,y) \equiv v_1 \), \( X \) in (1.1) will be positive recurrent and if \( b(x,y) \equiv v_2 \), \( X \) will be transient. Our results show that in a Markov modulated case where, for example, \( L = \{1,2\} \) and \( b(x,y) \equiv v_y \), the pair \((X,Y)\) will be positive recurrent (in fact geometrically ergodic) if \( b^* = q^*_1 v_1 + q^*_2 v_2 \in \mathcal{C}^o \) and transient if \( b^* \in \mathcal{C}^c \).

Regularity of the Skorohod map (Assumption 2.1) is a key ingredient in the proof of Theorems 2.2-2.4. Motivated by the study of diffusion approximations of multi-class queueing networks, in [18, 21] the authors have identified a necessary and sufficient condition on the reflection matrix for weak existence and uniqueness of a reflected Brownian motion in the positive orthant \( \mathbb{R}^K_+ \). This key condition, which requires the reflection matrix to be completely-S (see Section 2.2 for a precise definition) is substantially weaker than known sufficient conditions for Lipschitz continuity of the Skorohod map (Assumption 2.1). In [13], stability theory for such semimartingale reflected Brownian motions (SRBM) has been developed. The key stability condition, referred to in our work as the \( \text{DW-stability condition} \) (see Definition 2.5) is formulated in terms of certain \textit{fluid limit trajectories} associated with the SRBM model. Under this condition, the paper [13] shows that the SRBM is positive recurrent and admits a unique invariant probability measure. These results were strengthened in [5] by establishing geometric ergodicity. In the current work we consider a Markov modulated SRBM \( X \) in the nonnegative orthant \( \mathbb{R}^K_+ \). The modulating process \( Y \) is as before a Markov process with values in \( \mathbb{L} \) and satisfies Assumption 2.4. In particular, this corresponds to a setting where \( b(x,y) = b_2(y) \), \( \sigma(x,y) \equiv \sigma \), and \( G \equiv \mathbb{R}^K \). As in [13] we make the basic assumption that the matrix \( (d_1|d_2|\ldots|d_K) \) is completely-S. Using a standard argument based on Girsanov’s theorem and classical results of [21], one can establish the existence and uniquely characterize the probability law of such a process (see Theorem 2.5). Under the assumption that \( b^*(\equiv b^*_2) \) satisfies the DW-stability condition, we prove that \((X,Y)\) is positive recurrent and has a unique stationary distribution (Theorem 2.6). Furthermore, we show that \((X,Y)\) is \( V \)-uniformly ergodic where the Lyapunov function \( V \) grows exponentially, and establish various stability properties of Markov modulated SRBM (see Theorem 2.7) analogous to those in Theorem 2.4.

As noted earlier, constrained random environment diffusions studied in our work have been
shown in [7] to arise as diffusion limit models for certain generalized Jackson networks. Since such queueing networks are too complex to be analyzed directly, it is valuable to know that the steady state behavior of the limit diffusion model is a good approximation for that of the underlying queueing system. In the final part of this paper, we study the validity of such approximation. We consider a sequence of Markov modulated open queueing networks in heavy traffic. The external arrival processes and service processes are assumed to depend on the state of the system and an auxiliary finite state Markov process. The routing mechanism is governed by a $K \times K$ substochastic matrix $P$. In the $n^{th}$ network a Markov process $Y^n$ modulates the arrival and service rates. We assume that $Y^n$ has a finite state space $\mathbb{L}$ and infinitesimal generator $Q^n$ which converges to some matrix $Q$. Denote by $Q^n$ the queue length process and by $\hat{Q}^n$ a suitably scaled form of $Q^n$. It is shown in [7] that under suitable heavy traffic condition, $(\hat{Q}^n, Y^n)$ converges to $(X, Y)$ weakly, where $Y$ is a Markov process with infinitesimal generator $Q$ and $X$ is a constrained Markov modulated diffusion process defined as in (1.1). In Theorem 2.9 of the current work we show that, under appropriate heavy traffic and stability conditions, $(\hat{Q}^n, Y^n)$ admits a stationary distribution which converges to the unique stationary distribution of $(X, Y)$, as the scaling parameter $n \to \infty$.

The paper is organized as follows. We collect all the main results in Section 2. More precisely, the recurrence and transience properties for constrained Markov modulated diffusions are presented in Section 2.1. Section 2.2 considers stability properties for Markov modulated SRBM. Finally, in Section 2.3, we state our main result on the convergence of invariant measures for Markov modulated open queueing networks. Proofs of all results are given in Section 3. In Appendix, for completeness, we collect results, proofs of which are similar to arguments in existing literature.

The following notation will be used. For a metric space $U$, let $\mathcal{B}(U)$ be the Borel $\sigma$-field on $U$ and $\mathcal{P}(U)$ the collection of all probability measures on $U$. Denote the set of natural numbers by $\mathbb{N}$ and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Denote by $\mathbb{R}$ the set of real numbers and $\mathbb{R}_+$ the set of nonnegative real numbers. For $K \in \mathbb{N}$, let $\mathbb{R}^K = \{(x_1, x_2, \ldots, x_K)' : x_i \in \mathbb{R}, i = 1, 2, \ldots, K\}$ and $\mathbb{R}_+^K = \{(x_1, x_2, \ldots, x_K)' : x_i \in \mathbb{R}_+, i = 1, 2, \ldots, K\}$. Let $\{e_i\}_{i=1}^K$ be the standard basis in $\mathbb{R}^K$. For $x, y \in \mathbb{R}^K$, the usual inner product is denoted as $(x, y)$. For $d_i \in \mathbb{R}_+^K, i = 1, 2, \ldots, K$, denote $(d_1|d_2|\ldots|d_K)$ the matrix with $d_i$ as the $i^{th}$ column. For $B \in \mathcal{B}(\mathbb{R}^K)$, denote by $\partial B, B^\circ$ and $\overline{B}$, the boundary, the interior, and the closure of $B$, respectively. Space of real bounded measurable functions on some topological space $T$ will be denoted by $BM(T)$. For $f \in BM(T)$, we denote $\sup_{x \in T} |f(x)|$ by $|f|_\infty$. For $g : [0, \infty) \to \mathbb{R}^K$ and $t \geq 0$, we write $\sup_{0 \leq s \leq t} |g(s)|$ as $|g|^*_t$. Convergence in distribution of random variables (with values in some Polish space) $X_n$ to $X$ will be denoted as $X_n \Rightarrow X$. With an abuse of notation, weak convergence of probability measures (on a Polish space) $\mu_n$ to $\mu$ will also be denoted as $\mu_n \Rightarrow \mu$. For a Polish space $\mathcal{V}$, let $D([0, \infty), \mathcal{V})$ denote the class of right continuous functions with left limits defined from $[0, \infty)$ to $\mathcal{V}$ with the usual Skorohod topology and $C([0, \infty), \mathcal{V})$ the class of continuous functions from $[0, \infty)$ to $\mathcal{V}$ with the local uniform topology. A stochastic process $X = \{X(t) : t \geq 0\}$ is said to be RCLL if every sample path of $X$ is right continuous on $[0, \infty)$ with finite left limits on $(0, \infty)$. A countable set will be regarded as a metric space endowed with the discrete metric. We denote by $\iota : [0, \infty) \to [0, \infty)$ the identity map. Finally, we will denote generic positive constants by $c_1, c_2, \ldots$. Their values may change from one proof to another.
2. Main results

In this section we collect the main results of this work.

2.1. Recurrence and transience properties under a regular Skorohod map

Recall from Section 1 that \( G \) denotes a convex polyhedral cone with vertex at the origin given as the intersection of half spaces \( G_i, i = 1, \ldots, N \) and \( n_i \) is the inward normal vector associated with \( G_i \), i.e., \( G_i = \{ x \in \mathbb{R}^K : \langle x, n_i \rangle \geq 0 \} \). Denote the set \( \{ x \in \partial G : \langle x, n_i \rangle = 0 \} \) by \( F_i \).

With each face \( F_i \) we associate a unit vector \( d_i \) such that \( \langle d_i, n_i \rangle > 0 \). This vector defines the direction of constraint associated with the face \( F_i \). At points on \( \partial G \) where more than one faces meet, there are more than one allowed directions of constraint. For \( x \in \partial G \), define the set of directions of constraint

\[
d(x) = \left\{ d \in \mathbb{R}^K : d = \sum_{i \in I(x)} \zeta_i d_i, \zeta_i \geq 0, |d| = 1 \right\},
\]

where \( I(x) = \{ i \in \{1, 2, \ldots, N \} : \langle x, n_i \rangle = 0 \} \). Note that if \( I(x) = \{ j \} \) for some \( j \in \{1, 2, \ldots, N \} \), then \( d(x) = \{ d_j \} \).

We now introduce the Skorohod problem (SP) and the Skorohod map (SM) associated with \( G \) and \( d \). Define

\[
D_G([0, \infty) : \mathbb{R}^K) = \{ \xi \in D([0, \infty) : \mathbb{R}^K) : \xi(0) \in G \}.
\]

For \( \xi \in D([0, \infty) : \mathbb{R}^K) \) and \( T \geq 0 \), let \( |\xi|(T) \) denote the total variation of \( \xi \) on \([0, T]\) with respect to the Euclidean norm on \( \mathbb{R}^K \).

**Definition 2.1.** Let \( \psi \in D_G([0, \infty) : \mathbb{R}^K) \) be given. Then the pair \( (\phi, \eta) \in D([0, \infty) : G) \times D([0, \infty) : \mathbb{R}^K) \) solves the SP for \( \psi \) with respect to \( G \) and \( d \) if and only if \( \phi(0) = \psi(0) \) and for all \( t \in [0, \infty) \) the following hold:

(i) \( \phi(t) = \psi(t) + \eta(t) \), and \( \phi(t) \in G \).

(ii) \( |\eta(t)| < \infty \), and \( |\eta(t)| = \int_{[0,t]} 1_{\{\phi(s) \in \partial G\}} d|\eta|(s) \).

(iii) There exists Borel measurable map \( \gamma : [0, \infty) \rightarrow \mathbb{R}^K \) such that \( \gamma(t) \in d(\phi(t)) \) a.e. \( d|\eta| \) and \( \eta(t) = \int_{[0,t]} \gamma(s)d|\eta|(s) \).

Let \( D \subset D_G([0, \infty), \mathbb{R}^K) \) be the domain on which there is a unique solution to the SP. On \( D \) we define the SM \( \Gamma \) as \( \Gamma(\psi) = \phi \), if \((\phi, \phi - \psi)\) is the unique solution of the SP posed by \( \psi \). We will make the following assumption on the regularity of the SM defined by the data \( \{(d_i, n_i) : i \in \{1, 2, \ldots, N\}\} \).
Assumption 2.1. The SM is well defined on all of $D_G([0, \infty), \mathbb{R}^K)$, that is $D = D_G([0, \infty), \mathbb{R}^K)$, and the SM is Lipschitz continuous in the following sense. There exists $\kappa_1 \in (1, \infty)$ such that for all $\psi_1, \psi_2 \in D_G([0, \infty), \mathbb{R}^K)$:
\[
\sup_{t \geq 0} |\Gamma(\psi_1)(t) - \Gamma(\psi_2)(t)| \leq \kappa_1 \sup_{t \geq 0} |\psi_1(t) - \psi_2(t)|.
\]

We refer the reader to [15], [11], and [12] for sufficient conditions under which the above assumption holds.

We now introduce the Markov process $(X, Y)$ which will be studied here. The component $Y$ is a Markov process with a finite state space $\mathbb{L} = \{1, 2, \ldots, L\}$ and infinitesimal generator $Q$, while $X$ is a constrained diffusion with drift and diffusion coefficients that, in addition to depending on the current state, are modulated through the values of $Y$. More precisely, the process $X$ satisfies an integral equation of the form
\[
X(t) = \Gamma \left( x + \int_0^t b(X(s), Y(s)) \, ds + \int_0^t \sigma(X(s), Y(s)) \, dW(s) \right)(t), \quad t \geq 0, \tag{2.2}
\]
where $W$ is a standard Wiener process which is independent of $Y$, and $\sigma : \mathbb{L} \times \mathbb{L} \to \mathbb{R}^{K \times K}$, $b : \mathbb{L} \times \mathbb{L} \to \mathbb{R}^K$ are measurable maps. We will make the usual Lipschitz assumption on coefficients $b$ and $\sigma$ as follows.

Assumption 2.2. There exists $\kappa_2 \in (0, \infty)$ such that, for all $x_1, x_2 \in G$ and $y \in \mathbb{L}$,
\[
|\sigma(x_1, y) - \sigma(x_2, y)| + |b(x_1, y) - b(x_2, y)| \leq \kappa_2 |x_1 - x_2|.
\]

Let $S \equiv G \times \mathbb{L}$ and $Z \equiv (X, Y)$. Using the above Lipschitz property along with the regularity assumption on the SM $\Gamma$ (Assumption 2.1), it is easily seen that equation (2.2) is well posed. In particular, we have the following.

Theorem 2.1. Under Assumptions 2.1, 2.2, there is a filtered measurable space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0})$ on which are given a collection of probability measures $\{\mathbb{P}_z\}_{z \in S}$ and $\{\mathcal{F}_t\}$ adapted processes $(X, W, k)$ and $Y$ with sample paths in $C([0, \infty) : G \times \mathbb{R}^K \times \mathbb{R}^K)$ and $D([0, \infty) : \mathbb{L})$ respectively, such that $(Z, \{\mathbb{P}_z\}_{z \in S})$ is a Feller-Markov family and, for every $z \equiv (x, y) \in S$, $\mathbb{P}_z$-a.s., the following hold.

(i) $W$ is a $K$-dimensional standard $\mathcal{F}_t$ Brownian motion.

(ii) For all $t \in (0, \infty)$,
\[
X(t) = x + \int_0^t b(Z(s)) \, ds + \int_0^t \sigma(Z(s)) \, dW(s) + k(t), \tag{2.3}
\]
and $X(t) \in G$.

(iii) For all $t \in (0, \infty)$, $|k|(t) < \infty$ and $|k|(t) = \int_0^t 1_{\{X(s) \in \partial G\}} \, dk|(s)$.

(iv) There is a $\mathbb{R}^K$-valued $\{\mathcal{F}_t\}$ progressively measurable process $\gamma$ such that $\gamma(t) \in d(X(t))$ a.e. $d|k|$ and for all $t \in (0, \infty)$, $k(t) = \int_0^t \gamma(s) \, d|k|(s)$. 
(v) \( Y \) is a \( \mathbb{L} \)-valued \( \{ \mathcal{F}_t \} \)-Markov process with \( Y(0) = y \) and infinitesimal generator \( \mathbb{Q} \).

We will denote the Markov family \( (Z, \{ \mathbb{P}_z \}_{z \in \mathbb{S}}) \) merely as \( Z \) and denote the transition kernel of \( Z \) by \( P^t_z \), namely for \( z \in \mathbb{S} \) and \( B \in \mathcal{B}(\mathbb{S}) \), \( P^t_z(z, B) = \mathbb{P}_z(Z(t) \in B) \).

We recall the basic definitions of positive recurrence and transience of Markov process \( Z \).

**Definition 2.2.** The Markov process \( \{Z(t) : t \geq 0\} \) is said to be positive recurrent if for each \( A \in \mathcal{B}(\mathbb{G}) \) with positive Lebesgue measure, \( j \in \mathbb{L} \), and \( z \in \mathbb{S} \), we have \( E_z(\tau_{A \times \{j\}}) < \infty \), where \( \tau_{A \times \{j\}} = \inf\{t \geq 0 : Z(t) \in A \times \{j\}\} \) and \( E_z \) denotes the expectation under \( \mathbb{P}_z \).

**Definition 2.3.** The Markov process \( \{Z(t) : t \geq 0\} \) is said to be transient if there exist \( A \in \mathcal{B}(\mathbb{G}) \) with positive Lebesgue measure, \( j \in \mathbb{L} \), and \( z \in \mathbb{S} \) such that \( \mathbb{P}_z(\tau_{A \times \{j\}} < \infty) < 1 \).

We now introduce additional assumptions that will be needed for the main stability results. The second part of the following assumption will ensure irreducibility of the Markov process \( Z \) while the first will be needed in some moment estimates.

**Assumption 2.3.**

(i) For some \( \kappa_3 \in (0, \infty) \), \( |\sigma(z)| \leq \kappa_3 \) for all \( z \in \mathbb{S} \).

(ii) There exists \( \kappa_4 \in (0, \infty) \) such that for all \( z \in \mathbb{S} \) and \( \zeta \in \mathbb{R}^K \), \( \zeta'\sigma(z)\sigma'(z)\zeta \geq \kappa_4 \zeta'\zeta \).

We will make the following irreducibility assumption on the finite state Markov process associated with the generator \( \mathbb{Q} \). Let \( T_t = \exp(t\mathbb{Q}), t \geq 0 \).

**Assumption 2.4.** For every \( t > 0 \) and \( i, j \in \mathbb{L}, T_t(i, j) > 0 \).

This assumption ensures that the Markov process with the infinitesimal generator \( \mathbb{Q} \) has a unique stationary distribution \( q^* \equiv \{q^*_{ij}\}_{i,j \in \mathbb{L}} \). We now give the main stability assumption on the drift coefficient \( b \). Recall the cone \( \mathcal{C} \) and the set \( \mathcal{C}(\delta) \) introduced in (1.2) and (1.3). Also recall the maps \( b_1, b_2 \) from (1.4). Then our assumption on the drift \( b \) is as follows.

**Assumption 2.5.** There exist \( \delta_0 \in (0, \infty) \) and bounded set \( A \subset \mathbb{G} \) such that for all \( x \in \mathbb{G} \setminus A \) and \( y \in \mathbb{L} \), \( b^*(x, y) \in \mathcal{C}(\delta_0) \) where

\[
 b^*(x, y) = b_1(x, y) + b_2^* \quad \text{and} \quad b_2^* = \sum_{j \in \mathbb{L}} q_j^* b_2(j), \ (x, y) \in \mathbb{L}.
\]

The following theorem is the first main result of this work.

**Theorem 2.2.** Suppose that Assumptions 2.1-2.5 hold. Then the Markov process \( \{Z, \{\mathbb{P}_z\}_{z \in \mathbb{S}}\} \) is positive recurrent and has a unique invariant measure \( \pi \).

**Remark 2.1.** In [24], the authors consider a 1-dimensional Markov-modulated reflected Ornstein–Uhlenbeck process \( \{X(t) : t \geq 0\} \) defined as follows.

\[
 X(t) = - \int_0^t [\lambda_1(Y(s))X(s) + \lambda_2(Y(s))] ds + \int_0^t \sigma(Y(s)) dW(s) + k(t), \ t \geq 0,
\]
where \( \{Y(t) : t \geq 0\} \) is as in Theorem 2.1(v), \( \{B(t) : t \geq 0\} \) is a standard 1-dimensional Brownian motion, and \( \lambda_1, \lambda_2, \sigma \) are all strictly positive functions. The paper shows that \((X,Y)\) has a unique stationary distribution. Clearly \( b(x,y) = -[\lambda_1(y)x + \lambda_2(y)] \) satisfies Assumption 2.5 and thus Theorem 2.2 in particular covers the setting considered in [24]. In fact, Theorem 2.2, in addition to covering the much more general multidimensional setting, shows that the positive assumption on \( \lambda_1, \lambda_2 \) can be relaxed to the condition that \( \lambda_1, \lambda_2 \) are nonnegative and \( \lambda_2(j) > 0 \) for some \( j \in \mathbb{L} \).

For the case when \( b_1 = 0 \), we obtain a sharper result as follows.

**Theorem 2.3.** Suppose that \( b_1(z) = 0 \) for all \( z \in \mathbb{S} \). Also suppose that Assumptions 2.1-2.4 hold. Then the following hold:

1. If \( b_2^* \in \mathcal{C}^0 \), then \((Z, \{P_z\}_{z \in \mathbb{S}})\) is positive recurrent.
2. If \( b_2^* \notin \mathcal{C} \), then \((Z, \{P_z\}_{z \in \mathbb{S}})\) is transient.

In section 2.4, we will establish geometric ergodicity of the Markov family \((Z, \{P_z\}_{z \in \mathbb{S}})\). More precisely, the following result will be proved. Let \( f : \mathbb{S} \rightarrow \mathbb{R} \) be a measurable function such that, for some measurable \( g : \mathbb{S} \rightarrow \mathbb{R} \) and for all \( z \in \mathbb{S}, t \geq 0, \)

\[
\mathbb{E}_z \left[ |f(Z(t))| + \int_0^t |g(Z(s))|ds \right] < \infty, \quad \mathbb{E}_z[f(Z(t))] = f(z) + \mathbb{E}_z \left[ \int_0^t g(Z(s))ds \right].
\]

Denote by \( \mathcal{D}(A) \) the collection of all such measurable functions \( f \). For a pair \((f,g)\) as above, we write \((f,g) \in \mathcal{A}, \) or with abuse of terminology, \( g = Af. \) The (multi-valued) operator \( \mathcal{A} \) is referred to as the extended generator of \( Z \) and \( \mathcal{D}(\mathcal{A}) \) its domain.

**Theorem 2.4.** Suppose that Assumptions 2.1-2.5 hold. Then the following properties hold.

1. There exists \( \beta_1 \in (0, \infty) \) such that for all measurable \( f : \mathbb{S} \rightarrow \mathbb{R} \) which satisfy \( |f(z)| \leq e^{\beta_1|x|} \) for all \( z = (x,y) \in \mathbb{S}, \)

\[
\int_{\mathbb{S}} |f(z)|\pi(dz) < \infty.
\]

In particular, for all \( c \in \mathbb{R}^K \) with \( |c| \leq \beta_1, \)

\[
\int_{\mathbb{S}} e^{(c,x)}\pi(dx dy) < \infty.
\]

2. There are \( \beta_2, \beta_3, b_0 \in (0, \infty) \) such that for \( f \) as in (i), the following hold.
   
   a) For all \( z = (x,y) \in \mathbb{S} \) and \( t \in (0, \infty), \)

\[
|\mathbb{E}_z(f(Z(t)) - \pi(f))| \leq e^{\beta_2(|x|+1)}e^{-b_0t}.
\]

b) Defining for \( t \geq 0, S_t = \int_0^t f(Z(u))du, \) we have that \( f_t^e(z) \doteq \mathbb{E}_z(S_t - t\pi(f)) \) converges to a finite limit \( \hat{f}(z) \) for all \( z \in \mathbb{S}. \)

c) The convergence in (b) is exponentially fast, i.e.,

\[
|f_t^e(z) - \hat{f}(z)| \leq e^{\beta_3(|x|+1)}e^{-b_0t}
\]

for all \( z = (x,y) \in \mathbb{S} \) and \( t \in (0, \infty). \)
(d) The function \( \hat{f} \in D(A) \) and solves the Poisson equation: \( A\hat{f}(z) = \pi(f) - f(z), z \in \mathbb{S} \).

(iii) Let \( f : \mathbb{S} \to \mathbb{R} \) be a measurable function such that, with \( \beta_1 \) as in (i), \( f^2(z) \leq e^{\beta_1|x|} \), for all \( z = (x, y) \in \mathbb{S} \). Define for \( t \in [0,1] \),

\[
\xi^n(t) = \frac{1}{\sqrt{n}} \int_0^t [f(Z(s)) - \pi(f)] \, ds.
\]

Let \( \hat{f} \) be as in (ii)(b). Define

\[
\gamma_f^2 = 2 \int_\mathbb{S} \hat{f}(z)(f(z) - \pi(f))\pi(dz).
\]

Then \( |\gamma_f| < \infty \) and \( \xi^n \) converges weakly to \( \gamma_f B \) in \( C([0, 1], \mathbb{R}) \), where \( B \) is a 1-dimensional standard Brownian motion.

2.2. Markov modulated SRBM

In Section 4 we will consider a model with somewhat more restrictive conditions on the domain and the coefficients \( b \) and \( \sigma \) but significantly weaker assumptions on the constraint vector field \( d \). Suppose that \( G = \mathbb{R}_+^K \) and \( N = K \). For \( i \in \mathbb{K} \), let \( n_i = e_i \). Then the \( i \)th face \( F_i = \{x \in G : x_i = 0\} \). Define a \( K \times K \) matrix \( R = (d_1| \cdots |d_K) \) and a \( K \)-dimensional vector \( b_0 \in \mathbb{R}^K \). Let \( \sigma \) be a \( K \times K \) positive definite matrix. We recall from [21] the definition of a SRBM associated with \((G, b_0, \sigma, R)\).

**Definition 2.4.** For \( x \in G \), an SRBM associated with \((G, b_0, \sigma, R)\) that starts from \( x \) is a continuous, \( \{\tilde{F}_i\} \)-adapted \( K \)-dimensional process \( \tilde{X} \), defined on some filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) such that, \( P \)-a.s., the following hold.

(i) \( \tilde{X}(t) = x + b_0 t + \sigma \tilde{W}(t) + R \tilde{U}(t) \) and \( \tilde{X}(t) \in G \) for all \( t \geq 0 \).

(ii) \( \tilde{W} \) is a \( K \)-dimensional standard \( \{\tilde{F}_i\} \) Brownian motion.

(iii) \( \tilde{U} \) is an \( \{\tilde{F}_i\} \)-adapted \( K \)-dimensional process such that, for \( i = 1, \ldots, K \), \( \tilde{U}_i(0) = 0 \), \( \tilde{U}_i \) is continuous and nondecreasing, and \( \tilde{U}_i \) can increase only when \( \tilde{X} \) is on \( F_i \), i.e.,

\[
\int_0^\infty 1_{\{\tilde{X}_i(s) > 0\}} d\tilde{U}_i(s) = 0.
\]

An SRBM arises as the diffusion approximation limit for many multiclass queueing networks in heavy traffic (see [22]). The paper [21] shows that if \( R \) is completely-\( \mathcal{S} \), namely for every \( k \times k \) principle submatrix \( \tilde{R} \) of \( R \), there is a \( k \)-dimensional vector \( v_R \) such that \( v_R \geq 0 \) and \( \tilde{R} v_R > 0 \), then (weak) existence and uniqueness of SRBM hold. This condition, which is significantly weaker than Assumption 2.1 made in Section 2.2, is in fact known to be a necessary condition for existence of an SRBM ([18, Theorem 2]). We record this condition below for future reference.

**Assumption 2.6.** The matrix \( R \) is completely-\( \mathcal{S} \).

The following result follows from [21] along with a straightforward argument based on Girsanov’s theorem. Fix a measurable map \( b_2 : \mathcal{L} \to \mathbb{R}^K \).
Theorem 2.5. Suppose that Assumption 2.6 holds. Then there is a filtered measurable space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0})$ on which are given a collection of probability measures $\{P_z\}_{z \in S}$ and $\{\mathcal{F}_t\}$-adapted processes $(X, W, U)$ and $Y$ with sample paths in $C([0, \infty) : G \times \mathbb{R}^K \times G)$ and $D([0, \infty) : \mathbb{L})$, respectively, such that for every $z \equiv (x, y) \in S$, $P_z$-a.s., the following hold.

(i) $W$ is a $K$-dimensional standard $\{\mathcal{F}_t\}$ Brownian motion.

(ii) For all $t \geq 0$, 
$$X(t) = x + \int_0^t b_2(Y(s))ds + \sigma W(t) + RU(t),$$
and $X(t) \in G$.

(iii) For each $i = 1, \ldots, K$, $U_i(0) = 0$, $U_i$ is continuous and nondecreasing, and 
$$\int_0^\infty 1_{\{X_i(s) > 0\}}dU_i(s) = 0.$$

(iv) $Y$ is a $\mathbb{L}$-valued $\{\mathcal{F}_t\}$ Markov process with $Y(0) = y$ and infinitesimal generator $Q$.

Let $Z = (X, Y)$. Then $(Z, \{P_z\}_{z \in S})$ is a Feller-Markov family.

We now recall the key stability condition, introduced in [13], for positive recurrence of an SRBM in terms of the associated “fluid limit” trajectories.

Definition 2.5. We say a vector $b_0 \in \mathbb{R}^K$ satisfies the DW-stability condition if for all $\phi \in C([0, \infty) : G)$ satisfying the property (F) below, we have $\phi(t) \to 0$ as $t \to \infty$.

$$(F) \begin{cases} 
\text{For some } z \in G \text{ and } \eta \in C([0, \infty) : G), \phi(t) = z + b_0t + R\eta(t) \text{ for all } t \geq 0, \text{ where} \\
\text{for } i = 1, \ldots, K, \eta_i(0) = 0, \eta_i \text{ is nondecreasing, and } \int_0^\infty 1_{\{\phi_i(s) > 0\}}d\eta_i(s) = 0. 
\end{cases}$$

In [13], the authors showed that if $Z$ is a $(G, b_0, \sigma, R)$ SRBM, i.e., $b_2(y) = b_0$ for all $y \in \mathbb{L}$, and $b_0$ satisfies the DW-stability condition, then the SRBM is positive recurrent and consequently has a unique invariant probability distribution. In the current work we establish a similar result for the Markov modulated setting.

Theorem 2.6. Suppose that Assumptions 2.4 and 2.6 hold and the vector $b^* = \sum_{j \in \mathbb{L}} q_j b_2(j)$ satisfies the DW-stability condition. Then the family $(Z, \{P_z\}_{z \in S})$ is positive recurrent and admits a unique invariant probability measure $\pi$.

In fact, we establish geometric ergodicity properties similar to that in Theorem 2.4. Analogous result for the constant drift case (i.e. $b_2(y) \equiv b_0$) has been proved in [5].

Theorem 2.7. Under the assumptions made in Theorem 2.6, properties in Theorem 2.4 hold for the Markov process $Z$. 
2.3. Convergence of invariant measures for SPN

Consider a sequence of open queueing networks with the following structure. Each network has $K$ service stations each of which has an infinite capacity buffer. We denote the $i^{th}$ station by $P_i$, $i \in \mathbb{K} = \{1, 2, \ldots, K\}$. All customers/jobs at a station are “homogeneous” in terms of service requirement and routing decisions. Arrivals of jobs can be from outside the system and/or from internal routing. Upon completion of service at station $P_i$ a customer is routed to some other service station or exits the system. The external arrival processes and service processes are assumed to depend on the state of the system and an auxiliary finite state Markov process. The routing mechanism is governed by a $K \times K$ substochastic matrix $\mathbb{P}$. Roughly speaking, the conditional probability that a job completed at station $P_i$ is routed to station $P_j$ equals the $(i,j)^{th}$ entry of the matrix $\mathbb{P}$. The above formal description is made precise in what follows.

In the $n^{th}$ network, the Markov process modulating the arrival and service rates is denoted as $\{Y^n(t) : t \geq 0\}$. We assume that $Y^n$ has a finite state space $\mathbb{L}$ and infinitesimal generator $Q^n$ which converges to some matrix $Q$. Let $Q^n_{i}(t)$ denote the number of customers at station $P_i$ at time $t$. Then the evolution of $Q^n$ can be described by the following equation

$$Q^n_{i}(t) = Q^n_{i}(0) + A^n_{i}(t) - D^n_{i}(t) + \sum_{j=1}^{K} D^n_{ij}(t), \quad i \in \mathbb{K}. \quad (2.5)$$

Here $A^n_{i}(t)$ is the number of arrivals from outside at station $P_i$ by time $t$, $D^n_{i}(t)$ is the number of service completions by time $t$ at station $P_i$, and $D^n_{ij}(t)$ is the number of jobs that are routed to $P_i$ immediately upon completion at station $P_j$ by time $t$. Letting $D^n_{0i}(t)$ be the number of customers by time $t$ who leave the network after service at $P_i$, we have

$$D^n_{i}(t) = \sum_{j=0}^{K} D^n_{ij}(t). \quad (2.6)$$

The dependance of arrival and processing rates on the system state and $Y^n$ is modeled by requiring that $A^n_{i}$ and $D^n_{ij}$, $1 \leq i \leq K, 0 \leq j \leq K$, are counting processes given on a suitable filtered probability space $(\Omega^n, \mathcal{F}^n, \mathbb{P}^n, \{\mathcal{F}^n_t\}_{t \geq 0})$ such that for some measurable functions $\lambda^n_i, \tilde{\mu}^n_i : \mathbb{R}^K_+ \times \mathbb{L} \rightarrow \mathbb{R}_+$, the processes

$$\tilde{A}^n_i(\cdot) \equiv A^n_i(\cdot) - \int_{0}^{\cdot} \lambda^n_i(Q^n(u), Y^n(u))du,$$

$$\tilde{D}^n_{ij}(\cdot) \equiv D^n_{ij}(\cdot) - \int_{0}^{\cdot} \mathbb{P}_{ij} \tilde{\mu}^n_i(Q^n(u), Y^n(u))du$$

are locally square integrable $\{\mathcal{F}^n_t\}$ martingales. Here $\mathbb{P}_{i0} = 1 - \sum_{j=1}^{K} \mathbb{P}_{ij}$. We assume that processes $A^n_{i}$ and $D^n_{ij}$, $1 \leq i \leq K, 0 \leq j \leq K$, and $Y^n$ have no common jumps. We also require that $Y^n$ is a $\{\mathcal{F}^n_t\}$ Markov process. The functions $\lambda^n_i$ and $\tilde{\mu}^n_i$, $i \in \mathbb{K}$, represent the arrival and service rates. We denote by $\mathbb{K}_0 (\mathbb{K}_0 \subseteq \mathbb{K})$ the set of indices of stations which receive arrivals from outside. In particular, $\lambda^n_i(x,y) = 0$ for all $(x,y) \in \mathbb{R}^K_+ \times \mathbb{L}$ whenever $i \in \mathbb{K} \setminus \mathbb{K}_0$. Corresponding to the fact that no service occurs when the buffer is empty, $\tilde{\mu}^n_i(x,y) = 0$ if $x_i = 0$. Let $\lambda^n = \lambda^n_i(x,y)$ and $\tilde{\mu}^n_i(x,y)$.
We assume that, for each \( i \in \mathbb{K} \), \( \mu^n \) restricted to \( (\mathbb{R}_+^K \setminus \{x \in \mathbb{R}_+^K : x_i = 0\}) \times \mathbb{L} \) can be extended to a function \( \mu^n \) defined on \( \mathbb{R}_+^K \times \mathbb{L} \) (that satisfies additional properties as specified below), and write \( \mu^n = (\mu^n_1, \ldots, \mu^n_K)' \). Let

\[
    b^n = \frac{\lambda^n - [I - P] \mu^n}{\sqrt{n}}.
\]

We now introduce the main assumption on model parameters.

**Assumption 2.7.**

1. The spectral radius of \( P \) is strictly less than 1.
2. There exist some \( \theta_1, \theta_1 \in (0, \infty) \) such that, for all \( n \geq 1, i \in \mathbb{K}_0, j \in \mathbb{K} \) and \( (x, y) \in \mathbb{R}_+^K \times \mathbb{L} \),

\[
    n \theta_1 \leq |\lambda^n_i(x, y)| \leq n \bar{\theta}_1, \quad n \theta_1 \leq |\mu^n_j(x, y)| \leq n \bar{\theta}_1.
\]
3. For some \( \theta_2 \in (0, \infty) \),

\[
    \sup_{(x, y) \in \mathbb{R}_+^K \times \mathbb{L}} |b^n(x, y)| \leq \theta_2.
\]
4. There exists a bounded Lipschitz map \( b : \mathbb{R}_+^K \times \mathbb{L} \to \mathbb{R}^K \) such that

\[
    b^n(\sqrt{n}x, y) \to b(x, y)
\]

uniformly on \( \mathbb{R}_+^K \times \mathbb{L} \) as \( n \to \infty \).
5. There exist \( \mathbb{R}_+^K \)-valued bounded Lipschitz functions \( \lambda, \mu \) defined on \( \mathbb{R}_+^K \times \mathbb{L} \), such that

\[
    \frac{\lambda^n(\sqrt{n}x, y)}{n} \to \lambda(x, y), \quad \frac{\mu^n(\sqrt{n}x, y)}{n} \to \mu(x, y)
\]

uniformly for \( (x, y) \) in compact subsets of \( \mathbb{R}_+^K \times \mathbb{L} \) as \( n \to \infty \). Furthermore, \( \lambda = [I - P'] \mu \).
6. For each \( i \in \mathbb{K} \setminus \mathbb{K}_0 \), there exists \( j \in \mathbb{K}_0 \) such that \( P^n_{ji} > 0 \) for some \( m \in \mathbb{N} \).

For \( t \geq 0 \), let

\[
    \hat{Q}^n(t) = \frac{Q^n(t)}{\sqrt{n}}.
\]

Denote the \( i^{th} \) column of \( [I - P] \) by \( d_i \) and let \( G = \mathbb{R}_+^K \) and \( N = K \). Recall from Section 2.1 the definition of the SP and SM \( \Gamma \) associated with \( G \) and \( d \) (\( d \) is defined through (2.1)). Under Assumption 2.7 (i), it follows from [15] that Assumption 2.1 is satisfied.

Theorem 3.2 of [7] shows that under Assumption 2.7, as \( n \to \infty \), \( (\hat{Q}^n, Y^n) \) converges weakly to a Markov process \((X, Y)\), where \( Y \) is a Markov process with infinitesimal generator \( \mathcal{Q} \) and \( X \) is a reflected diffusion process with state dependent and Markov modulated coefficients, defined as in (1.1). To state this result precisely, define for \( z \in \mathbb{S} \), a \( K \times [K + K(K + 1)] \)-dimensional matrix \( \Sigma(z) \) as

\[
    \Sigma(z) = (A(z), B_1(z), \ldots, B_K(z)),
\]

where \( A \) and \( B_i, i \in \mathbb{K} \), are \( K \times K \) and \( K \times (K + 1) \) matrices given as follows. For \( z \in \mathbb{S} \),

\[
    A(z) = \text{diag} \left( \sqrt{\lambda_1(z)}, \ldots, \sqrt{\lambda_K(z)} \right),
\]

\[
    B_i(z) = \left( B_{i1}^0(z), B_{i1}^1(z), \ldots, B_{iK}^1(z) \right),
\]
where \( B_i^j(z) = -1_i \sqrt{p_{ij} \mu_i(z)} \), \( B_i^j(z) = 0 \) and for \( j \in \mathbb{K} \) and \( j \neq i \), \( B_i^j(z) = 1_{ij} \sqrt{p_{ij} \mu_i(z)} \). Here \( 1_i \) is a \( K \)-dimensional vector with 1 at the \( i^\text{th} \) coordinate and 0 elsewhere, \( 0 \) is the \( K \)-dimensional 0 vector, and \( 1_{ij} \) is a \( K \)-dimensional vector with \(-1\) at the \( i^\text{th} \) coordinate, 1 at the \( j^\text{th} \) coordinate, and 0 elsewhere. It is easy to see that due to Assumption (ii) and (vi), \( \Sigma(z) \Sigma(z)' \) is uniformly nondegenerate (see [4, Appendix]). More precisely, there exists a \( \theta_3 \in (0, \infty) \) such that, for all \( \zeta \in \mathbb{R}^K \) and \( z \in S \),

\[
\zeta'(\Sigma(z)\Sigma(z)')\zeta \geq \theta_3 \zeta' \zeta.
\]

One can then find a Lipschitz function \( \sigma : S \to \mathbb{R}^{K \times K} \) (cf. [20, Theorem 5.2.2]) such that

\[
\Sigma(z)\Sigma(z)' = \sigma(z)\sigma(z)'.
\]

Note that \( b \) given in Assumption 2.7(iv) and \( \sigma \) introduced above satisfy Assumptions 2.2 and 2.3. Denote \( Z^n = (Q^n, Y^n) \).

From Theorem 3.2 of [7], we have the following conclusion.

**Theorem 2.8.** [7] Let \( (\Omega, \mathcal{F}, \{\mathbb{P}_z\}_{z \in S}, \{\mathcal{F}_t\}_{t \geq 0}) \) and \( Z \) be as in Theorem 2.1. Suppose \( P^n \circ Z^n(0)^{-1} \) converges to \( \nu \in \mathcal{P}(\Omega) \) weakly as \( n \to \infty \). Then

\[
P^n \circ (Z^n)^{-1} \Rightarrow \int_S \mathbb{P}_z \circ Z^{-1} \nu(dz), \text{ as } n \to \infty.
\]

The following is the main result of the section.

**Theorem 2.9.** Suppose that Assumption 2.7 and 2.4 holds and that \( b \) in Assumption 2.7(iv) can be expressed as in (1.4) in terms of functions \( b_1 \) and \( b_2 \) that satisfy Assumption 2.5. Then there exists \( N \in \mathbb{N} \) such that for any \( n \geq N \), the Markov process \( Z^n \) admits a stationary distribution. Let \( \pi_n \) be a stationary distribution of \( Z^n \). Then \( \pi_n \Rightarrow \pi \) as \( n \to \infty \), where \( \pi \) is as in Theorem 2.2.

In the following, we provide an explicit example, where assumptions of the above theorem hold.

**Example 2.1.** Let \( K = 2, \mathbb{L} = \{1, 2\} \), and \( \mathbb{P} = \left( \begin{array}{cc} 0 & 1/2 \\ 1/3 & 0 \end{array} \right) \). The arrival and service rate \( \lambda^n \) and \( \mu^n \) are defined as follows. For \( x = (x_1, x_2) \in \mathbb{R}_+^2 \) and \( y \in \mathbb{L} \),

\[
\lambda^n(x, y) = \left( \sqrt{n}(e^{-x_1/\sqrt{n}} + 4) + ny, \sqrt{n}(e^{-x_2/\sqrt{n}} + 4) + 2ny \right),
\]

\[
\mu^n(y) = \left( 24/5 \sqrt{ny} + 2ny, 27/5 \sqrt{ny} + 3ny \right).
\]

Therefore,

\[
b^n(x, y) = \left( e^{-x_1/\sqrt{n}} + 4 - 3y, e^{-x_2/\sqrt{n}} + 4 - 3y \right),
\]

and

\[
b(x, y) = b^n(\sqrt{n}x, y) = \left( e^{-x_1} + 4 - 3y, e^{-x_2} + 4 - 3y \right). \tag{2.8}
\]
Let $q^* = (\frac{1}{4}, \frac{3}{4})'$. Let $Y^n$ be a Markov process with state space $\mathbb{L}$ such that it converges to a Markov process $Y$ with the stationary distribution $q^*$. With the above model parameters and letting $(\hat{Q}^n(0), Y^n(0)) = z_0 \in \mathbb{S}$, we have from Theorem 2.8 that $(\hat{Q}^n, Y^n) \Rightarrow (X, Y)$, where $X$ is defined as in (1.1) with drift $b$ defined in (2.8) and diffusion coefficient $\sigma$ constructed as above Theorem 2.8.

Note that the constraint directions in this example are $d_1 = (1, -\frac{1}{2})'$ and $d_2 = (-\frac{1}{3}, 1)'$ and the cone $C = \{-\zeta_1 d_1 - \zeta_2 d_2 : \zeta_1 \geq 0, \zeta_2 \geq 0\}$. Also observe that, for all $x \in \mathbb{R}^2_+$

$$b(x, 1) = (e^{-x_1} + 1, e^{-x_2} + 1) \in C^c, \quad b(x, 2) = (e^{-x_1} - 2, e^{-x_2} - 2) \in C^o,$$
and the averaged drift

$$b^*(x) = \left( e^{-x_1} - \frac{5}{4}, e^{-x_2} - \frac{5}{4} \right) \in C^o.$$

In fact, for any $0 < \delta_0 < \frac{1}{4}$, we have that for all $x \in \mathbb{R}^2_+$, $b^*(x) \in C(\delta_0)$. By Theorem 2.2, $(X, Y)$ is positive recurrent and has a unique invariant measure $\pi$. Finally, from Theorem 2.9, $(\hat{Q}^n, Y^n)$ admits an invariant probability measure $\pi^n$ for $n$ large enough and $\pi^n \Rightarrow \pi$ as $n \to \infty$.

3. Stability properties under a regular Skorohod map

3.1. Positive Recurrence

In this section we prove Theorem 2.2. Assumptions 2.1-2.5 will be assumed throughout this section. Recall the parameter $\delta_0$ introduced in Assumption 2.5. Let $v : [0, \infty) \to \mathbb{R}^K$ be a measurable map such that,

$$\int_0^t |v(s)| ds < \infty, \text{ for all } t \geq 0. \quad (3.1)$$

For $x \in G$ and $v$ as above, let

$$x(t) = \Gamma \left( x + \int_0^t v(s) ds \right), \quad t \geq 0. \quad (3.2)$$

For $x \in G$, let $A(x) \equiv A(x, \delta_0)$ be the set of all absolutely continuous functions $x$ defined by (3.2) for some $v : [0, \infty) \to C(\delta_0)$ that satisfies (3.1). Define the “hitting time to the origin” function as follows.

$$T(x) = \sup_{x \in A(x)} \inf \{ t \in (0, \infty) : x(t) = 0 \}, \quad x \in G. \quad (3.3)$$

Note that $T(0) = 0$. The following lemma from [1](cf. Lemma 3.1 therein) is a key ingredient in our analysis.

**Lemma 3.1.** [1] The function $T$ defined by (3.3) satisfies the following properties.
(i) For some \( \Theta_1 \equiv \Theta_1(\delta_0) \in (0, \infty) \), \( |T(x_1) - T(x_2)| \leq \Theta_1|x_1 - x_2| \) for all \( x_1, x_2 \in G \).
(ii) For some \( \Theta_2 \equiv \Theta_2(\delta_0) \in (0, \infty) \), \( \Theta_2|x| \leq T(x) \leq \Theta_1|x| \), for all \( x \in G \).
(iii) Fix \( x \in G \) and let \( x \in A(x) \). Then for all \( t > 0 \), \( T(x(t)) \leq (T(x) - t)^+ \).

Define for \((x, y) \in \mathbb{S}\), \( b^c(y) = b(x, y) - b^*(x, y) = b_2(y) - b_2^* \). The following lemma is an immediate consequence of Lemma 3.1 and the Lipschitz property of \( \Gamma \). The proof is quite similar to that of Lemma 4.1 of [1], however for completeness we provide the arguments in Appendix. Recall the filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \{P_z\}_{z \in \mathbb{S}})\) and processes \( X, W, Y, Z \) introduced in Theorem 2.1.

**Lemma 3.2.** Let \( \Delta > 0 \) and \( u > 0 \) be arbitrary. Fix \( z \in \mathbb{S} \). Then, \( P_z\)-a.s., on the set \( \{\omega: X(t, \omega) \in G \setminus A \text{ for all } t \in (u, u + \Delta]\} \)

\[
T(X(u + \Delta)) \leq (T(X(u)) - \Delta)^+ + \kappa_1 \Theta_1 v^u_{\Delta},
\]

where \( \Theta_1 \) and \( \kappa_1 \) are as in Lemma 3.1(i) and Assumption 2.1 respectively, and

\[
v^u_{\Delta} \equiv \sup_{u \leq t \leq u + \Delta} \left| \int_u^t b^c(Y(s))ds + \int_u^t \sigma(Z(s))dW(s) \right|. \tag{3.4}
\]

**Lemma 3.3.** There exists a \( \Theta_3 \in (0, \infty) \) such that for all \( \alpha, t \in (0, \infty) \) and \( z \in \mathbb{S} \),

\[
E_z(\exp\{\Theta_3 \alpha(1 + \alpha t)\}) \leq 8 \exp\{\Theta_3 \alpha(1 + \alpha t)\}, \tag{3.5}
\]

where \( v^0_{\Delta} \) is defined as in (3.4) with \( u, \Delta \) replaced by \( 0, \Delta \), respectively.

**Proof:** By Holder’s inequality,

\[
\left[ E_z\left( \exp\left\{ \alpha \sup_{0 \leq s \leq t} \left| \int_0^s b^c(Y(u))du + \int_0^s \sigma(Z(u))dW(u) \right| \right\} \right) \right]^2 \leq E_z\left( \exp\left\{ 2\alpha \sup_{0 \leq s \leq t} \left| \int_0^s b^c(Y(u))du \right| \right\} \right) E_z\left( \exp\left\{ 2\alpha \sup_{0 \leq s \leq t} \left| \int_0^s \sigma(Z(u))dW(u) \right| \right\} \right). \tag{3.6}
\]

Consider the first expectation on the right hand side of the above inequality. For \( f \in BM(\mathbb{L}) \), \( s \geq 0 \) and \( y \in \mathbb{L} \), let \( P^s_Y f(y) = E(f(Y(s))|Y(0) = y) \). Let \( g(\cdot) \) be a solution of the Poisson equation for \( b^c(\cdot) \) corresponding to the Markov semigroup \( \{P^s_Y\}_{s \geq 0} \), i.e., for \( y \in \mathbb{L} \) and \( s \geq 0 \),

\[
P^s_Y g(y) - g(y) - \int_0^s P^u_Y b^c(y)du = 0.
\]

Then, under \( P_z \),

\[
M_s = g(Y(s)) - g(Y(0)) - \int_0^s b^c(Y(u))du \tag{3.7}
\]

is an \( \{\mathcal{F}_s\} \) martingale.

We next show that, for all \( s \geq 0 \) and \( y \geq 0 \),

\[
P_z(|M_s| \geq y) \leq 2 \exp\left\{ \frac{-2y^2}{(1 + v^2)^2(s + 1)} \right\}, \tag{3.8}
\]
where \( v = 2(|g|_\infty + |b^c|_\infty) < \infty \). For fixed \( s \geq 0 \), let
\[
\xi_k = \begin{cases} M_{k+1} - M_k, & 0 \leq k \leq \lfloor s \rfloor - 1, \\ M_s - M_{\lfloor s \rfloor}, & k = \lfloor s \rfloor. \end{cases}
\]
Then \( M_s = \sum_{i=0}^{\lfloor s \rfloor} \xi_i \) and for \( 0 \leq k \leq \lfloor s \rfloor \), \( \mathbb{E}_z (\xi_k | \mathcal{F}_k) = 0 \) and \( |\xi_k| \leq v \). Using well known concentration inequalities for martingales with bounded increments (see e.g. Corollary 2.4.7 in [9]), for \( 0 \leq k \leq \lfloor s \rfloor \) and \( y \geq 0 \),
\[
\mathbb{P}_z \left( \sum_{i=0}^{k} \xi_i \geq y \sqrt{k+1} \right) \leq \exp \left\{ -\frac{2y^2}{(1+v^2)^2} \right\}.
\]
Therefore,
\[
\mathbb{P}_z \left( \sum_{i=0}^{\lfloor s \rfloor} \xi_i \geq y \right) \leq \exp \left\{ -\frac{2y^2}{(1+v^2)^2(\lfloor s \rfloor + 1)} \right\} \leq \exp \left\{ -\frac{2y^2}{(1+v^2)^2(s+1)} \right\}.
\]
Similarly,
\[
\mathbb{P}_z \left( -\sum_{i=0}^{\lfloor s \rfloor} \xi_i \geq y \right) \leq \exp \left\{ -\frac{2y^2}{(1+v^2)^2(s+1)} \right\}.
\]
The inequality in (3.8) follows on combining the above two estimates.

Denoting \( \frac{2}{(1+v^2)^2} \) by \( c_1 \), we have,
\[
\mathbb{E}_z (\exp \{2\alpha |M_s|\}) \leq \int_0^{\infty} 2 \exp \left\{ -\frac{c_1 (\log y)^2}{4\alpha^2(s+1)} \right\} dy \leq 2 \sqrt{\frac{4\pi \alpha^2(s+1)}{c_1}} \exp \left\{ \frac{\alpha^2(s+1)}{c_1} \right\}
\]
\[
\leq 2 \exp \left\{ \frac{(1+4\pi)\alpha^2(s+1)}{c_1} \right\}.
\]
An application of Doob’s inequality now yields that
\[
\mathbb{E}_z \left( \exp \left\{ 2\alpha \sup_{0 \leq s \leq t} |M_s| \right\} \right) \leq 4\mathbb{E}_z (\exp \{2\alpha |M_t|\}) \leq 8 \exp \left\{ \frac{(1+4\pi)\alpha^2(t+1)}{c_1} \right\}.
\]
Combining this with (3.7), we have that
\[
\mathbb{E}_z \left( \exp \left\{ 2\alpha \sup_{0 \leq s \leq t} \int_0^s b^c(Y(u))du \right\} \right) \leq 8 \exp \left\{ 4\alpha |g|_\infty + \frac{(1+4\pi)\alpha^2(t+1)}{c_1} \right\}.
\]
(3.9)
For the second expectation on the right side of (3.6), using Assumption 2.3 (i), we have by standard estimates (see e.g. Lemma 4.2 of [1])
\[
\mathbb{E}_z \left( \exp \left\{ 2\alpha \sup_{0 \leq s \leq t} \int_0^s \sigma(Z(u))dW(u) \right\} \right) \leq 8 \exp \left\{ 2\alpha^2 \kappa_2^2 K^2 t \right\}.
\]
(3.10)
Using (3.9) and (3.10), we now have that the left side of (3.5) is bounded above by
\[ 8 \exp \left\{ 2 \frac{|g|}{\infty} \alpha + \frac{1 + 4\pi}{2c_l} \alpha^2 + \left( \frac{1 + 4\pi}{2c_l} + \kappa_3^2 K^2 \right) \alpha^2 t \right\}. \]

The result follows. 

**Remark 3.1.** Note that, in the above proof, \( b^c \) only depends on \( y \) and satisfies the condition \( \sum_{j \in L} q_{j}^b b(x,j) = 0 \). We can then derive a solution of the Poisson equation for \( b^c \) corresponding to the Markov semigroup \( \{P_{s}^{Y}\}_{s \geq 0} \) and use the martingale property to get the desirable result. However, if we consider a general measurable drift \( b(x,y) \) and define "averaged" drift \( \bar{b}^*(x) = \sum_{j \in L} q_{j}^{i} b(x,j) \), then \( \bar{b}^* = b - b^c \) would depend on both \( x \) and \( y \). In such a case, the above method will become difficult to be applied.

Using the fact that \((X,Y)\) is a \( \{F_{t}\} \) Markov process and that \( W \) is a \( \{F_{t}\} \) Brownian motion (cf. Lemma 4.3 of [1]) we have the following lemma. Proof is omitted.

**Lemma 3.4.** Let \( z \in S \) and \( \Delta > 0 \) be fixed. For \( n \in \mathbb{N} \), let \( \nu_{n} \equiv \nu_{\Delta}^{(n-1)\Delta} \), where \( \nu_{\Delta}^{(n-1)\Delta} \) is defined as in (3.4) with \( u \) replaced by \((n-1)\Delta\). Then for any \( \alpha \in (0, \infty) \) and \( m, n \in \mathbb{N} ; m \leq n \),
\[
\mathbb{E}_{z} \left( \exp \left\{ \alpha \sum_{i=m}^{n} \nu_{i} \right\} \right) \leq \left[ 8 \exp \{ \Theta_{3}(1 + \alpha(\alpha \Delta)) \} \right]^{n-m+1},
\]
where \( \Theta_{3} \) is as in Lemma 3.3.

Given a compact set \( C \subset S \), let
\[
\tau_{C} \equiv \inf\{ t \geq 0 : Z(t) \in C \}.
\]
(3.11)

For \( M > 0 \), let \( B_{M} \equiv \{(x,y) \in S : T(x) \leq M\} \) and \( C_{M} = \{(x,y) \in S : |x| \leq M\} \).

**Theorem 3.1.** There exist \( \Delta, a_{0} \in (0, \infty) \) and \( \varsigma \in (0, 1) \) such that for any \( z = (x,y) \in S \) and \( t \in (0, \infty) \),
\[
\mathbb{P}_{z}(\tau_{B_{\Delta}} > t) \leq \exp \{ \varsigma T(x) + (a_{0} - \varsigma) \Delta - a_{0} t \}. \]
(3.12)

In particular, for every \( M \in (0, \infty) \) and \( a < a_{0} \),
\[
\sup_{z \in C_{M}} \mathbb{E}_{z}(\exp\{a\tau_{B_{\Delta}}\}) < \infty.
\]

**Proof.** Proof is similar to that of Theorem 4.1 of [1], so only a sketch is provided. Fix \( z = (x,y) \in S \). Recall the set \( A \) from Assumption 2.5. Choose \( \Delta > 0 \) large enough so that \( A \times \mathbb{R} \subset B_{\Delta} \). Additional restrictions on \( \Delta \) will be imposed later in the proof. Let
\[
\Omega_{n} \equiv \{ \omega : \tau_{B_{\Delta}} > n \Delta \} = \left\{ \omega : \inf_{0 \leq s \leq n \Delta} T(X(s,\omega)) > \Delta \right\}.
\]
Then for \( z \in S \), \( \mathbb{P}_{z}(\Omega_{n}) \leq \mathbb{P}_{z}(T(X(n \Delta)) > \Delta) \). By Lemma 3.2 we have, for \( \omega \in \Omega_{n} \),
\[
T(X(n \Delta)) \leq T(x) - n \Delta + \kappa_{1} \Theta_{1} \sum_{j=1}^{n} \nu_{j},
\]
where \( \nu_j \) is as in Lemma 3.4. Using Lemma 3.4 and a calculation similar to that in the proof of Theorem 4.1 of [1] we now have, for any \( \varsigma \in (0, 1) \),

\[
P_z(\Omega_n) \leq \exp\{\varsigma(T(x) - \Delta)\} \exp\{n [\Theta_{3s\kappa_1} \Theta_1 (1 + \varsigma\kappa_1 \Theta_1) + \log 8 + \Theta_{3s^2\kappa_1^2} \Theta_1^2 \Delta - \varsigma\Delta]\}.
\]

Take \( \varsigma = (2\Theta_{3s\kappa_1^2} \Theta_1^2 + 1)^{-1} \). Choose \( \Delta \) sufficiently large so that, in addition to the property \( A \times \mathcal{L} \subseteq B_\Delta \), we have \( \Delta^{-1} [\Theta_{3s\kappa_1} \Theta_1 (1 + \varsigma\kappa_1 \Theta_1) + \log 8] < \varsigma/2 \). Then

\[
\Delta^{-1} [\Theta_{3s\kappa_1} \Theta_1 (1 + \varsigma\kappa_1 \Theta_1) + \log 8 + \Theta_{3s^2\kappa_1^2} \Theta_1^2 \Delta - \varsigma\Delta] < \Theta_{3s^2\kappa_1^2} \Theta_1^2 - \frac{\varsigma}{2} \equiv -a_0 < 0,
\]

and

\[
P_z(\Omega_n) \leq \exp\{\varsigma(T(x) - \Delta)\} \exp\{-a_0 n\Delta\}.
\]

The proof of (3.12) now follows from the above estimate, exactly as in the proof of Theorem 4.1 of [1]. Second part of the theorem is an immediate consequence of (3.12). \( \blacksquare \)

The lemma below gives the tightness of the family \( \{P_z \circ Z(t)^{-1} : z \in C_M, t \geq 0\} \) for any \( M > 0 \). The proof is similar to Lemma 4.4 of [1]. A sketch is given in Appendix.

**Lemma 3.5.** There exists \( \kappa \) in \((0, \infty)\) such that for all \( M > 0 \),

\[
\sup_{z \in C_M} \sup_{t \geq 0} \mathbb{E}_z(\exp\{\kappa |X(t)|\}) < \infty.
\]

The following irreducibility property is used in showing uniqueness of the invariant measure. For \( j \in \mathbb{L}, z \in S, t > 0 \), define \( m_{z}^{\cdot t}(E) = P_z(Z(t) \in E \times \{j\}) \), \( E \in \mathcal{B}(G) \).

**Lemma 3.6.** For every \( j \in \mathbb{L}, z \in S \) and \( t > 0 \), \( m_{z}^{\cdot t} \) is mutually absolutely continuous with respect to the Lebesgue measure \( \lambda \) on \( G \).

**Proof:** Without loss of generality we can assume that on the filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0})\), introduced in Theorem 2.1 we have, for each \( z = (x, j) \in G \times \mathbb{L} \), probability measures \( P^z_x \) under which (i) - (iv) of Theorem 2.1 hold, with (2.3) replaced by

\[
X(t) = x + \int_0^t b(X(s), j)ds + \int_0^t \sigma(X(s), j)dW(s) + k(t), \text{ a.s.}
\]

As argued in the proof of Lemma 5.7 of [5],

\[
\text{for all } (x, j) \in S, t > 0, P^z_x \circ X(t)^{-1} \text{ is mutually absolutely continuous to } \lambda. \tag{3.13}
\]

Fix \( z = (x, i) \in S \). Denote by \( \{\tau_k\}_{k \in \mathbb{N}_0} \) the sequence of transition times of the pure jump process \( Y \), namely, \( \tau_0 = 0, \tau_{k+1} = \inf\{t > \tau_k : Y(t) \neq Y(t-)\}, k \in \mathbb{N}_0 \). Then \( P_z \) a.s., \( \tau_k \) is strictly increasing to \( \infty \). Also, the law of \( \tau_{k+1} - \tau_k \), conditioned on \( \mathcal{F}_{\tau_k} \) (under \( P_z \)) has density \( \varphi_{\tau_k} \), where for \( i \in \mathbb{L}, \varphi_i \) is the Exponential density with rate \( \sum_{j \neq i} Q_{ij} \). For \( k \geq 0 \), let \( m^z_k \in \mathcal{P}([0, \infty) \times G \times \mathbb{L}) \) be the probability law of \( (\tau_k, X(\tau_k), Y(\tau_k)) \). Also for \( j \in \mathbb{L} \), define sub-probability measures \( m^z_k \) on \([0, \infty) \times G \) by the relation

\[
m^z_k(E) = m^z_k(E \times \{j\}), E \in \mathcal{B}([0, \infty) \times G).
\]
Then, for $A \in B(G)$, $j \in \mathbb{L}$, $t > 0$,

$$P_z(X(t) \in A, Y(t) = j) = \sum_{k=0}^{\infty} \int_{[0,t] \times G} \left( \int_{t}^{\infty} \varphi_j(v)dv \right) P_{\tilde{x}}^j(X(t-u) \in A) m_{k}^{\tau_j}(dud\tilde{\xi}).$$

From (3.13) and the above display, if $\lambda(A) = 0$, then $P_z(X(t) \in A, Y(t) = j) = 0$. Conversely suppose that $\lambda(A) > 0$. From Assumption 2.4, for some $k_0 \in \mathbb{N}_0$, $P(Y(\tau_{k_0})|Y(0) = i) > 0$ and therefore $m_{k_0}^{\tau_j}([0, t] \times G)$ is nonzero for every $t > 0$. Finally, from the above display and using (3.13) once again we obtain

$$P_z(X(t) \in A, Y(t) = j) \geq \int_{[0,t] \times G} \left( \int_{t}^{\infty} \varphi_j(v)dv \right) P_{\tilde{x}}^j(X(t-u) \in A) m_{k_0}^{\tau_j}(dud\tilde{\xi}) > 0.$$ 

The result follows.

**Proof of Theorem 2.2.** Let $S$ be a compact subset of $G$ with a positive Lebesgue measure. For the proof of positive recurrence, it suffices to show that for every $M > 0$ and $j \in \mathbb{L}$,

$$\sup_{z \in C_M} E_z(\tau^{(j)}) < \infty,$$

where $\tau^{(j)} = \inf\{t \geq 0 : Z(t) \in S^{(j)}\}$ and $S^{(j)} = S \times \{j\}$. Let $\Delta$ be as in Theorem 3.1. From Assumptions 2.3 and 2.4, it follows that $p_1 = \inf_{z \in B_\Delta} P_z(Z(1) \in S^{(j)}) > 0$.

Since the family $\{P_z \circ Z(1)^{-1}, z \in B_\Delta\}$ is tight, there exists $c_1 \in (0, \infty)$ such that

$$\inf_{z \in B_\Delta} P_z(Z(1) \in S^{(j)}, |X|^*_1 \leq c_1) \geq \frac{p_1}{2}.$$

Arguing as in the proof of Theorem 2.2 of [1], we now have that for all $M > c_1$,

$$\sup_{z \in C_M} E_z(\tau^{(j)}) \leq \sup_{z \in C_M} E_z(\tau^\Delta) + \frac{2}{p_1} \left( 1 + \sup_{z \in C_M} E_z(\tau^\Delta) \right).$$

This, in view of Theorem 3.1, proves (3.14) and positive recurrence of $Z$ follows. Existence of a unique invariant probability measure is an immediate consequence of Lemma 3.5, the Feller property of $Z$, and the irreducibility property in Lemma 3.6.

### 3.2. Transience

In this section, we prove Theorem 2.3. We will assume through this section that Assumption 2.1-2.4 hold and that $b_1(z) = 0, z \in \mathbb{S}$. Let $\iota$ be the identity map from $[0, \infty)$ to $[0, \infty)$. The following lemma is taken from [3] (cf. Lemma 3.1 and Theorem 3.10 therein).
Lemma 3.7. [3] For each $\zeta \in \mathbb{R}^K$, there is a $\tilde{\zeta} \in \mathbb{R}^K_+$ such that $\Gamma(\zeta t)(t) = \tilde{\zeta} t$ for all $t \geq 0$. Furthermore, $\tilde{\zeta} \neq 0$ if and only if $\zeta \not\in C$.

Proof of Theorem 2.3: Part (i) is immediate from Theorem 2.2. Consider now Part (ii). Since $b_1 \equiv 0$, the process $Z = (X, Y)$ satisfies, for every $z = (x, y) \in \mathbb{S}$, $P_z$-a.s.,

$$X(t) = \Gamma \left( x + \int_0^t b_2(Y(s))ds + \int_0^t \sigma(Z(s))dW(s) \right), \quad t \geq 0.$$

An application of triangle inequality shows that

$$\frac{|X(t)|}{t} = \frac{1}{t} \left| \Gamma \left( x + \int_0^t b_2(Y(s))ds + \int_0^t \sigma(Z(s))dW(s) \right) \right|$$

$$\geq \frac{1}{t} \left| \Gamma(b_2^*(t)) \right| - \frac{1}{t} \left| \Gamma \left( x + \int_0^t b_2(Y(s))ds + \int_0^t \sigma(Z(s))dW(s) \right) - \Gamma(b_2^*(t)) \right|.$$

Noting that $b_2^*(y) = b_2(y) - b_2^*$, we have from the Lipschitz property of $\Gamma$ and Lemma 3.7,

$$\frac{|X(t)|}{t} \geq |\tilde{\beta}| - \frac{\kappa_1}{t} \sup_{0 \leq s \leq t} \left| \int_0^s \sigma(Z(u))dW(u) \right| - \frac{\kappa_1}{t} \sup_{0 \leq s \leq t} \left| \int_0^s b_2^*(Y(u))du \right| - \frac{\kappa_1|x|}{t}, \quad (3.15)$$

where $\tilde{\beta} = \Gamma(b_2^*)'(1)$. Since $b_2^* \not\in C$, from Lemma 3.7, $\tilde{\beta} \neq 0$. Let $\tilde{W}_i(t) = \langle e_i, \int_0^t \sigma(Z(s))dW(s) \rangle$, where $\{e_i\}_{i=1}^K$ is the standard basis in $\mathbb{R}^K$. Then the quadratic variation of the martingale $\tilde{W}_i$ equals $\langle \tilde{W}_i \rangle_t = \langle e_i, \int_0^t \sigma(Z(s))\sigma(Z(s))'e_i ds \rangle$. By Assumption 2.3, $\kappa_4 t \leq \langle \tilde{W}_i \rangle_t \leq \kappa_3^2 t$. Using a standard time change argument and the law of iterated logarithm for a scalar Brownian motion, we now have for some $c_1 \in (0, \infty)$,

$$\limsup_{t \to \infty} \frac{1}{t} \sup_{0 \leq s \leq t} \left| \int_0^s \sigma(Z(u))dW(u) \right| \leq c_1 \limsup_{t \to \infty} \frac{1}{t} \sup_{0 \leq s \leq t} \sum_{i=1}^K |\tilde{W}_i(s)| = 0, \quad P_z$-a.s.$$

Next consider the martingale $\{M_t : t \geq 0\}$ defined by (3.7). From (3.8), there exists $c_2 \in (0, \infty)$ such that for all $t, y \in (0, \infty)$ and $z \in \mathbb{S}$,

$$P_z(\{|M_t| \geq y\} \leq 2 \exp \left\{ -\frac{c_2 y^2}{t + 1} \right\}.$$

Consequently, using Markov’s inequality and $L^p$ maximal inequality, for any $\epsilon > 0$,

$$P_z \left( \frac{1}{t} \sup_{0 \leq s \leq t} |M_s| > \epsilon \right) \leq \frac{1}{\epsilon^{4t}} E_z \left[ \left( \sup_{0 \leq s \leq t} |M_s| \right)^4 \right] \leq \frac{4^4}{3^4 \epsilon^{4t}} E_z \left( |M_t|^4 \right)$$

$$\leq 2 \frac{4^4}{3^4 \epsilon^{4t}} \int_0^\infty \exp \left\{ -\frac{c_2 \sqrt{y}}{t + 1} \right\} dy = \frac{4^5(t + 1)^2}{3^4 c_2^2 \epsilon^{4t}}.$$

An application of Borel-Cantelli lemma now shows that, $P_z$-a.s.

$$\limsup_{t \to \infty} \left( \frac{1}{t} \sup_{0 \leq s \leq t} |M(s)| \right) = 0.$$
Combining this with (3.7), we have
\[
\limsup_{t \to \infty} \left( \frac{1}{t} \sup_{0 \leq s \leq t} \left| \int_0^s b(Y(u)) du \right| \right) \leq \limsup_{t \to \infty} \left( \frac{1}{t} \sup_{0 \leq s \leq t} |M(s)| + \frac{2|g|_\infty}{t} \right) = 0.
\]

Recalling that \( \tilde{\beta} \neq 0 \), we now have from (3.15) that,
\[
\liminf_{t \to \infty} \frac{|X(t)|}{t} > 0, \mathbb{P}_z \text{-a.s.} \tag{3.16}
\]
Finally, we argue that, for some \( z \in \mathbb{S} \),
\[
\mathbb{P}_z(\tau_{\mathbb{C}_1} < \infty) < 1, \tag{3.17}
\]
where \( \tau_{\mathbb{C}_1} \) is as in (3.11) with \( C_1 \) defined as below (3.11) with \( M \) replaced by 1. Suppose that (3.17) is false. Then by a straightforward application of the strong Markov property, we have that
\[
\mathbb{P}_z(Z(t_n) \in C_1 \text{ for some sequence } \{t_n\}, \text{ s.t. } t_n \uparrow \infty) = 1.
\]
However, this contradicts (3.16) and the result follows. \( \blacksquare \)

### 3.3. Geometric ergodicity

In this section we prove Theorem 2.4. Assumption 2.1-2.5 will be assumed throughout this section. The following drift inequality is at the heart of Theorem 2.4.

**Lemma 3.8.** For some \( \varpi > 0 \), the \( \varpi \)-skeleton chain \( \{\tilde{Z}_n = Z(n\varpi) : n \in \mathbb{N}\} \) satisfies the following drift inequality: There exist \( \alpha_0, \beta_0 \in (0,1), \gamma_0 \in (0,\infty) \) and a compact set \( S \subset G \) such that
\[
\mathbb{E}_z(V(\tilde{Z}_1)) \leq (1 - \beta_0)V(z) + \gamma_0 1_S(z), \quad z \in \mathbb{S}, \tag{3.18}
\]
where, for \( z = (x,y) \in \mathbb{S} \), \( V(z) = e^{\alpha_0 T(x)} \).

**Proof:** Recall \( \tau_{A \times L} = \inf\{t \geq 0 : Z(t) \in A \times L\} \). From Lemma 3.2, for \( \alpha_0, \varpi \in (0,\infty) \),
\[
\mathbb{E}_z(V(\tilde{Z}_1)1_{\{\tau_{A \times L}>\varpi\}}) \leq \mathbb{E}_z \left( \exp\left\{ \alpha_0(T(x) - \varpi)^+ + \alpha_0 \kappa_2 \Theta \nu_0^\varpi \right\} \right) \tag{3.19}
\]
where \( \nu_0^\varpi \) is defined by (3.4) with \( \Delta \) and \( u \) replaced by \( \varpi \) and 0, respectively. Recall \( B_\varpi = \{z = (x,y) \in \mathbb{S} : T(x) \leq \varpi\} \). Thus for \( z \in (B_\varpi)^c \), by Lemma 3.3,
\[
\mathbb{E}_z(\frac{V(\tilde{Z}_1)1_{\{\tau_{A \times L}>\varpi\}}}{V(z)}) \leq \mathbb{E}_z \left( \exp\left\{ \alpha_0 \kappa_2 \Theta \nu_0^\varpi - \alpha_0 \varpi \right\} \right) \leq 8 \exp\{c_1 \alpha_0 + c_2 \alpha_0^2 + c_2 \alpha_0 \varpi - \alpha_0 \varpi\},
\]
where \( c_1 = \kappa_2 \Theta \kappa_1 \) and \( c_2 = \Theta_2 \kappa_2^2 \Theta_1^2 \). Now fix \( \alpha_0 \) small enough and \( \varpi \) large enough so that
\[
8 \exp\{c_1 \alpha_0 + c_2 \alpha_0^2 + c_2 \alpha_0 \varpi - \alpha_0 \varpi\} \leq (1 - 2\beta_0) < 1.
\]
Then for \( z \in (B_\omega)^c \),
\[
E_z \left( V(\hat{Z}_1)1_{\{r_{AXL} > \omega\}} \right) \leq (1 - 2\beta_0)V(z).
\]
From the strong Markov property of \( Z \), we see that for all \( z \in S \),
\[
E_z \left( V(\hat{Z}_1)1_{\{r_{AXL} \leq \omega\}} \right) = E_z \left[ E_z \left( V(\hat{Z}_1) | \mathcal{F}_{r_{AXL}} \right) 1_{\{r_{AXL} \leq \omega\}} \right] \\
= E_z \left[ E_{Z(\{r_{AXL} \leq \omega\})} \left( V(Z(\omega - r_{AXL})) \right) 1_{\{r_{AXL} \leq \omega\}} \right].
\]
Therefore, by Assumptions 2.1, 2.2 and 2.3, there exists \( c_1 \in (0, \infty) \) such that for all \( z \in S \),
\[
E_z \left( V(\hat{Z}_1)1_{\{r_{AXL} \leq \omega\}} \right) \leq \sup_{z \in A_{XL}} E_z \left( \sup_{0 \leq t \leq \omega} V(Z(t)) \right) \leq c_1. \tag{3.20}
\]
Choose \( M > \omega \) such that for all \( z \in (B_M)^c, \beta_0 V(z) \geq c_1 \). Then on \((B_M)^c\),
\[
E_z \left( V(\hat{Z}_1) \right) \leq (1 - \beta_0)\tilde{V}(z). \quad \text{For } z \in B_M, \quad (T(x) - \omega)^+ \leq M \quad \text{and from (3.19) and (3.20),}
\]
\[
E_z \left( V(\hat{Z}_1) \right) \leq E_z \left( \exp \left\{ a_0(T(x) - \omega)^+ + a_0 \kappa_1 \Theta_1 \nu_\omega \right\} \right) + c_1 \\
\leq 8 \exp \left\{ M \alpha_0 + c_1 a_0 + c_2 \alpha_0^2 + c_2 \alpha_0^2 \omega \right\} + c_1 \equiv \gamma_0.
\]
The lemma follows on setting \( S \times \mathbb{L} = B_M \). \( \blacksquare \)

For a signed measure \( \mu \) on \((S, \mathcal{B}(S))\) and a measurable function \( f : S \to \mathbb{R} \), let \( \mu(f) = \int_S f(z) \mu(dz) \) if \( f \) is \(|\mu|\)-integrable. If \( f : S \to (0, \infty) \) is a \(|\mu|\)-integrable map, we define the \( f \)-norm of \( \mu \) as \(|\mu|_f = \sup_{|g| \leq f} |\mu(g)|\). We set \(|\mu|_f = \infty \) if \( f \) is not \(|\mu|\)-integrable. As an immediate consequence of Lemma 3.8 and Theorems 14.0.1, 16.0.1 in [19], we have the following corollary. Denote by \( \{ P^n \}_{n \in \mathbb{N}} \) the transition kernel of the chain \( \{ \hat{Z}_n : n \in \mathbb{N} \} \), namely, for \( z \in S \) and \( B \in \mathcal{B}(S) \), \( P^n(z, B) = P_z(\hat{Z}_n \in B) \). From Lemma 3.8, it follows that \( P^n(z, V) < \infty, \forall n \in \mathbb{N} \) and \( z \in S \).

**Corollary 3.1.** The invariant measure \( \pi \) satisfies \( \pi(V) < \infty \). Furthermore, the \( \omega \)-skeleton chain \( \{ \hat{Z}_n \} \) is \( V \)-uniformly ergodic, i.e., there exist \( \rho_0 \in (0, 1) \) and \( B_0 \in (0, \infty) \) such that for all \( z \in S \),
\[
\| P^n(z, \cdot) - \pi \|_V \leq B_0 \rho^n_0 V(z). \tag{3.21}
\]

**Proof of Theorem 2.4:**

(i) This is immediate from Corollary 3.1 and Lemma 3.1(ii), on taking \( \beta_1 \leq \Theta_2 a_0 \), where \( \Theta_2 \) and \( a_0 \) are as in Lemma 3.1(ii) and Lemma 3.8, respectively.

(ii) (a). For a map \( \nu \) from \( S \) to the space of signed measures on \((S, \mathcal{B}(S))\), let
\[
\| \nu \|_V = \sup_{z \in S} \| \nu(z) \|_V / V(z).
\]
Recall that \( \hat{P}_t^\nu \) denote the transition kernel of \( Z \). Denoting the signed measure \( \hat{P}_t^\nu(z, \cdot) - \pi(\cdot) \) as \( \tilde{P}_t^\nu(z) \), we have from Corollary 3.1, \| \tilde{P}^\omega \|_V \leq B_0 \rho_0^n. \) Fix \( t \in (0, \infty) \) and let \( n_0 \in \mathbb{N} \) be such that \( t \in [n_0 \omega, (n_0 + 1) \omega) \). It is easy to check that
\[
\| \tilde{P}_t \|_V \leq \| \tilde{P}^{n_0 \omega} \|_V \| \tilde{P}^{t-n_0 \omega} \|_V \leq B_0 \rho_0^{n_0} \sup_{z \in S} \sup_{0 \leq r \leq \omega} E_z \left[ V(Z(r)) \right] / V(z) + \pi(V).
\]
From Assumptions 2.1, 2.2, and 2.3, we have for some \( \tilde{B}_0 \equiv \tilde{B}_0(\varpi) \in (0, \infty) \),

\[
\sup_{0 \leq r \leq \varpi} E_x[V(Z(r))] \leq \tilde{B}_0 V(z). \tag{3.22}
\]

Let \( \tilde{\rho} = \rho_0^{1/\varpi} \) and \( \tilde{B} = B_0(\tilde{B}_0 + \pi(V)) \). Then \( \|\tilde{P}^t\|^V \leq \tilde{B} \tilde{\rho}^t \). This proves (a).

(b) & (c). By (a), for all \( z \in S \), \( f_t^c(z) = E_z(S_t - t\pi(f)) \) is well defined. We observe that for \( 0 \leq t < T < \infty \),

\[
|f_t^c(z) - f_T^c(z)| \leq \int_t^T |P_Z^s(z, f) - \pi(f)| \, ds \leq V(z) \tilde{B} \int_t^T \tilde{\rho}^s \, ds \leq B_1 V(z) (\tilde{\rho}^t - \tilde{\rho}^T). \tag{3.23}
\]

where \( B_1 = -\tilde{B} / \log \tilde{\rho} \). Noting that \( |f_t^c(z) - f_T^c(z)| \to 0 \) as \( t, T \to \infty \), \( \lim_{t \to \infty} f_t^c(z) \) exists. In particular, denoting the limit by \( \hat{f}(z) \) and letting \( t = 0 \) and \( T \to \infty \) in (3.23), we have for all \( z \in S \),

\[
|\hat{f}(z)| \leq V(z) B_1. \tag{3.24}
\]

Then fixing \( t \) and letting \( T \to \infty \), we have from (3.23), that

\[
|f_t^c(z) - \hat{f}(z)| \leq V(z) B_1 \tilde{\rho}^t. \tag{3.25}
\]

This proves (b)&(c).

(d). From (3.24), for \( t > 0 \), \( E_z(|\hat{f}(Z(t))|) \leq B_1 E_z[V(Z(t))] \). Also

\[
\int_0^t E_z(|\pi(f) - f(Z(s))|) \, ds \leq \int_0^t E_z(|f(Z(s))|) \, ds + t\pi(f) \leq \int_0^t E_z[V(Z(s))] \, ds + t\pi(f).
\]

Similar to (3.22), we have for all \( t \geq 0 \), \( \sup_{0 \leq s \leq t} E_z[V(Z(s))] < \infty \). Consequently, for all \( t \in [0, \infty) \),

\[
E_z \left( |\hat{f}(Z(t))| + \int_0^t |\pi(f) - f(Z(s))| \, ds \right) < \infty.
\]

Also note that

\[
E_z[\hat{f}(Z(t))] = \int_0^\infty E_z[P_Z^s(Z(t), f) - \pi(f)] \, ds = \int_0^\infty [P_Z^{s+t}(z, f) - \pi(f)] \, ds
\]

\[
= \int_t^\infty [P_Z^s(z, f) - \pi(f)] \, ds = \hat{f}(z) - \int_0^t [P_Z^s(z, f) - \pi(f)] \, ds
\]

\[
= \hat{f}(z) + \int_0^t E_z[\pi(f) - f(Z(s))] \, ds.
\]

This proves (d).

(iii) The proof is an immediate consequence of [14, Theorem 4.4] and [10, Theorem 5.1(f)].
4. Markov modulated SRBM

This section is devoted to proofs of Theorems 2.6 and 2.7. We use the notation introduced in Section 2.2. In particular, throughout this section, $G = \mathbb{R}_+^K$, $N = K$, and for $i = 1, 2, \ldots, K$, $G_i = \{ x \in \mathbb{R}_+^K : \langle x, e_i \rangle \geq 0 \}$. Also, $R = (d_1| \ldots |d_K)$ and $\sigma$ is a $K \times K$ positive definite matrix.

The proof of Theorem 2.6 crucially makes use of the Lyapunov function $F$, which was constructed in [13]. Recall the DW-stability condition introduced in Definition 2.5.

**Theorem 4.1**. [13] Suppose that $b_2^*$ satisfies the DW-stability condition. Then there exists a continuous map $F : \mathbb{R}_+^K \to \mathbb{R}$ such that the following hold.

(i) $F \in C^2([0, \infty) \times \mathbb{R}_+^K \setminus \{0\})$.

(ii) Given $\epsilon \in (0, \infty)$, there exists an $M \in (0, \infty)$ such that, for all $\bar{x} \in \mathbb{R}_+^K$ and $|\bar{x}| \geq M$, $F(\bar{x}) \leq \epsilon$.

(iii) There exists $c \in (0, \infty)$ such that
   (a) for all $\bar{x} \in G \setminus \{0\}$, $\langle \nabla F(\bar{x}), b_2^* \rangle \leq -c$,
   (b) for all $\bar{x} \in \partial G \setminus \{0\}$ and $d \in d(\bar{x})$, $\langle \nabla F(\bar{x}), d \rangle \leq -c$.

(iv) $F$ is radially homogeneous, i.e., $F(\zeta \bar{x}) = \zeta F(\bar{x})$ for all $\zeta \geq 0$ and $\bar{x} \in \mathbb{R}_+^K$.

(v) $\nabla F$ is uniformly bounded on $G \setminus \{0\}$. We denote

$$\Lambda \doteq \sup_{\bar{x} \in G \setminus \{0\}} |\nabla F(\bar{x})| < \infty.$$

(vi) There exist $a_1, a_2 \in (0, \infty)$ such that, for all $\bar{x} \in G$, $a_1 |\bar{x}| \leq F(\bar{x}) \leq a_2 |\bar{x}|$.

With an abuse of notation, we set $\nabla F(0) = 0$ and $\nabla^2 F(0) = 0$. Fix $z = (x, y) \in S$ and recall the martingale $\{ M_t : t \geq 0 \}$ introduced in (3.7). Denote

$$\Upsilon(t) \doteq X(t) - g(Y(t)) + g(Y(0)), \quad t \geq 0. \quad (4.1)$$

Then from (2.4) and (3.7), for all $t \geq 0$, $P_z$-a.s.,

$$\Upsilon(t) = x + b_2^* t - M_t + \sigma W(t) + RU(t).$$

By Ito’s formula, we have that

$$F(\Upsilon(t)) = F(x) + \int_0^t \left( \frac{1}{2} tr \left[ \nabla^2 F(\Upsilon(s)) \sigma \sigma' \right] + \langle \nabla F(\Upsilon(s)), b_2^* \rangle \right) ds$$

$$+ \int_0^t \langle \nabla F(\Upsilon(s)), \sigma dW(s) \rangle - \int_{0+}^t \langle \nabla F(\Upsilon(s^-)), dM_s \rangle$$

$$+ \sum_{i=1}^K \int_0^t \langle \nabla F(\Upsilon(s)), d_i \rangle dU_i(s) + R_t, \quad (4.2)$$

where

$$R_t = \sum_{0 < s \leq t} \left[ F(\Upsilon(s)) - F(\Upsilon(s^-)) - \langle \nabla F(\Upsilon(s^-)), g(Y(s)) - g(Y(s^-)) \rangle \right]. \quad (4.3)$$
Applying Ito’s formula,

By Theorem 3.2 in [8], there exists

By (4.6), we have

Next by Theorem 4.1(ii) and (iii)(a) and for 0

Similarly, by Theorem 4.1(ii) and (iii)(b) and for 0

where \( \varsigma_1 \equiv \varsigma_1(s, \omega) \in [0, 1] \). Taking expectation, we have for some \( c_1 \in (0, \infty) \),

Next by Theorem 4.1(ii) and (iii)(a) for 0 \( \leq s \leq t \wedge \bar{\tau} \), there exists \( \varsigma_2 \equiv \varsigma_2(s, \omega) \in (0, 1) \)

such that

\[
\langle \nabla F(\Upsilon(s)), b^*_2 \rangle = \langle \nabla F(X(s)), b^*_2 \rangle + \langle \nabla F(\Upsilon(s)) - \nabla F(X(s)), b^*_2 \rangle \\
\leq -c + \langle \nabla^2 F(\varsigma_2 \Upsilon(s) + (1 - \varsigma_2)X(s))[g(Y(s)) - g(Y(0))], b^*_2 \rangle \\
\leq -c + 2\epsilon |g|_{\infty} |b^*_2|.
\]

Similarly, by Theorem 4.1(ii) and (iii)(b) and for 0 \( \leq s \leq t \wedge \bar{\tau} \) and \( \varsigma_3 \equiv \varsigma_3(s, \omega) \in (0, 1) \), whenever \( X(s) \in F_i \),

\[
\langle \nabla F(\Upsilon(s)), d_i \rangle = \langle \nabla F(X(s)), d_i \rangle + \langle \nabla F(\Upsilon(s)) - \nabla F(X(s)), d_i \rangle \\
\leq \langle \nabla^2 F(\varsigma_3 \Upsilon(s) + (1 - \varsigma_3)X(s))[g(Y(s)) - g(Y(0))], d_i \rangle \\
\leq 2\epsilon |g|_{\infty}.
\]

From Theorem 4.1(ii) and (4.5), there exists \( c_2 \in (0, \infty) \), such that for all \( t > 0 \)

\[
\frac{1}{2} tr \left[ \nabla^2 F(Y(t \wedge \bar{\tau})) \sigma \sigma' \right] + \langle \nabla F(\Upsilon(t \wedge \bar{\tau})), b^*_2 \rangle \leq c_2 \epsilon - c. \tag{4.7}
\]

By (4.6), we have

\[
\sum_{i=1}^{K} \int_{0}^{t \wedge \check{\tau}} \langle \nabla F(\Upsilon(s)), d_i \rangle dU_i(s) \leq 2\epsilon |g|_{\infty} \sum_{i=1}^{K} U_i(t \wedge \check{\tau}). \tag{4.8}
\]

By Theorem 3.2 in [8], there exists \( h \in C^2(G) \) such that for \( i \in \mathbb{K} \) and \( \check{x} \in F_i \), \( \langle \nabla h(\check{x}), d_i \rangle \geq 1 \).

Applying Ito’s formula,

\[
E_{\check{z}}[h(X(t \wedge \bar{\tau}))] = h(\check{x}) + E_{\check{z}} \left( \int_{0}^{t \wedge \check{\tau}} \left( \langle \nabla h(X(s)), b(Y(s)) \rangle + \frac{1}{2} tr[\nabla^2 h(X(s)) \sigma \sigma'] \right) ds \right) \\
+ \sum_{i=1}^{K} E_{\check{z}} \left( \int_{0}^{t \wedge \check{\tau}} \langle \nabla h(X(s)), d_i \rangle dU_i(s) \right).
\]
Thus we have for $c_3 \in (0, \infty)$,

$$
\sum_{i=1}^{K} \mathbb{E}_z(U_i(t \wedge \bar{\tau}_r)) \leq \sum_{i=1}^{K} \mathbb{E}_z \left( \int_0^{t \wedge \bar{\tau}_r} \langle \nabla h(X(s)), d_i \rangle dU_i(s) \right) \\
\leq 2|h|_\infty + \mathbb{E}_z \left( \int_0^{t \wedge \bar{\tau}_r} \left( |\nabla h(X(s)), b(Y(s))| + \frac{1}{2} tr[\nabla^2 h(X(s)) \sigma \sigma'] \right) ds \right) \quad (4.9)
$$

Using (4.8) and (4.9), we now have

$$
\mathbb{E}_z \left( \sum_{i=1}^{K} \int_0^{t \wedge \bar{\tau}_r} \langle \nabla F(Y(s)), d_i \rangle dU_i(s) \right) \leq 2\epsilon |g|_\infty c_3 (1 + \mathbb{E}_z(t \wedge \bar{\tau}_r)). \quad (4.10)
$$

We note that the constants $c_1, c_2$, and $c_3$ only depend on bounds of $\sigma, g, h$, and $b_r^*$. In particular, they are independent of $\epsilon$ and $t$. Combining (4.4), (4.7), (4.10) and applying (4.2), we have

$$
\mathbb{E}_z[F(Y(t \wedge \bar{\tau}_r))] - F(x) \leq 2\epsilon |g|_\infty c_3 + [\epsilon (2|g|_\infty c_3 + c_1 + c_2) - c] \mathbb{E}_z(t \wedge \bar{\tau}_r). \quad (4.11)
$$

Again applying the Lagrange remainder form of Taylor’s expansion and Theorem 4.1(ii), (v) and (vi), there exists $\varsigma \in (0, 1)$ such that for $0 \leq s \leq t \wedge \bar{\tau}_r$,

$$
F(Y(s)) = F(X(s)) + [g(Y(s)) - g(Y(0))] \nabla F(X(s)) \\
+ \frac{1}{2} [g(Y(s)) - g(Y(0))]^T \nabla^2 F(\varsigma Y(s) + (1 - \varsigma) X(s))[g(Y(s)) - g(Y(0))]
$$

$$
\geq F(X(t)) - 2|g|_\infty \Lambda - 2|g|_\infty^2 \epsilon
$$

Choosing

$$
\epsilon = \frac{c}{2(2|g|_\infty c_3 + c_1 + c_2)},
$$

we have

$$
\mathbb{E}_z(t \wedge \bar{\tau}_r) \leq \frac{2}{c} (F(x) + 2\epsilon |g|_\infty c_3 + 2|g|_\infty \Lambda) < \infty.
$$

Letting $t \to \infty$, we have $\mathbb{E}_z(\bar{\tau}_r) < \infty$. If $|x| \leq r$, $\mathbb{E}_z(\bar{\tau}_r) < \infty$ automatically. Therefore, $\mathbb{E}_z(\bar{\tau}_r) < \infty$ for all $z \in \mathbb{S}$. The rest of the argument is as in the proof of Theorem 2.6 in [13]. Details are left to the reader. ■

We next establish geometric ergodicity for $Z$. We begin with some preliminary estimates. Arguments similar to those used in Lemmas 3.3 and 3.4 yield the following result. Proof is provided in Appendix.

**Lemma 4.1.** Let $z \in \mathbb{S}$ and $\Delta > 0$ be fixed. For $n \in \mathbb{N}$, let $\bar{\nu}_n$ be defined as follows:

$$
\bar{\nu}_n = \sup_{(n-1)\Delta \leq s \leq n\Delta} \left| \int_{(n-1)\Delta}^{s} \langle \nabla F(Y(s)), \sigma dW(s) \rangle - \int_{(n-1)\Delta}^{s} \langle \nabla F(Y(s-)), dM_s \rangle \right|.
$$

(4.12)
Then there exists $\Theta_4 \in (0, \infty)$ such that, for any $z \in \mathcal{S}, \alpha \in (0, \infty)$ and $m, n \in \mathbb{N}; m \leq n$,

$$E_z \left( \exp \left\{ \alpha \sum_{i=m}^{n} \tilde{\nu}_i \right\} \right) \leq \left( 8 \exp \left\{ \Theta_4 \alpha^2 (1 + \Delta) \right\} \right)^{n-m+1}.$$ 

For $r \in (0, \infty)$, define

$$C(r) = \{ \tilde{x} \in G : F(\tilde{x}) \leq r \}, \tau_r = \inf \{ t \geq 0 : X(t) \in C(r) \}. \tag{4.13}$$

**Lemma 4.2.** There exist $r_0, \beta, \gamma_1, \gamma_2 \in (0, \infty)$ such that, for all $z = (x, y) \in \mathcal{S}$,

$$E_z (\exp \{ \beta \tau \}) \leq \gamma_1 \exp \{ \gamma_2 |x| \},$$

where $\tau = \tau_{r_0}$ and $\tau_{r_0}$ is defined as in (4.13) with $r$ replaced by $r_0$.

**Proof:** By Theorem 4.1(ii), given $\epsilon > 0$, there exists a $r > 2|g|_\infty$ such that $|\nabla^2 F(\tilde{x})| \leq \epsilon$ whenever $\tilde{x} \in \mathbb{R}^K$ and $|\tilde{x}| \geq r - 2|g|_\infty$. By Theorem 4.1(vi), we can choose $r_0$ such that $C(r_0) \supset \{ \tilde{x} \in G : |\tilde{x}| \leq r \}$. Let $\alpha \in (0, \infty)$ and $z = (x, y) \in \mathcal{S}$ with $|x| > r$. Similar to the arguments for (4.9) and using Lemma 4.2 in [1], there exist $c_1 \in (0, \infty)$ such that for all $t \geq 0$,

$$E_z \left( \exp \left\{ \alpha \sum_{i=1}^{K} U_i(t) \right\} \right) \leq E_z \left( \exp \left\{ \alpha \sum_{i=1}^{K} \int_{0}^{t} \langle \nabla h(X(s)), d_i \rangle dU_i(s) \right\} \right)$$

$$\leq E_z \left( \exp \left\{ \alpha |h(X(t))| + \alpha |h(x)| + \alpha \left| \int_{0}^{t} \langle \nabla h(X(s)), \sigma dW(s) \rangle + \frac{1}{2} tr [\nabla^2 h(X(s)) \sigma \sigma'] ds \right| \right)$$

$$\leq \exp \{ c_1 \alpha (1 + t + at) \}. \tag{4.14}$$

Fix $\Delta \in (0, \infty)$ and $n \in \mathbb{N}$. Using (4.2) and (4.3), we have

$$F(\mathcal{Y}(n\Delta)) = F(\mathcal{Y}((n-1)\Delta)) + \int_{(n-1)\Delta}^{n\Delta} \left( \frac{1}{2} tr \left[ \nabla^2 F(\mathcal{Y}(s)) \sigma \sigma' \right] + \langle \nabla F(\mathcal{Y}(s)), b^* \rangle \right) ds$$

$$+ \int_{(n-1)\Delta}^{n\Delta} \langle \nabla F(\mathcal{Y}(s)), \sigma dW(s) \rangle - \int_{(n-1)\Delta}^{n\Delta} \langle \nabla F(\mathcal{Y}(s-)), dM_s \rangle$$

$$+ \sum_{i=1}^{K} \int_{(n-1)\Delta}^{n\Delta} \langle \nabla F(\mathcal{Y}(s)), d_i \rangle dU_i(s) + \mathcal{R}_{n\Delta} - \mathcal{R}_{(n-1)\Delta}.$$ 

For $m \geq n$, define $A_m = \{ \omega \in \Omega : \inf_{0 \leq \xi \leq m\Delta} F(X(s)) > r_0 \}$. For $\omega \in A_m$ we have that $|\mathcal{Y}(n\Delta)| > r - 2|g|_\infty$. Using arguments similar to that for (4.7), we have for some $c_2 \in (0, \infty)$, on $A_m$

$$\int_{(n-1)\Delta}^{n\Delta} \left( \frac{1}{2} tr \left[ \nabla^2 F(\mathcal{Y}(s)) \sigma \sigma' \right] + \langle \nabla F(\mathcal{Y}(s)), b^* \rangle \right) ds \leq (c_2 \epsilon - c) \Delta,$$
Therefore, on the set $A_m$ and for $n \leq m$,

$$F(\Upsilon(n\Delta)) \leq F(x) + (c_2 \epsilon - c)n\Delta + \sum_{i=1}^{K} \int_{0}^{n\Delta} \langle \nabla F(\Upsilon(s)), d_{i}\rangle dU_{i}(s) + R_n\Delta + \sum_{i=1}^{n} \tilde{\nu}_{i},$$

where $\tilde{\nu}_{n}$ is as in (4.12). From (4.14) and (4.6), on $A_m$, for $t \leq m\Delta$

$$E_{z} \left( \exp \left\{ \alpha \sum_{i=1}^{K} \int_{0}^{t} \langle \nabla F(\Upsilon(s)), d_{i}\rangle dU_{i}(s) \right\} \right) \leq \exp \{ 2|g|_{\infty} c_{1} \alpha (1 + t + \alpha t) \}. $$

Also note that on $A_m$ for some $c_3 \in (0, \infty)$ (independent of $\alpha, t, \epsilon, \Delta, m$),

$$E_{z}(\exp \{ \alpha R_{t} \}) \leq E \left( \exp \left\{ \alpha \epsilon \sum_{0<s\leq t} |g(Y(s)) - g(Y(s-))|^{2} \right\} \right) \leq \exp \{ c_3 \epsilon \alpha t \}. $$

Noting that on $A_m$, $F(\Upsilon(n\Delta)) > r_0$, we have

$$P_{z}(A_m)$$

$$\leq P_{z} \left( A_m; \sum_{i=1}^{K} \int_{0}^{n\Delta} \langle \nabla F(\Upsilon(s)), d_{i}\rangle dU_{i}(s) + R_n\Delta + \sum_{i=1}^{n} \tilde{\nu}_{i} > r_0 - (c_2 \epsilon - c)n\Delta - F(x) \right)$$

$$\leq E_{z} \left( 1_{A_m} \exp \left\{ \alpha \sum_{i=1}^{K} \int_{0}^{n\Delta} \langle \nabla F(\Upsilon(s)), d_{i}\rangle dU_{i}(s) + R_n\Delta + \sum_{i=1}^{n} \tilde{\nu}_{i} \right\} \right)$$

$$\times \exp \{ -\alpha [r_0 - (c_2 \epsilon - c)n\Delta - F(x)] \}$$

$$\leq \exp \left\{ n\Delta \left[ 2|g|_{\infty} c_{1} \frac{\epsilon \alpha}{n\Delta} + (2|g|_{\infty} c_{1} + c_{3}) \epsilon \alpha + 6|g|_{\infty} c_{1} \epsilon \alpha^{2} + 3\Theta_{4} \frac{\alpha^{2}}{\Delta} + 3\Theta_{4} \alpha^{2} \right. \right.$$

$$+ \left. \log \frac{8}{3\Delta} + c_2 \epsilon \alpha - r_0 \frac{\alpha}{n\Delta} - \alpha c \right] + \alpha F(x) \right\}. $$

Let $\epsilon$ and $\alpha$ be small enough and $\Delta$ large enough, we have

$$2|g|_{\infty} c_{1} \frac{\epsilon \alpha}{n\Delta} + (2|g|_{\infty} c_{1} + c_{3}) \epsilon \alpha + 6|g|_{\infty} c_{1} \epsilon \alpha^{2} + 3\Theta_{4} \frac{\alpha^{2}}{\Delta} + 3\Theta_{4} \alpha^{2} \left. + \log \frac{8}{3\Delta} + c_2 \epsilon \alpha - r_0 \frac{\alpha}{n\Delta} - \alpha c \right] \equiv -\eta < 0.$$  

Then

$$P_{z}(A_m) \leq \exp \{ -n\Delta \eta + \alpha F(x) \}. $$

Let $t \in (0, \infty)$ be arbitrary and let $n_0 \in \mathbb{N}_0$ be such that $t \in [n_0\Delta, (n_0 + 1)\Delta)$. Then

$$P_{z}(\tau > t) \leq P_{z}(A_{n_0}) \leq \exp \{ -(t - \Delta) \eta + \alpha F(x) \}. $$

The result follows. ■

As an immediate consequence of the above lemma, we have the following. For $\theta_0 \in (0, \infty)$ and a compact set $S \subset G$, define a stopping time $\tau_S(\theta_0) = \inf \{ t \geq \theta_0 : X(t) \in S \}$. 

**Lemma 4.3.** Fix $\theta_0 \in (0, \infty)$ and let $\beta, \gamma_2, r_0$ be as in Lemma 4.2 and $C \equiv C(r_0)$ be as in (4.13) with $r$ replaced by $r_0$. Then there exists $\gamma_3 \in (0, \infty)$ such that for $z = (x, y) \in S$,

$$E_z(\exp\{\beta \tau C(\theta_0)\}) \leq \gamma_3 \exp\{\gamma_2 |x|\}.$$ 

**Proof:** An application of the strong Markov property yields

$$E_z(\exp\{\beta \tau C(\theta_0)\}) = \exp\{\beta \theta_0\} E_z(\exp\{\beta (\tau C(\theta_0) - \theta_0)\} | F_{\theta_0})$$

$$= \exp\{\beta \theta_0\} E_z(\exp\{\beta \tau\} | F_{\theta_0}).$$

where $\tau$ is as in Lemma 4.2. Now by Lemma 4.2,

$$E_z(\exp\{\beta \tau C(\theta_0)\}) \leq \gamma_1 E_z(\exp\{\gamma_2 |X(\theta_0)|\}). \tag{4.15}$$

Using the oscillation estimate from [23] (also see [2]), we have the following result: There exists $c_1 \in (0, \infty)$ such that for all $z = (x, y) \in S$ and $0 \leq t_1 < t_2 < \infty$, $P_z$-a.s.,

$$\sup_{t_1 \leq s \leq t \leq t_2} |X(t) - X(s)| \leq c_1 \left( \sup_{t_1 \leq s \leq t \leq t_2} |W(t) - W(s)| + (t_2 - t_1) \right). \tag{4.16}$$

Combining the estimates in (4.15) and (4.16), we have

$$E_z(\exp\{\beta \tau C(\theta_0)\}) \leq \gamma_1 \exp\{\gamma_2 (x + c_1 \theta_0)\} E_z(\exp\left\{ c_1 \gamma_2 \sup_{0 \leq s \leq \theta_0} |W(s)| \right\}).$$

The result follows via an application of Doob's inequality. ■

A key step in the proof of geometric ergodicity is the following result from [10]. For $\theta_0, \beta$, and $C$ as in Lemma 4.3, let

$$V_0(z) \doteq \frac{E_z(\exp\{\beta \tau C(\theta_0)\}) - 1}{\beta} + 1.$$ 

Define, for $\theta > 0$,

$$V_\theta(z) = R_\theta V_0(z) \doteq \int_0^\infty E_z[V_0(Z(t)) | \theta \exp\{-\theta t\}] dt.$$

By Lemma 4.3 (a), Theorem 6.2 (b), and Theorem 5.1 (a) in [10], we have the following result.

**Theorem 4.2.** [10] For all $\theta > 0$, $AV_\theta = \theta(V_\theta - V_0)$, where $A$ is the extended generator of $Z$ introduced below Theorem 2.3. Furthermore, there exist $\kappa_0, h_0 \in (0, \infty)$ such that for all $z \in S$,

$$AV_\theta(z) \leq -\kappa_0 V_\theta(z) + h_0 1_{C_{X,L}}(z).$$

The following lemma is proved exactly as Lemma 4.8 of [5]. Proof is omitted.
Lemma 4.4. There exist $a_1, a_2, A_1, A_2 \in (0, \infty)$ such that for all $z = (x, y) \in S$,
\begin{equation}
    a_1 e^{a_2|x|} \leq V_0(z) \leq A_1 e^{A_2|x|}.
\end{equation}
Furthermore, there exists a constant $\tilde{\theta} \in (0, \infty)$ such that for every $\theta \in (\tilde{\theta}, \infty)$ there are $\tilde{a}_1, \tilde{a}_2, \tilde{A}_1, \tilde{A}_2 \in (0, \infty)$ such that for $z = (x, y) \in S$,
\begin{equation}
    \tilde{a}_1 e^{\tilde{a}_2|x|} \leq V_\theta(z) \leq \tilde{A}_1 e^{\tilde{A}_2|x|}.
\end{equation}

We will fix $\theta \in (\tilde{\theta}, \infty)$ and denote $V \equiv V_\theta$. Then the following corollary is an immediate consequence of Theorem 4.2.

Corollary 4.1. $V$ is in $D(A)$ and $AV = \theta(V - V_0)$. Furthermore, there exist $\kappa_0, h_0 \in (0, \infty)$ such that for all $z \in S$,
\begin{equation}
    AV(z) \leq -\kappa_0 V(z) + h_0 1_{C \times L}(z).
\end{equation}

Corollary 4.2. Let $\pi$ be the unique invariant measure of $Z$. Then $\pi(V) < \infty$.

Proof: By Corollary 4.1 and Theorem 5.1(d) of [10], for all $s \geq 0$, there exist $\varsigma_0(s) \in (0, 1)$ and $h_1 > 0$ such that for all $z \in S$, $E_z[V(Z(s))] \leq \varsigma_0(s)V(z) + h_1 1_{C \times L}(z)$. Integrating both sides with respect to $\pi$, we have $\pi(V) \leq h_1 \pi(C \times L)/(1 - \varsigma_0(s)) < \infty$. □

As a consequence of the above corollaries and Theorem 5.2(c) of [10], we have the following geometric ergodicity result.

Theorem 4.3. The Markov process $Z \equiv (X, Y)$ is $V$-uniformly ergodic, i.e., there exist constants $B_0 \in (0, \infty), \rho_0 \in (0, 1)$ such that for all $t \in (0, \infty)$ and $z \in S$,
$$
\|P^t(z, \cdot) - \pi\|_V \leq B_0 \rho_0^t V(z).
$$

Proof of Theorem 2.7: Part (i) of the theorem is immediate from Corollary 4.2 and Lemma 4.4. Part (ii)(a) is a consequence of Theorem 4.3 and Lemma 4.4. The rest of the proof is same as that for Theorem 2.4. □

5. Convergence of invariant measures for Markov modulated open queueing networks in heavy traffic.

In this section we prove Theorem 2.9. Recall the network description and the processes $Q^n, \tilde{A}^n, \tilde{D}^n$ defined in (2.5) and (2.7). Define $\mathbb{R}^K$ valued stochastic processes $M^n, B^n, \eta^n$ as follows. For
Noting that 

With this notation, equation (5.2) can be written as

\[
\begin{align*}
M^n_i(t) &= \frac{1}{\sqrt{n}} \left( \widetilde{A}^n_i(t) - \sum_{j=0}^{K} \widetilde{D}^n_{ij}(t) + \sum_{j=1}^{K} \bar{D}^n_{ji}(t) \right), \\
B^n_i(t) &= \int_0^t b^n_i(\sqrt{n}\bar{Q}^n(u), Y^n(u))du, \\
\eta^n_i(t) &= \frac{1}{\sqrt{n}} \int_0^t \mu^n_i(\sqrt{n}\bar{Q}^n(u), Y^n(u))(\bar{Q}^n(u)_{ij} = 0)du.
\end{align*}
\]

(5.1)

Noting that \(\tilde{\mu}^n_i(x,y) = \mu^a_i(x,y)1_{\{x,y\}}\) for all \((x,y) \in \mathbb{R}^K_+ \times \mathbb{L}\), we have from (2.5) that

\[
\bar{Q}^n(t) = \tilde{Q}^n(0) + M^n(t) + B^n(t) + [\mathbb{I} - \mathbb{P}']\eta^n(t).
\]

(5.2)

With this notation, equation (5.2) can be written as

\[
\tilde{Q}^n(t) = \Gamma \left( \tilde{Q}^n(0) + M^n(\cdot) + B^n(\cdot) \right)(t),
\]

(5.3)

where \(\Gamma\) is the Skorohod map with reflection matrix \(\mathbb{I} - \mathbb{P}'\). As noted in Section 2.3, due to Assumption 2.7(i), \(\Gamma\) is Lipschitz continuous, namely Assumption 2.1 is satisfied. The following stability estimate is a key ingredient in our proof. Denote by \(D_n = \{ z = (x,y) \in \mathbb{S} : nx \in \mathbb{N}_0 \}\).

**Proposition 5.1.** There exist \(N_1 \in \mathbb{N}\) and \(t_0, a_1 \in (0, \infty)\) such that for all \(t \geq t_0\) and \(n \geq N_1\),

\[
\sup_{n \geq N_1} \mathbb{E}_z \left( |\tilde{Q}^n(t|x)|^2 \right) \leq a_1(1 + |x|), \text{ for all } z = (x,y) \in D_n.
\]

**Proof:** Fix \(z = (x,y) \in D_n\) such that \(x \in G \setminus A\). Let \(M > 0\) be large enough such that \(G_M = \{ x \in G : |x| < M \} \supseteq \bar{A}\). Suppressing \(n\) in the notation, define a sequence of stopping times \(\sigma_k \in \mathbb{N}_0\) as \(\sigma_0 = 0\),

\[
\sigma_{2k+1} = \inf\{ t \geq \sigma_{2k} : \tilde{Q}^n(t) \in A \}, \sigma_{2k+2} = \inf\{ t \geq \sigma_{2k+1} : \tilde{Q}^n(t) \notin G_M \}, \quad k \in \mathbb{N}_0.
\]

If \(t \in [\sigma_{2k}, \sigma_{2k+1}]\) for some \(k \in \mathbb{N}_0\), then

\[
|\tilde{Q}^n(t)| \leq M + 1.
\]

(5.4)

Suppose now \(t \in [\sigma_{2k}, \sigma_{2k+1})\) for some \(k \in \mathbb{N}_0\). Then

\[
\tilde{Q}^n(t) = \Gamma \left( \tilde{Q}^n(\sigma_{2k}) + M^n(\cdot + \sigma_{2k}) - M^n(\sigma_{2k}) + B^n(\cdot + \sigma_{2k}) - B^n(\sigma_{2k}) \right)(t - \sigma_{2k}).
\]

From convergence of \(Q^n\) to \(Q\), it follows that for some \(n_0 \in \mathbb{N}\), the Markov process \(Y^n\) has a unique invariant measure \(q^a\), whenever \(n \geq n_0\). Furthermore, \(q^n \to q^*\) as \(n \to \infty\). We will assume \(n \geq n_0\). Define for \(s \geq 0\),

\[
\tilde{X}^n(s) = \Gamma \left( \tilde{Q}^n(\sigma_{2k}) + B^n_*(\cdot + \sigma_{2k}) - B^n_*(\sigma_{2k}) \right)(s - \sigma_{2k}),
\]

where

\[
B^n_*(s) = \int_0^s b^n_*(\sqrt{n}\tilde{Q}^n(u), Y^n(u))du.
\]
A similar argument, using Lemma 3.1(i) and (iii), shows that in the case
\[ k > 0, \]
Thus using Assumption 2.1, we have that if \( s < \sigma \)
\[ b_n^i(x, y) = b^n(x, y) - b_2(y) + \sum_{j \in L} b_2(j)q^n(j). \]
Using Assumption 2.7 (iv) and the property \( q^n \to q^* \), we see that as \( n \to \infty \), \( b_n^i(\sqrt{n}x, y) \to b^*(x, y) \) uniformly on \( S \). Using Assumption 2.5 we now have that for some \( n_1 \in \mathbb{N} \) and \( n \geq n_1 \), \( b_n^i(\sqrt{n}x, y) \in C(\delta_0/2) \) for all \( (x, y) \in D_n \). Thus
\[ \Gamma \left( \int_{\sigma_{2k}^+}^{\sigma_{2k+1}^-} b_n^i(\sqrt{n}\hat{Q}^n(u \wedge \sigma_{2k+1}), Y^n(u \wedge \sigma_{2k+1}))du \right) \leq \sigma_{2k} \in A(0, \delta_0/2), \]
where \( A \) is as defined below (3.2). Applying Lemma 3.1 (iii), we now have that for all \( \sigma_{2k} \leq s < \sigma_{2k+1} \),
\[ \Gamma(B_n^i(- + \sigma_{2k}) - B_n^i(\sigma_{2k}))(s - \sigma_{2k}) = \Gamma \left( \int_{\sigma_{2k}}^{\sigma_{2k+1}} b_n^i(\sqrt{n}\hat{Q}^n(u \wedge \sigma_{2k+1}), Y^n(u \wedge \sigma_{2k+1}))du \right) (s - \sigma_{2k}) = 0. \]
Thus using Assumption 2.1, we have that if \( k > 0 \), for \( n \geq \max(n_0, n_1) \),
\[ |\hat{X}^n(t)| \leq \kappa_1|\hat{Q}^n(\sigma_{2k})| \leq \kappa_1(M + 1), \forall t \in [\sigma_{2k}, \sigma_{2k+1}]. \]
A similar argument, using Lemma 3.1(i) and (iii), shows that in the case \( k = 0 \), i.e. \( t \in [\sigma_0, \sigma_1] \)
and \( t \geq \Theta_1(\delta_0/2)|x|, |\hat{X}^n(t)| = 0, \mathbb{P}_x\)-a.s., where \( \Theta_1(\delta_0/2) \) is as in Lemma 3.1. Next note that \( B^n = B_n^i + B_n^c \), where
\[ B_n^c(s) = \int_0^s \left( b_2(Y^n(u)) - \sum_{j \in L} b_2(j)q^n(j) \right) du. \]
Lipschitz property of \( \Gamma \) yields that
\[ |\hat{Q}^n(t) - \hat{X}^n(t)| \leq 2\kappa_1 \sup_{0 \leq s \leq t} |M^n(s) + B^n_c(s)|. \]
Combining the above estimates, for all \( t \geq \Theta_1|x| \),
\[ |\hat{Q}^n(t)| \leq 2\kappa_1 \sup_{0 \leq s \leq t} |M^n(s)| + 2\kappa_1 \sup_{0 \leq s \leq t} |B^n_c(s)| + \kappa_1(M + 1). \]
By martingale properties of processes in (2.7), Doob’s inequality and Assumption 2.7(ii), we have that for some \( c_1 \in (0, \infty) \),
\[ \text{E}_z \left( \sup_{0 \leq s \leq t} |M^n(s)| \right)^2 \leq 4 \sum_{i=1}^{K} \text{E}_z(|M_i^n(t)|^2) \]
\[ \leq 4 \sum_{i=1}^{K} \text{E}_z \left( \int_0^t \lambda_i^n \left( \sqrt{n}\hat{Q}^n_x(u), Y^n(u) \right) du \right) \]
\[ \leq c_1 t. \]
Next we consider \( \mathbb{E}_z \left( \sup_{0 \leq s \leq t} |B^n_c(s)| \right)^2 \). Let \( g^n(\cdot) \) be a solution of the Poisson equation for \( b^n_c(\cdot) \) corresponding to the Markov semigroup \( \{P^n_t\} \) of \( Y^n \). Then

\[
M^n_s = g^n(Y^n(s)) - g^n(Y^n(0)) - \int_0^s b^n_c(Y^n(u))du
\]
is a \( \{\mathcal{F}_n^n\} \) martingale and \( \Theta = \sup_n |g^n|_\infty < \infty \). Therefore, another application of Doob's inequality yields

\[
\mathbb{E}_z \left( \sup_{0 \leq s \leq t} |B^n_c(s)| \right)^2 = \mathbb{E}_z \left( \sup_{0 \leq s \leq t} |g^n(Y^n(s)) - g^n(Y^n(0)) - M^n_s| \right)^2 \leq 4\Theta^2 + 4\mathbb{E}_z \left( |M^n_t|^2 \right)
\]

Analogous to (3.8), we have for some \( c_2 \in (0, \infty) \) and \( n_2 \in \mathbb{N} \),

\[
\sup_{n \geq n_2} \mathbb{P}_z \left( |M^n_t|^2 \geq v \right) \leq 2 \exp \left\{ - \frac{c_2v}{t + 1} \right\}.
\]

Therefore, for \( n \geq n_2 \),

\[
\mathbb{E}_z \left( \sup_{0 \leq s \leq t} |B^n_c(s)| \right)^2 \leq 4\Theta^2 + 4 \int_0^\infty 2 \exp \left\{ - \frac{c_2v}{t + 1} \right\} dv
\]

\[
\leq 4\Theta^2 + \frac{8(t + 1)}{c_2}.
\]

Combing (5.5), (5.6) and (5.7), we have for some \( c_3 \in (0, \infty) \) and all \( t \geq \Theta_1 \) and \( n \geq \max(n_0, n_1, n_2) \),

\[
\mathbb{E}_z \left( \left| \tilde{Q}^n(t|x) \right|^2 \right) \leq c_3(1 + t|x), z = (x, y) \in D_n.
\]

The lemma now follows on setting \( t_0 = \Theta_1 \) and \( N_1 = \max(n_0, n_1, n_2) \). 

The following proposition yields the tightness of \( \{\mathbb{P}_z^n \circ Z^n(t)^{-1} : z \in C_M, t \geq 0, n \geq N\} \) for all \( M > 0 \) and \( N \) sufficiently large, where \( C_M \) is defined as below (3.11). Proof is similar to that of Lemma 3.5. For completeness, a sketch is given in Appendix.

**Proposition 5.2.** There exist \( N_2 \in \mathbb{N} \) and \( \kappa \in (0, \infty) \) such that for \( M > 0 \),

\[
\sup_{n \geq N_2} \sup_{z \in C_M \cap D_n} \sup_{t \geq 0} \mathbb{E}_z \left( e^{\kappa \tilde{Q}^n(t)} \right) < \infty.
\]

The following two propositions will be needed in the proof of Theorem 2.9. Proof of the next proposition is identical to that of Proposition 4.2 of [6] and thus is omitted. For \( \varrho \in (0, \infty) \) and a compact set \( F \subset \mathbb{S} \), let

\[
\tau^n_{\varrho}(\varrho) \equiv \inf \{ t \geq \varrho : Z^n(t) \in F \}.
\]

**Proposition 5.3.** Let \( f : \mathbb{S} \to \mathbb{R}_+ \) be a measurable map. Define for \( \varrho \in (0, \infty) \),

\[
G_n(z) = \mathbb{E}_z \left( \int_0^{\tau^n_{\varrho}(\varrho)} f(Z^n(t))dt \right), z \in D_n.
\]
Assume
$$\sup_{n} \sup_{z \in C_{n} \cap D_{n}} G_{n}(z) < \infty \text{ for every } M > 0. \tag{5.9}$$

Then there exists a $\bar{\kappa} \in (0, \infty)$ such that, for all $n \in \mathbb{N}, t \in [\varrho, \infty)$ and $z \in D_{n},$
$$\frac{1}{t}E_{z} [G_{n}(Z^{n}(t))] + \frac{1}{t} \int_{0}^{t} E_{z} [f(Z^{n}(s))] ds \leq \frac{1}{t} G_{n}(z) + \bar{\kappa}. \tag{5.10}$$

By Proposition 5.1, there exists $\Lambda_{0} \in (0, \infty)$ such that for $|x| \geq \Lambda_{0}, z = (x, y) \in D_{n},$ and $n \geq N_{1},$
$$E_{z} \left( \left| \hat{Q}^{n}(t_{0}|x|) \right|^{2} \right) \leq \frac{1}{2} |x|^{2},$$
where $t_{0}$ and $N_{1}$ are as in Proposition 5.1. The following proposition is proved exactly as Proposition 4.2 of [4] and thus the proof is omitted.

**Proposition 5.4.** There exists $N_{3} \in \mathbb{N}$ and $c_{0} \in (0, \infty)$ such that for all $n \geq N_{3}$ and $z \in D_{n},$
$$\sup_{n \geq N_{3}} E_{z} \left( \int_{0}^{t_{n}} \left( 1 + \left| \hat{Q}^{n}(t) \right| \right) dt \right) \leq c_{0} \left( 1 + |x|^{2} \right),$$
where $t_{n} = \tau_{C_{\Lambda_{0}}}(t_{0}\Lambda_{0})$ (see (5.8)), $t_{0}, \Lambda_{0}$ are as introduced above and $C_{\Lambda_{0}}$ is defined as below (3.11) with $M$ replaced by $\Lambda_{0}.$

**Proof of Theorem 2.9:** From Proposition 5.2, it follows that for all $n \geq N_{2}, Z^{n}$ has an invariant probability measure on $D_{n}.$ Denote by $\{\pi_{n}\}_{n \geq N}$ one such sequence of invariant measures, where $N = \max(N_{2}, N_{3})$ and $N_{2}, N_{3}$ are as in Propositions 5.2, 5.4, respectively. Since $\pi$ is the unique invariant measure of the Feller-Markov process $(Z, \{P_{z}\}_{z \in S}),$ we have from $Z^{n} \Rightarrow Z$ (Theorem 2.8), that it suffices to establish the tightness of the family $\{\pi_{n}\}$ (regarded as a sequence of probability measures on $S.$) We apply Proposition 5.3 with $f(z) = 1 + |x|, z = (x, y) \in S$ and $\varrho = t_{0}\Lambda_{0}, F = C_{\Lambda_{0}},$ where $t_{0}$ and $\Lambda_{0}$ as in Proposition 5.4. Note that condition (5.9) in Proposition 5.3 is satisfied as a consequence of Proposition 5.4. To prove the desired tightness we only need to show that, for all $n \geq N, \langle \pi_{n}, f \rangle \leq c_{1} < \infty.$ Note that for any nonnegative, real measurable function $\psi$ on $S$ and $n \geq N,$
$$\int_{D_{n}} E_{z} [\psi(Z^{n}(t))] \pi_{n}(dz) = \langle \pi_{n}, \psi \rangle. \tag{5.10}$$

Fix $k \in \mathbb{N}$ and $t \in (\varrho, \infty).$ Let for $z \in D_{n},$
$$\Phi_{n}(z) = \frac{1}{t} G_{n}(z) - \frac{1}{t} E_{z} [G_{n}(Z^{n}(t))].$$

By (5.10), $\int_{D_{n}} \Phi_{n}(z) \pi_{n}(dz) = 0.$ From Proposition 5.3,
$$0 = \int_{D_{n}} \Phi_{n}(z) \pi_{n}(dz) \geq \int_{D_{n}} \left( \frac{1}{t} \int_{0}^{t} E_{z} [f(Z^{n}(s))] ds - \bar{\kappa} \right) \pi_{n}(dz).$$

Recalling (5.10), we have that $\langle \pi_{n}, f \rangle \leq \bar{\kappa}.$ The result follows. ■
6. Appendix

Proof of Lemma 3.2: For $t \geq 0$, let $\tilde{X}(t) = X(t + u)$. Then $P_z$-a.s.,

$$\tilde{X}(t) = \Gamma \left( X(u) + \int_0^t b(Z(s + u))ds + \int_0^t \sigma(Z(s + u))dW_u(s) \right)(t),$$

where $W_u(s) = W(s + u) - W(u)$. Let

$$\tilde{X}(t) = \Gamma \left( X(u) + \int_0^t b^*(Z(s + u))ds \right)(t),$$

where $b^*$ is as defined in Assumption 2.5. By Lipschitz property of $\Gamma$ (Assumption 2.1),

$$\sup_{0 \leq t \leq \Delta} \left| \tilde{X}(t) - \tilde{X}(t) \right| \leq \kappa_1 \sup_{0 \leq t \leq \Delta} \left| \int_0^t b^*(Y(s + u))ds + \int_0^t \sigma(Z(s + u))dW_u(s) \right|$$

Recalling the assumption on $b^*_2$ (Assumption 2.5), we have applying Lemma 3.1, that on the set $\{ \omega : X(t, \omega) \in G \setminus A \text{ for all } t \in (u, u + \Delta) \}$

$$T(X(u + \Delta)) = T(\tilde{X}(\Delta)) \leq (T(\tilde{X}(\Delta)) + \kappa_1 \Theta_1 \nu_{2}^u \leq (T(X(u)) - \Delta)^+ + \kappa_1 \Theta_1 \nu_{2}^u.$$

Proof of Lemma 3.5 (sketch): Recall $\Delta$ and $\nu_n$ defined as in Lemma 3.4. By arguing as in the proof of Lemma 4.4 of [1], we can show, for $M_0 \in (0, \infty)$ such that

$$P_z(T(X(n\Delta) \geq M_0)) \leq \sum_{l=1}^{n} P_z \left( 2\kappa_1 \Theta_1 \sum_{j=l}^{n} \nu_j \geq M_0 + (n - l - 1)\Delta - T(x) \right)$$

$$\leq \frac{\exp\{\alpha(T(x) + 2\Delta)\}}{\exp\{\alpha M_0\}} \sum_{l=1}^{n} \frac{\exp\{2\alpha \kappa_1 \Theta_1 \sum_{j=l}^{n} \nu_j\}}{\exp\{\alpha(n - l + 1)\Delta\}},$$

where $\alpha \in (0, \infty)$ is arbitrary. From Lemma 3.4 we now have

$$P_z(T(X(n\Delta)) \geq M_0) \leq \frac{\exp\{\alpha(T(x) + 2\Delta)\}}{\exp\{\alpha M_0\}} \sum_{l=1}^{n} \frac{(8 \exp\{2\Theta_3 \kappa_1 \Theta_1 \alpha(1 + 2\kappa_1 \Theta_1 \alpha + 2\kappa_1 \Theta_1 \alpha \Delta)\})^{n-l+1}}{\exp\{\alpha(n - l + 1)\Delta\}}$$

$$\leq \frac{\exp\{\alpha(T(x) + 2\Delta)\}}{\exp\{\alpha M_0\}} \sum_{l=1}^{n} \exp\{(n - l + 1)\log 8 + 2\Theta_3 \kappa_1 \Theta_1 \alpha(1 + 2\kappa_1 \Theta_1 \alpha + 2\kappa_1 \Theta_1 \alpha \Delta) - \alpha \Delta\}. $$

Similar to the proof of Theorem 3.1, we can choose $\alpha$ and $\Delta$ so that

$$\log 8 + 2\Theta_3 \kappa_1 \Theta_1 \alpha(1 + 2\kappa_1 \Theta_1 \alpha + 2\kappa_1 \Theta_1 \alpha \Delta) - \alpha \Delta = -\bar{\theta} < 0.$$


An application of Lemma 3.1 yields that for every \( \kappa \in (0, \alpha \Theta) \) and \( M > 0 \),

\[
\sup_{|z| \leq M} \sup_{n \in \mathbb{N}} \mathbb{E}_z(e^{\kappa|X_{(n\Delta)}|}) < \infty.
\]

The result follows from the above estimate, using the Lipschitz property of \( \Gamma \), in a straightforward manner (see Lemma 4.4 of [1]). ■

**Proof of Lemma 4.1 (sketch):** By the strong Markov property of \( Z \), it suffices to show

\[
\mathbb{E}_z(\exp\{\alpha \tilde{\nu}_1\}) \leq 8 \exp\{\Theta_4 \alpha^2 (1 + \Delta)\}.
\]

By Holder’s inequality,

\[
\mathbb{E}_z \left( \exp \left\{ \alpha \sup_{0 \leq t \leq \Delta} \left| \int_0^t \langle \nabla F(Y(s)), \sigma dW(s) \rangle - \int_0^t \langle \nabla F(Y(s-)), dM_s \rangle \right| \right\} \right)^2 \\
\leq \mathbb{E}_z \left( \exp \left\{ 2\alpha \sup_{0 \leq t \leq \Delta} \left| \int_0^t \langle \nabla F(Y(s)), \sigma dW(s) \rangle \right| \right\} \right) \\
\times \mathbb{E}_z \left( \exp \left\{ 2\alpha \sup_{0 \leq t \leq \Delta} \left| \int_0^t \langle \nabla F(Y(s-)), dM_s \rangle \right| \right\} \right).
\]

Using Lagrange remainder form of Taylor expansion we have for \( t \geq 0 \),

\[
\nabla F(Y(t)) = \nabla F(X(t)) + \nabla^2 F(\varsigma Y(t) + (1 - \varsigma)X(t))(g(Y(t)) - g(Y(0))),
\]

where \( \varsigma \equiv \varsigma(s, \omega) \in (0, 1) \). From Theorem 4.1 (ii) and (v), there exists some \( c_1 \in (0, \infty) \) for all \( t \geq 0 \), \( |\nabla F(Y(t))| \leq c_1 \). We have by standard estimates (see e.g. Lemma 4.2 of [1]) for some \( c_2 \in (0, \infty) \),

\[
\mathbb{E}_z \left( \exp \left\{ 2\alpha \sup_{0 \leq t \leq \Delta} \left| \int_0^t \langle \nabla F(Y(s)), dW(s) \rangle \right| \right\} \right) \leq 8 \exp \{ c_2 \alpha^2 \Delta \}. \quad (6.1)
\]

Applying arguments similar to those between (3.8) and (3.9) in the proof of Lemma 3.3, there exist \( c_3 \in (0, \infty) \) such that

\[
\mathbb{E}_z \left( \exp \left\{ 2\alpha \sup_{0 \leq s \leq t} \left| \int_0^s \langle \nabla F(Y(s)), dM_s \rangle \right| \right\} \right) \leq 8 \exp \{ c_3 \alpha^2 (1 + \Delta) \}.
\]

Result follows on combining the above estimates. ■

**Proof of Proposition 5.2 (sketch):** Define, for \( j \in \mathbb{N} \),

\[
\nu_j^n = \sup_{(j-1)\Delta \leq s \leq j\Delta} |M^n(s) - M^n((j-1)\Delta) + B^n_c(s) - B^n_c((j-1)\Delta)|.
\]

Along the lines of proof of Lemma 4.4 in [1], we have that for all \( q \in \mathbb{N} \),

\[
T(X^n(q\Delta)) \leq T(x) + 2\Delta + \max_{1 \leq l \leq q} \sum_{j=l}^q (2\kappa_1 \Theta_1 \nu_j^n - \Delta).
\]
Hence for $\alpha, M_0 \in (0, \infty)$,

$$
\mathbf{P}_z \left( T(X^n(q\Delta)) \geq M_0 \right) \leq \sum_{i=1}^{q} \mathbf{P}_z \left( 2\kappa_1 \sum_{j=1}^{q} \nu_j^n \geq M_0 + (q - l - 1)\Delta - T(x) \right)
$$

$$
\leq \exp\{\alpha(T(x) + 2\Delta - M_0)\} \sum_{i=1}^{q} \mathbf{E}_z \left( \exp\{2\alpha \kappa_1 \sum_{j=1}^{q} \nu_j^n\} \right) \exp\{\alpha(q - l + 1)\Delta\}.
$$

Let $c_1 = 2\kappa_1 \Theta_1$. We claim that there exist constants $\alpha_0, \Delta_0, \eta \in (0, \infty)$ and $N \in \mathbb{N}$ such that

$$
\sup_{n \geq N} \sup_{j \in \mathbb{N}} e^{-\alpha_0 \Delta_0} \mathbf{E}_z \left( e^{c_1 \alpha_0 \nu_j^n} | F^n_{(j-1)\Delta_0} \right) \leq e^{-\eta \Delta_0}, \tag{6.2}
$$

where $F^n_t$ is as introduced below (2.6). Suppose, for now, that the claim holds. Then by the Markov properties of $Z^n$ and $Y^n$, we have that, for $n \geq N$ and $q \in \mathbb{N},$

$$
\mathbf{P}_z \left( T(X^n(q\Delta_0)) \geq M_0 \right) \leq \exp\{\alpha_0(T(x) + 2\Delta_0 - M_0)\} \sum_{i=1}^{q} \exp\{-(q - l + 1)\eta \Delta_0\}
$$

$$
\leq \frac{\exp\{\alpha_0(T(x) + 2\Delta_0 - M_0)\}}{1 - \exp\{-\eta \Delta_0\}}.
$$

Consequently, there exists $\kappa_1 \in (0, \infty)$ such that for all $M \in (0, \infty),$

$$
\sup_{n \geq N} \sup_{|z| \leq M} \mathbf{E}_z \left( \exp\{\kappa_1 |X^n(q\Delta_0)|\} \right) < \infty.
$$

The result now follows by a standard argument, using the Lipschitz property of $\Gamma$. Finally we prove the claim in (6.2). Note that

$$
\nu_j^n \leq \sup_{(j-1)\Delta \leq s \leq j\Delta} |M^n(s) - M^n((j-1)\Delta)| + \sup_{(j-1)\Delta \leq s \leq j\Delta} |B^n_c(s) - B^n_c((j-1)\Delta)|. \tag{6.3}
$$

Following the proof of Lemma 3.3 (see arguments between (3.7) and (3.9)), we can find $c_2 \in (0, \infty)$ such that for all $j \in \mathbb{N}, \alpha, \Delta \in (0, \infty),$

$$
\mathbf{E}_z \left( \exp \left\{ \alpha \sup_{(j-1)\Delta \leq s \leq j\Delta} |B^n_c(s) - B^n_c((j-1)\Delta)| \right\} | F^n_{(j-1)\Delta} \right) \leq 8 \exp\{c_2 \alpha(1 + \alpha + \alpha \Delta)\}.
$$

Furthermore, following the proof of Proposition 3.2 in [4] (see arguments below (7.4) therein). We can find $c_3 \in (0, \infty)$ such that for $j \in \mathbb{N}, \alpha, \Delta \in (0, \infty),$

$$
\mathbf{E}_z \left( \exp \left\{ \alpha \sup_{(j-1)\Delta \leq s \leq j\Delta} |M^n(s) - M^n((j-1)\Delta)| \right\} | F^n_{(j-1)\Delta} \right) \leq 8 \exp\{c_3 \alpha^2 \Delta\}.
$$

By Holder’s inequality, for $j \in \mathbb{N}, \alpha, \Delta \in (0, \infty),$

$$
\exp\{-\alpha \Delta\} \mathbf{E}_z \left( \exp\{\alpha c_1 \nu_j^n\} | F^n_{(j-1)\Delta} \right) \leq 8 \exp\{c_1 c_2 \alpha + 2c_1^2 c_2 \alpha^2 + 2(c_1^2 c_2 + c_1^2 c_3)\alpha^2 + \alpha \Delta - \alpha \Delta_0\}.
$$

Finally, choose appropriate (small) $\alpha_0$ and (large) $\Delta_0$ such that

$$
\frac{1}{\Delta_0} (\log 8 + c_1 c_2 \alpha_0 + 2c_1^2 c_2 \alpha_0^2 + 2(c_1^2 c_2 + c_1^2 c_3)\alpha_0^2 \Delta_0 - \alpha_0 \Delta_0) \equiv -\eta < 0.
$$

The claim follows. ■
References


