Set the stage

- Study the link between international stochastic discount factors and the depreciation of the US dollar:

\[
\frac{M_{t+1}^{uk}}{M_{t+1}^{us}} = \frac{e_{t+1}}{e_t}
\]

where

\[
E_t \left[ M_{t+1}^i R_{t+1}^i \right] = 1, \quad \forall i \in \{us, uk\}
\]
Set the stage

- Study the link between international stochastic discount factors and the depreciation of the US dollar:

\[
\frac{M_{t+1}^{uk}}{M_{t+1}^{us}} = \frac{e_{t+1}}{e_t}
\]

where

\[
E_t \left[ M_{t+1}^i R_{t+1}^i \right] = 1, \quad \forall i \in \{us, uk\}
\]

- How to evaluate \( M_{t+1}^i \)?

  1. From prices: \( \sigma \left( M_{t+1}^i \right) \geq \frac{E[R_{t+1}^i - R_{t+1}^f]}{\sigma(R_{t+1}^i - R_{t+1}^f)} \)
Set the stage

- Study the link between international stochastic discount factors and the depreciation of the US dollar:

$$\frac{M_{t+1}^{uk}}{M_{t+1}^{us}} = \frac{e_{t+1}}{e_{t}}$$

where

$$E_{t} \left[ M_{t+1}^{i} R_{t+1}^{i} \right] = 1, \quad \forall i \in \{us, uk\}$$

- How to evaluate $M_{t+1}^{i}$?

  1. From prices: $\sigma \left( M_{t+1}^{i} \right) \geq \frac{E \left[ R_{t+1}^{i} - R_{t+1}^{f} \right]}{\sigma \left( R_{t+1}^{i} - R_{t+1}^{f} \right)}$

    - e.g. if $E \left[ R^{i} - R^{f} \right] \approx 7\%$, $\sigma \left( R^{i} - R^{f} \right) \approx 17\%$ then $\sigma \left( M^{i} \right) \approx 40\%$
Set the stage

- Study the link between international stochastic discount factors and the depreciation of the US dollar:

\[
\frac{M_{t+1}^{uk}}{M_{t+1}^{us}} = \frac{e_{t+1}}{e_t}
\]

where

\[
E_t \left[ M_{t+1}^i R_{t+1}^i \right] = 1, \quad \forall i \in \{us, uk\}
\]

- How to evaluate \( M_{t+1}^i \)?

1. From prices: \( \sigma \left( M_{t+1}^i \right) \geq \frac{E \left[ R_{t+1}^i - R_{t+1}^f \right]}{\sigma \left( R_{t+1}^i - R_{t+1}^f \right)} \)
   - e.g. if \( E \left[ R^i - R^f \right] \approx 7\%, \sigma \left( R^i - R^f \right) \approx 17\% \) then \( \sigma \left( M^i \right) \approx 40\% \)

2. From quantities: \( M_{t+1}^i = \frac{\partial U^i / \partial C_{t+1}^i}{\partial U^i / \partial C_t^i} \)
Set the stage

- Study the link between international stochastic discount factors and the depreciation of the US dollar:

\[
\frac{M_{t+1}^{uk}}{M_{t+1}^{us}} = \frac{e_{t+1}}{e_t}
\]

where

\[
E_t \left[ M_{t+1}^i R_{t+1}^i \right] = 1, \quad \forall i \in \{us, uk\}
\]

- How to evaluate \( M_{t+1}^i \)?

1. From prices: \( \sigma \left( M_{t+1}^i \right) \geq \frac{E \left[ R_{t+1}^i - R_{t+1}^f \right]}{\sigma \left( R_{t+1}^i - R_{t+1}^f \right)} \)
   - e.g. if \( E \left[ R^i - R^f \right] \approx 7\% \), \( \sigma \left( R^i - R^f \right) \approx 17\% \) then \( \sigma \left( M^i \right) \approx 40\% \)

2. From quantities: \( M_{t+1}^i = \frac{\partial U^i / \partial C_{t+1}^i}{\partial U^i / \partial C_t^i} \)
   - e.g. with CRRA preferences \( M_{t+1}^i = \delta \left( \frac{C_{t+1}^i}{C_t^i} \right)^{-\gamma} \)
Puzzle

By no arbitrage

$$E_t \left[ M_{t+1}^f R_{t+1}^f \right] = 1 = E_t \left[ M_{t+1}^h R_{t+1}^h \right]$$
By no arbitrage

\[ E_t \left[ M_{t+1}^f R_{t+1}^f \right] = 1 = E_t \left[ M_{t+1}^h \frac{e_{t+1}}{e_t} R_{t+1}^f \right] \]
By no arbitrage

\[ E_t \left[ M_{t+1}^f R_{t+1}^f \right] = 1 = E_t \left[ M_{t+1}^h \frac{e_{t+1}}{e_t} R_{t+1}^f \right] \]
By no arbitrage

\[ \log M_{t+1}^f - \log M_{t+1}^h = \log \frac{e_{t+1}}{e_t} \]
Puzzle

- By no arbitrage

\[ \log M_{t+1}^f - \log M_{t+1}^h = \log \frac{e_{t+1}}{e_t} \]

- Correlation of stochastic discount factors

\[ \sigma_{mf}^2 + \sigma_{mh}^2 - 2\rho_{mf,mh} \sigma_{mf} \sigma_{mh} = \sigma_\pi^2 \]
Puzzle

- By no arbitrage

\[ \log M_{t+1}^f - \log M_{t+1}^h = \log \frac{e_{t+1}}{e_t} \]

- Correlation of stochastic discount factors

\[
\begin{align*}
\sigma^2_{m^f} + \sigma^2_{m^h} - 2 \rho_{m^f,m^h} \sigma_{m^f} \sigma_{m^h} &= \sigma^2_\pi \\
20\% &+ 20\% - 2 \times 0.96 \times 20\% &1.5\%
\end{align*}
\]
Puzzle

- By no arbitrage

\[
\log M_{t+1}^f - \log M_{t+1}^h = \log \frac{e_{t+1}}{e_t}
\]

- Correlation of stochastic discount factors

\[
\sigma_{m_f}^2 + \sigma_{m_h}^2 - 2 \rho_{m_f,m_h} \sigma_{m_f} \sigma_{m_h} = \sigma^2_{\pi}
\]

  \[
\begin{array}{c}
\sigma_{m_f}^2 = 20\% \\
\sigma_{m_h}^2 = 20\% \\
\rho_{m_f,m_h} = 0.96 \\
\sigma_{m_f} \sigma_{m_h} = 20\%
\end{array}
\]

  \[
\sigma^2_{\pi} = 1.5\%
\]

- Assuming identical CRRA preferences:

\[
\log M_{t+1}^i = -\gamma \Delta c_{t+1}^i
\]

\[
\gamma^2 \sigma_{\Delta c_f}^2 + \gamma^2 \sigma_{\Delta c_h}^2 - 2 \rho_{\Delta c_f,\Delta c_h} \gamma^2 \sigma_{\Delta c_f} \sigma_{\Delta c_h} = \sigma^2_{\pi}
\]
By no arbitrage

\[
\log M_t^f - \log M_t^h = \log \frac{e_{t+1}}{e_t}
\]

Correlation of stochastic discount factors

\[
\sigma_{m^f}^2 + \sigma_{m^h}^2 - 2 \rho_{m^f, m^h} \sigma_{m^f} \sigma_{m^h} \frac{20\%}{20\%} 0.96 \frac{20\%}{1.5\%} = \sigma_\pi^2
\]

Assuming identical CRRA preferences:

\[
\log M_t^i = -\gamma \Delta c_t^i
\]

\[
\gamma^2 \sigma_{\Delta c^f}^2 + \gamma^2 \sigma_{\Delta c^h}^2 - 2 \rho_{\Delta c^f, \Delta c^h} \gamma^2 \sigma_{\Delta c^f} \sigma_{\Delta c^h} \frac{20\%}{20\%} 20\% = \sigma_\pi^2
\]
Puzzle

- By no arbitrage

\[
\log M^f_{t+1} - \log M^h_{t+1} = \log \frac{e_{t+1}}{e_t}
\]

- Correlation of stochastic discount factors

\[
\sigma^2_{m^f} + \sigma^2_{m^h} - 2\rho_{m^f, m^h} \sigma_{m^f} \sigma_{m^h} = \sigma^2_{\pi}
\]

- 20%  
- 20%  
- 0.96  
- 20%  
- 1.5%

- Assuming identical CRRA preferences:

\[
\log M^i_{t+1} = -\gamma \Delta c^i_{t+1}
\]

\[
\gamma^2 \sigma^2_{\Delta c^f} + \gamma^2 \sigma^2_{\Delta c^h} - 2\rho_{\Delta c^f, \Delta c^h} \gamma \sigma_{\Delta c^f} \sigma_{\Delta c^h} = \sigma^2_{\pi}
\]

- 20%  
- 20%  
- 0.3  
- 20%  
- 28%
The puzzle (cont’d)

- Brandt, Cochrane and Santa-Clara (2005):

<table>
<thead>
<tr>
<th></th>
<th>$\rho_{m_f,m_h}$</th>
<th>$\sigma^2_\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prices</td>
<td>high</td>
<td>low</td>
</tr>
<tr>
<td>Quantities</td>
<td>low</td>
<td>high</td>
</tr>
</tbody>
</table>
The puzzle (cont’d)

- Brandt, Cochrane and Santa-Clara (2005):

<table>
<thead>
<tr>
<th>$\rho_{m^f,m^h}$</th>
<th>$\sigma^2_\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prices</td>
<td>high low</td>
</tr>
<tr>
<td>Quantities</td>
<td>low high</td>
</tr>
</tbody>
</table>

- How does this puzzle look in a cross section of countries?
The puzzle in a cross section of countries

- HJ bound and depreciation rate: high correlation of SDF and low variance of depreciation rate.
The puzzle in a cross section of countries

- Consumption data and CRRA preferences: low correlation of SDF and high variance of depreciation rate.

![Graph showing correlations](image)

- Prices
- Quantities
- UK
- France
- Germany
- Netherlands
- Sweden

Risks for the Long Run and the Real Exchange Rate, UCLA, 2.22.06 – p. 5/29
The puzzle in a cross section of countries

- This paper: high correlation of SDF, low correlation of $\Delta c$ and low variance of depreciation rate.
1. We want:
   - to reconcile the correlation of SDF from
     1. prices
     2. quantities
Rules of the game and outline

1. We want:
   - to reconcile the correlation of SDF from
     (a) prices
     (b) quantities
   - low volatility of depreciation rate
   - low volatility of consumption growth
   - low correlation of consumption growths
1. We want:
   - to reconcile the correlation of SDF from
     (a) prices
     (b) quantities
   - low volatility of depreciation rate
   - low volatility of consumption growth
   - low correlation of consumption growths

2. Can we match other key features of financial markets?
1. We want:
   - to reconcile the correlation of SDF from
     - (a) prices
     - (b) quantities
   - low volatility of depreciation rate
   - low volatility of consumption growth
   - low correlation of consumption growths
2. Can we match other key features of financial markets?
3. Can we estimate this model?
Rules of the game and outline

1. We want:
   - to reconcile the correlation of SDF from
     (a) prices
     (b) quantities
   - low volatility of depreciation rate
   - low volatility of consumption growth
   - low correlation of consumption growths

2. Can we match other key features of financial markets?
3. Can we estimate this model?
4. Extensions and future research...
Setup of the economy

- Endowment economy.
- Two country specific goods.
- Complete home bias in consumption.
Setup of the economy

- Endowment economy.
- Two country specific goods.
- Complete home bias in consumption.
- Epstein, Zin and Weil preferences:

\[
U_t^i = \left\{ (1 - \delta)(C_t^i)^{\frac{1 - \gamma}{\sigma}} + \delta \left[ E_t(U_{t+1}^i)^{1 - \gamma} \right]^{\frac{1}{\theta}} \right\}^{\frac{\theta}{1 - \gamma}}, \forall i \in \{h, f\}
\]

where

\[
\theta = \frac{1 - \gamma}{1 - 1/\psi}
\]
Setup of the economy

- Endowment economy.
- Two country specific goods.
- Complete home bias in consumption.
- Epstein, Zin and Weil preferences:

\[ U_t^i = \left\{ (1 - \delta)(C_t^i)^{\frac{1-\gamma}{\delta}} + \delta \left[ E_t(U_{t+1}^i)^{1-\gamma} \right]^{\frac{1}{\theta}} \right\}^{\frac{\theta}{1-\gamma}}, \forall i \in \{h, f\} \]

where

\[ \theta = \frac{1 - \gamma}{1 - 1/\psi} \]

- What do stochastic discount factors look like?
Stochastic discount factors

Assume $\psi = 1$:

$$U_t^i = (1 - \delta) \log C_t^i + \frac{\delta}{1 - \gamma} \log E_t \left[ \exp \left( 1 - \gamma \right) U_{t+1}^i \right]$$

The stochastic discount factors are

$$\log M_{t+1}^i = \log \frac{\partial U^i / \partial C_{t+1}^i}{\partial U^i / \partial C_t^i}$$

$$= \log \delta + \log \frac{C_t^i}{C_{t+1}^i} + \log \frac{\exp \left\{ (1 - \gamma) U_{t+1}^i \right\}}{E_t \left[ \exp \left\{ (1 - \gamma) U_{t+1}^i \right\} \right]}$$

Brandt, Cochrane and Santa-Clara use:

$$\log M_{t+1}^i = \log \delta + \log \frac{C_t^i}{C_{t+1}^i}$$
Remainder of the economy

- **Home country**

  \[
  \Delta c_h^t = \mu_c + x_{t-1}^h + \sigma \varepsilon_{c,t}^h
  \]

  \[
  x_t^h = \rho x_{t-1}^h + \sigma \varphi \varepsilon_{x,t}^h
  \]

- **Foreign country**

  \[
  \Delta c_f^t = \mu_c + x_{t-1}^f + \sigma \varepsilon_{c,t}^f
  \]

  \[
  x_t^f = \rho x_{t-1}^f + \sigma \varphi \varepsilon_{x,t}^f
  \]

- Shocks are *i.i.d.* within each country

- Shocks are correlated across countries

  - \( \rho_c = \text{corr}(\varepsilon_{c,t}^h, \varepsilon_{c,t}^f) \)

  - \( \rho_x = \text{corr}(\varepsilon_{x,t}^h, \varepsilon_{x,t}^f) \)
### Calibration

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$\gamma$</th>
<th>$\psi$</th>
<th>$\theta$</th>
<th>$\mu_c$</th>
<th>$\sigma$</th>
<th>$\rho$</th>
<th>$\varphi_e$</th>
<th>$\rho_x$</th>
<th>$\rho_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.998</td>
<td>4.25</td>
<td>2</td>
<td>-6.5</td>
<td>0.0015</td>
<td>0.0068</td>
<td>0.987</td>
<td>0.048</td>
<td>1</td>
<td>0.3</td>
</tr>
</tbody>
</table>

\[
m_t^{i+1} = \theta \log \delta - \frac{\theta}{\psi} \Delta c_t^i + (\theta - 1) \log R_{c,t+1}^i
\]

\[
\Delta c_t^i = \mu_c + x_{t-1}^i + \sigma \varepsilon_{c,t}^i
\]

\[
x_t^i = \rho x_{t-1}^i + \sigma \varphi_e \varepsilon_{x,t}^i
\]

**Preferences:**

- Low risk aversion ($\gamma$)
- IES from Bansal, Gallant and Tauchen (2004)
- Monthly model: high discounting
## Calibration

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$\gamma$</th>
<th>$\psi$</th>
<th>$\theta$</th>
<th>$\mu_c$</th>
<th>$\sigma$</th>
<th>$\rho$</th>
<th>$\varphi_e$</th>
<th>$\rho_x$</th>
<th>$\rho_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>.998</td>
<td>4.25</td>
<td>2</td>
<td>-6.5</td>
<td>.0015</td>
<td>.0068</td>
<td>.987</td>
<td>.048</td>
<td>1</td>
<td>.3</td>
</tr>
</tbody>
</table>

$$m_{t+1}^i = \theta \log \delta - \frac{\theta}{\psi} \Delta c_t^i + (\theta - 1) \log R_{c,t+1}^i$$

$$\Delta c_t^i = \mu_c + x_{t-1}^i + \sigma \varepsilon_{c,t}^i$$

$$x_t^i = \rho x_{t-1}^i + \sigma \varphi_e \varepsilon_{x,t}^i$$

### Consumption process:
- Average consumption growth $\approx 2\%$
- Standard deviation of consumption growth $\approx 2.5\%$
- Variance explained by long run risk $\approx 7 - 8\%$
Calibration

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$\gamma$</th>
<th>$\psi$</th>
<th>$\theta$</th>
<th>$\mu_c$</th>
<th>$\sigma$</th>
<th>$\rho$</th>
<th>$\varphi_e$</th>
<th>$\rho_x$</th>
<th>$\rho_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>.998</td>
<td>4.25</td>
<td>2</td>
<td>−6.5</td>
<td>.0015</td>
<td>.0068</td>
<td>.987</td>
<td>.048</td>
<td>1</td>
<td>.3</td>
</tr>
</tbody>
</table>

$$m_{t+1}^i = \theta \log \delta - \frac{\theta}{\psi} \Delta c_t^i + (\theta - 1) \log R_{c,t+1}^i$$

$$\Delta c_t^i = \mu_c + x_{t-1}^i + \sigma \varepsilon_{c,t}^i$$

$$x_t^i = \rho x_{t-1}^i + \sigma \varphi_e \varepsilon_{x,t}^i$$

Cross correlations of shocks:

- Correlation of consumption growths $\approx 0.3$
Three ingredients

- We can solve the puzzle by appropriately combining three ingredients:
We can solve the puzzle by appropriately combining three ingredients:

1. Use Epstein and Zin preferences:
   - BCSC (2005): \( m_{t+1}^i = E \left[ g(\Delta c_{t+1}^i) | I_{t+1} \right] = -\gamma \Delta c_{t+1}^i \)
   - This paper: \( m_{t+1}^i = E \left[ \tilde{g}(\Delta c_{t+1}^i, \Delta c_{t+2}^i, \Delta c_{t+3}^i, ...) | I_{t+1} \right] \)
Three ingredients

- We can solve the puzzle by appropriately combining three ingredients:

1. Use Epstein and Zin preferences:
   - BCSC (2005): \( m_{t+1}^i = E \left[ g(\Delta c_{t+1}^i) | I_{t+1} \right] = -\gamma \Delta c_{t+1}^i \)
   - This paper: \( m_{t+1}^i = E \left[ \tilde{g}(\Delta c_{t+1}^i, \Delta c_{t+2}^i, \Delta c_{t+3}^i, \ldots) | I_{t+1} \right] \)

   Alter the conditional distribution of \((\Delta c^h, \Delta c^f)\):

   \[
   \Delta c_{t+1}^i = \mu_c + x_t^i + \sigma \varepsilon_{c,t+1}^i \\
   x_{t+1}^i = \rho^i x_t^i + \sigma \varphi \varepsilon_{x,t+1}^i
   \]

by assuming

2. High persistence \(\rho^i\)

3. High cross country correlation \(corr \left( \varepsilon_{x,t+1}, \varepsilon_{x,t+1}^f \right)\)
Stochastic discount factors

\[ m_{t+1}^i = \theta \log \delta - \frac{1}{\psi} x_t^i - \gamma \sigma \varepsilon_{c,t+1}^i + \frac{\delta (1 - \gamma \psi)}{\psi (1 - \rho \delta)} \sigma \varphi \varepsilon_x^i, t+1 \]
\[ m_{t+1}^i = \theta \log \delta - \frac{1}{\psi} x_t^i - \gamma \sigma \varepsilon_{c,t+1}^i + \frac{\delta(1-\gamma \psi)}{\psi(1-\rho \delta)} \sigma \varphi \varepsilon_{x,t+1}^i \]
\[ m_{t+1}^i = \theta \log \delta - \frac{1}{\psi} x_t^i - \gamma \sigma \varepsilon_{c,t+1}^i + \frac{\delta (1-\gamma \psi)}{\psi (1-\rho \delta)} \sigma \varphi e \varepsilon_{x,t+1}^i \]
\[ m_{i+1}^t = \theta \log \delta - \frac{1}{\psi} x_t^i - \gamma \sigma \varepsilon_{c,t+1}^i + \frac{\delta (1-\gamma \psi)}{\psi (1-\rho \delta)} \sigma \varphi \varepsilon_x^i + 1 \]
$m_{t+1}^i = \theta \log \delta - \frac{1}{\psi} x_t^i - \gamma \sigma \varepsilon_{c,t+1}^i + \frac{\delta(1-\gamma \psi)}{\psi(1-\rho \delta)} \sigma \varphi \varepsilon_{x,t+1}^i$
\[ m_{t+1}^i = \theta \log \delta - \frac{1}{\psi} x_{t}^i - \gamma \sigma \varepsilon_{c,t+1}^i + \frac{\delta(1-\gamma \psi)}{\psi(1-\rho \delta)} \sigma \psi \varepsilon_{x,t+1}^i \]
Exchange rate depreciation

\[
Var \left( \frac{e_{t+1}}{e_t} \right) = \frac{2(1-\rho_x)}{\psi^2} \left\{ \frac{1}{1-\rho^2} + \left[ \frac{\delta(1-\gamma\psi)}{1-\rho\delta} \right]^2 \right\} \varphi_c^2 \sigma^2 + 2\gamma^2 (1 - \rho_c) \sigma^2
\]
\[
Var \left( \frac{e_{t+1}}{e_t} \right) = \frac{2(1-\rho_x)}{\psi^2} \left\{ \frac{1}{1-\rho^2} + \left[ \frac{\delta(1-\gamma\psi)}{(1-\rho\delta)} \right]^2 \right\} \varphi_e^2 \sigma^2 + 2\gamma^2 (1-\rho_c) \sigma^2
\]
\[ \text{Var} \left( \frac{e_{t+1}}{e_t} \right) = \frac{2(1-\rho_x)}{\psi^2} \left\{ \frac{1}{1-\rho^2} + \left[ \frac{\delta(1-\gamma\psi)}{(1-\rho\delta)} \right]^2 \right\} \phi_e^2 \sigma^2 + 2\gamma^2 (1 - \rho_c) \sigma^2 \]
$$\text{Var} \left( \frac{e_{t+1}}{e_t} \right) = \frac{2(1-\rho_x)}{\psi^2} \left\{ \frac{1}{1-\rho^2} + \left[ \frac{\delta(1-\gamma \psi)}{1-\rho \delta} \right]^2 \right\} \varphi e^2 \sigma^2 + 2\gamma^2 (1-\rho_c) \sigma^2$$
Every assumption counts

Ingredients needed to solve the puzzle:

1. Disentangle elasticity of substitution from risk aversion
2. Highly persistent predictable component
3. Highly correlated predictable components
Every assumption counts

- Ingredients needed to solve the puzzle:
  1. Disentangle elasticity of substitution from risk aversion
  2. Highly persistent predictable component
  3. Highly correlated predictable components

- Can we match key moments of international financial markets?
Introducing dividends

- The system becomes

\[ \Delta c_t^i = \mu_c + x_{t-1}^i + \sigma \varepsilon_{c,t}^i \]
\[ \Delta d_t^i = \mu_d + \lambda x_{t-1}^i + \sigma \varphi \varepsilon_{d,t}^i \]
\[ x_t^i = \rho x_{t-1}^i + \sigma \varphi \varepsilon_{x,t}^i \]

\[ \forall i \in \{h, f\} \]

- Shocks are \textit{i.i.d.} within each country
- Shocks are correlated across countries
Introducing dividends

The system becomes

\[
\Delta c_t^i = \mu_c + x_{t-1}^i + \sigma \varepsilon_{c,t}^i \\
\Delta d_t^i = 0.0007 + 3 \cdot x_{t-1}^i + \sigma \cdot 5 \cdot \varepsilon_{d,t}^i \\
x_t^i = \rho x_{t-1}^i + \sigma \varphi \varepsilon_{x,t}^i
\]

\( \forall i \in \{h, f\} \)

- Shocks are i.i.d. within each country
- Shocks are correlated across countries
- Calibrate coefficients of dividend growth to match:
  - Average dividend growth \( \approx 1\% \)
  - Standard deviation of dividend growth \( \approx 12\% \)
  - Leverage is 5
  - Small correlation of dividend growths: \( corr \left( \varepsilon_{d,t}^h, \varepsilon_{d,t}^f \right) \approx 0 \)
### Introducing dividends: results

<table>
<thead>
<tr>
<th></th>
<th>US</th>
<th>UK</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho(m^h, m^f)$</td>
<td>-</td>
<td>-</td>
<td>0.93</td>
</tr>
<tr>
<td>$\sigma(e_{t+1}/e_t)$</td>
<td>11.21</td>
<td>11.83</td>
<td></td>
</tr>
<tr>
<td>$E(r_d - r_f)$</td>
<td>7.02</td>
<td>9.17</td>
<td>7.01</td>
</tr>
<tr>
<td>$\sigma(r_d - r_f)$</td>
<td>17.13</td>
<td>22.83</td>
<td>19.60</td>
</tr>
<tr>
<td>$\rho(r_d^h - r_f^h, r_d^f - r_f^f)$</td>
<td>0.60</td>
<td>0.58</td>
<td></td>
</tr>
<tr>
<td>$E(r_f)$</td>
<td>1.47</td>
<td>1.62</td>
<td>1.33</td>
</tr>
<tr>
<td>$\sigma(r_f)$</td>
<td>1.53</td>
<td>2.92</td>
<td>1.19</td>
</tr>
<tr>
<td>$\rho(r_f^h, r_f^f)$</td>
<td>0.65</td>
<td>1.00</td>
<td></td>
</tr>
</tbody>
</table>
Introducing stochastic volatility

- The system becomes $\forall i \in \{h, f\}$

\[
\begin{align*}
\Delta c_t^i &= \mu_c + x_{t-1}^i + \sigma_t \varepsilon_{c,t}^i \\
\Delta d_t^i &= \mu_d + \lambda x_{t-1}^i + \sigma_t \varphi_d \varepsilon_{c,t}^i \\
x_t^i &= \rho x_{t-1}^i + \sigma_t \varphi e_{x,t}^i \\
(\sigma_t^2)^i &= \overline{\sigma}^2 + \nu_1 \left[ (\sigma_{t-1}^2)^i - \overline{\sigma}^2 \right] + \sigma_w \varepsilon_{\sigma,t}^i
\end{align*}
\]

- Shocks are i.i.d. within each country
- Shocks are correlated across countries
- Guidelines to calibrate stochastic volatility given by

\[
Var_t \left( r_{t+1}^d \right) = (1 - \nu_1)k_0 + \nu_1 Var_{t-1} \left( r_t^d \right) + k_1 \sigma_w \varepsilon_{\sigma,t}
\]

Risks for the Long Run and the Real Exchange Rate, UCLA, 2.22.06 – p. 17/29
Introducing stochastic volatility

- The system becomes $\forall i \in \{h, f\}$

\[
\begin{align*}
\Delta c_t^i &= \mu_c + x_{t-1}^i + \sigma_t \varepsilon_{c,t}^i \\
\Delta d_t^i &= \mu_d + \lambda x_{t-1}^i + \sigma_t \varphi_d \varepsilon_{c,t}^i \\
x_t^i &= \rho x_{t-1}^i + \sigma_t \varphi_e \varepsilon_{x,t}^i \\
(\sigma_t^2)^i &= .0068^2 + .96 \left[ (\sigma_{t-1}^2)^i - .0068^2 \right] + .23 e^{-5} \varepsilon_{\sigma,t}^i
\end{align*}
\]

- Shocks are i.i.d. within each country
- Shocks are correlated across countries
- Guidelines to calibrate stochastic volatility given by

\[
Var_t \left( r_{t+1}^d \right) = \left( 1 - \nu_1 \right) k_0 + \nu_1 Var_{t-1} \left( r_t^d \right) + k_1 \sigma_w \varepsilon_{\sigma,t}
\]
Introducing stochastic volatility

The system becomes \( \forall i \in \{h, f\} \)

\[
\begin{align*}
\Delta c^i_t &= \mu_c + x^i_{t-1} + \sigma_t \varepsilon^i_{c,t} \\
\Delta d^i_t &= \mu_d + \lambda x^i_{t-1} + \sigma_t \varphi_d \varepsilon^i_{c,t} \\
x^i_t &= \rho x^i_{t-1} + \sigma_t \varphi_e \varepsilon^i_{x,t} \\
(\sigma^2_t)^i &= \cdot068^2 + .96 \left((\sigma^2_{t-1})^i - \cdot068^2\right) + .23e^{-5} \varepsilon^i_{\sigma,t}
\end{align*}
\]

- Shocks are \textit{i.i.d.} within each country
- Shocks are correlated across countries
- Guidelines to calibrate stochastic volatility given by

\[
Var_t \left(r^d_{t+1}\right) = (1 - \nu_1)k_0 + \nu_1 Var_{t-1} \left(r^d_t\right) + k_1 \sigma_w \varepsilon_{\sigma,t}
\]

- Cross correlation of \( \varepsilon_{\sigma,t} \) has small impact on results.
## Results

<table>
<thead>
<tr>
<th></th>
<th>US</th>
<th>UK</th>
<th>No stoch vol</th>
<th>W/Stoch vol</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho (m^h, m^f)$</td>
<td>-</td>
<td>-</td>
<td>0.93</td>
<td>0.92</td>
</tr>
<tr>
<td>$\sigma \left( \frac{e_{t+1}}{e_t} \right)$</td>
<td>11.21</td>
<td>11.83</td>
<td>12.67</td>
<td></td>
</tr>
<tr>
<td>$E (r_d - r_f)$</td>
<td>7.02</td>
<td>9.17</td>
<td>7.01</td>
<td>7.03</td>
</tr>
<tr>
<td>$\sigma (r_d - r_f)$</td>
<td>17.13</td>
<td>22.83</td>
<td>19.60</td>
<td>19.41</td>
</tr>
<tr>
<td>$\rho \left( r^h_d - r^h_f, r^f_d - r^f_f \right)$</td>
<td>0.60</td>
<td>0.58</td>
<td>0.57</td>
<td></td>
</tr>
<tr>
<td>$E (r_f)$</td>
<td>1.47</td>
<td>1.62</td>
<td>1.33</td>
<td>1.33</td>
</tr>
<tr>
<td>$\sigma (r_f)$</td>
<td>1.53</td>
<td>2.92</td>
<td>1.19</td>
<td>1.22</td>
</tr>
<tr>
<td>$\rho (r^h_f, r^f_f)$</td>
<td>0.65</td>
<td></td>
<td>1.00</td>
<td>0.98</td>
</tr>
<tr>
<td>$\sigma (r_c)$</td>
<td>-</td>
<td>-</td>
<td>4.74</td>
<td>4.75</td>
</tr>
<tr>
<td>$\rho (r^h_c, r^f_c)$</td>
<td>-</td>
<td>-</td>
<td>0.85</td>
<td>0.85</td>
</tr>
</tbody>
</table>
Can we justify high persistence and high correlation:

\[
\begin{bmatrix}
\Delta c^h_t \\
\Delta c^f_t
\end{bmatrix} =
\begin{bmatrix}
\mu^h_c & \\
\mu^f_c
\end{bmatrix} +
\begin{bmatrix}
x^h_{t-1} \\
x^f_{t-1}
\end{bmatrix} +
\begin{bmatrix}
1 & 0 \\
\rho_c & \sqrt{1 - \rho^2_c}
\end{bmatrix}
\begin{bmatrix}
\sigma^h_{\varepsilon^h_{c,t}} \\
\sigma^f_{\varepsilon^f_{c,t}}
\end{bmatrix}
\]

\[
\begin{bmatrix}
x^h_t \\
x^f_t \\
x_t
\end{bmatrix} =
\begin{bmatrix}
\rho^h & 0 \\
0 & \rho^f
\end{bmatrix}
\begin{bmatrix}
x^h_{t-1} \\
x^f_{t-1} \\
x_{t-1}
\end{bmatrix} +
\begin{bmatrix}
1 & 0 \\
\rho_x & \sqrt{1 - \rho^2_x}
\end{bmatrix}
\begin{bmatrix}
\sigma^h_{\varepsilon^h_{x,t}} \\
\sigma^f_{\varepsilon^f_{x,t}}
\end{bmatrix}
\]
Estimating long run risks

Can we justify high persistence and high correlation:

\[
\begin{bmatrix}
\Delta c^h_t \\
\Delta c^f_t
\end{bmatrix}
= \begin{bmatrix}
\mu^h_c \\
\mu^f_c
\end{bmatrix}
+ \begin{bmatrix}
x^h_{t-1} \\
x^f_{t-1}
\end{bmatrix}
+ \begin{bmatrix}
1 & 0 \\
\rho_c & \sqrt{1 - \rho_c^2}
\end{bmatrix}
\begin{bmatrix}
\sigma^h \epsilon^h_{c,t} \\
\sigma^f \epsilon^f_{c,t}
\end{bmatrix}
\]

\[
\begin{bmatrix}
x^h_t \\
x^f_t
\end{bmatrix}
= \begin{bmatrix}
\rho^h & 0 \\
0 & \rho^f
\end{bmatrix}
\begin{bmatrix}
x^h_{t-1} \\
x^f_{t-1}
\end{bmatrix}
+ \begin{bmatrix}
1 & 0 \\
\rho_x & \sqrt{1 - \rho_x^2}
\end{bmatrix}
\begin{bmatrix}
\sigma^h \varphi^h_{e,x,t} \\
\sigma^f \varphi^f_{e,x,t}
\end{bmatrix}
\]

Roadmap:

1. Use consumption data only
   - Use Kalman filter to get a recursive representation of the likelihood function
   - Multi-country provide inconclusive evidence
Estimating long run risks

- Can we justify high persistence and high correlation:

\[
\begin{bmatrix}
\Delta c^h_t \\
\Delta c^f_t
\end{bmatrix} = \begin{bmatrix}
\mu^h_c \\
\mu^f_c
\end{bmatrix} + \begin{bmatrix}
x^h_{t-1} \\
x^f_{t-1}
\end{bmatrix} + \begin{bmatrix}
1 & 0 \\
\rho_c & \sqrt{1 - \rho_c^2}
\end{bmatrix} \begin{bmatrix}
\sigma^h e^h_{c,t} \\
\sigma^f e^f_{c,t}
\end{bmatrix}
\]

\[
\begin{bmatrix}
x^h_t \\
x^f_t
\end{bmatrix} = \begin{bmatrix}
\rho^h & 0 \\
0 & \rho^f
\end{bmatrix} \begin{bmatrix}
x^h_{t-1} \\
x^f_{t-1}
\end{bmatrix} + \begin{bmatrix}
1 & 0 \\
\rho_x & \sqrt{1 - \rho_x^2}
\end{bmatrix} \begin{bmatrix}
\sigma^h \phi^h e^h_{x,t} \\
\sigma^f \phi^f e^f_{x,t}
\end{bmatrix}
\]

- Roadmap:
  1. Use consumption data only
     - Use Kalman filter to get a recursive representation of the likelihood function
     - Multi-country provide inconclusive evidence
  2. Use consumption and price data
     - Focus on a set of moments of interest
     - Sharply identify departure from i.i.d.
Consumption data only: results

- Home = US and Foreign = UK
Consumption data only: results

- Home = US and Foreign = UK

<table>
<thead>
<tr>
<th></th>
<th>$\rho^h$</th>
<th>$\rho^f$</th>
<th>$\rho_x$</th>
<th>$\rho_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calibrated</td>
<td>0.987</td>
<td>0.987</td>
<td>1.000</td>
<td>0.300</td>
</tr>
<tr>
<td>Real Data (T=120)</td>
<td>0.909</td>
<td>0.940</td>
<td>0.897</td>
<td>0.208</td>
</tr>
<tr>
<td></td>
<td>[0.547,0.995]</td>
<td>[0.308,0.995]</td>
<td>[0.696,1.000]</td>
<td>[0.004,0.406]</td>
</tr>
</tbody>
</table>
Home = US and Foreign = UK


<table>
<thead>
<tr>
<th></th>
<th>$\rho^h$</th>
<th>$\rho^f$</th>
<th>$\rho_x$</th>
<th>$\rho_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calibrated</td>
<td>0.987</td>
<td>0.987</td>
<td>1.000</td>
<td>0.300</td>
</tr>
<tr>
<td>Real Data (T=120)</td>
<td>[0.547,0.995]</td>
<td>[0.308,0.995]</td>
<td>[0.696,1.000]</td>
<td>[0.004,0.406]</td>
</tr>
<tr>
<td>Simulations (T=120)</td>
<td>[0.731,1.000]</td>
<td>[0.630,1.000]</td>
<td>[0.467,1.0]</td>
<td>[0.066,0.530]</td>
</tr>
</tbody>
</table>
Consumption data only: results

- Home = US and Foreign = UK

<table>
<thead>
<tr>
<th></th>
<th>$\rho^h$</th>
<th>$\rho^f$</th>
<th>$\rho_x$</th>
<th>$\rho_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calibrated</td>
<td>0.987</td>
<td>0.987</td>
<td>1.000</td>
<td>0.300</td>
</tr>
<tr>
<td>Real Data</td>
<td>0.909</td>
<td>0.940</td>
<td>0.897</td>
<td>0.208</td>
</tr>
<tr>
<td>(T=120)</td>
<td>[0.547,0.995]</td>
<td>[0.308,0.995]</td>
<td>[0.696,1.000]</td>
<td>[0.004,0.406]</td>
</tr>
<tr>
<td>Simulations</td>
<td>0.941</td>
<td>0.934</td>
<td>0.844</td>
<td>0.312</td>
</tr>
<tr>
<td>(T=120)</td>
<td>[0.731,1.000]</td>
<td>[0.630,1.000]</td>
<td>[0.467,1.0]</td>
<td>[0.066,0.530]</td>
</tr>
<tr>
<td>Simulations</td>
<td>0.987</td>
<td>0.987</td>
<td>0.986</td>
<td>0.302</td>
</tr>
<tr>
<td>(T=10000)</td>
<td>[0.983,0.990]</td>
<td>[0.983,0.991]</td>
<td>[0.953,1.0]</td>
<td>[0.285,0.318]</td>
</tr>
</tbody>
</table>
Consumption data only: results (cont’d)

... other parameters

<table>
<thead>
<tr>
<th></th>
<th>$\varphi^h_e$</th>
<th>$\varphi^f_e$</th>
<th>$\sigma^h$</th>
<th>$\sigma^f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calibrated</td>
<td>0.048</td>
<td>0.048</td>
<td>68</td>
<td>68</td>
</tr>
<tr>
<td>Real Data (T=120)</td>
<td>0.351</td>
<td>0.184</td>
<td>38.1</td>
<td>77.9</td>
</tr>
<tr>
<td></td>
<td>[0.000,0.424]</td>
<td>[0.000,1.092]</td>
<td>[28.6,38.1]</td>
<td>[72.3,78.0]</td>
</tr>
<tr>
<td>Simulations</td>
<td>0.142</td>
<td>0.119</td>
<td>66.1</td>
<td>66.1</td>
</tr>
<tr>
<td>(T=120)</td>
<td>[0.000,0.475]</td>
<td>[0.000,1.165]</td>
<td>[51.2,71.6]</td>
<td>[37.4,68.2]</td>
</tr>
<tr>
<td>Simulations</td>
<td>0.049</td>
<td>0.049</td>
<td>67.9</td>
<td>68.0</td>
</tr>
<tr>
<td>(T=10000)</td>
<td>[0.039,0.059]</td>
<td>[0.041,0.057]</td>
<td>[67.1,69.1]</td>
<td>[67.3,68.7]</td>
</tr>
</tbody>
</table>
Does adding other countries help?

<table>
<thead>
<tr>
<th></th>
<th>US, UK and Germany</th>
<th>US, UK and Japan</th>
<th>US, UK, Germany and Japan</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho^{US}$</td>
<td>0.911</td>
<td>0.906</td>
<td>0.919</td>
</tr>
<tr>
<td></td>
<td>[.530, .990]</td>
<td>[.552, .984]</td>
<td>[.796, .996]</td>
</tr>
<tr>
<td>$\rho^{UK}$</td>
<td>0.928</td>
<td>0.947</td>
<td>0.934</td>
</tr>
<tr>
<td>$\rho^{Ger}$</td>
<td>0.932</td>
<td>-</td>
<td>0.993</td>
</tr>
<tr>
<td></td>
<td>[.274, .985]</td>
<td>-</td>
<td>[.614, .996]</td>
</tr>
<tr>
<td>$\rho^{Jpn}$</td>
<td>-</td>
<td>0.989</td>
<td>0.99</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>[.215, .989]</td>
<td>[.504, .998]</td>
</tr>
<tr>
<td>$\rho_{US,UK}$</td>
<td>0.920</td>
<td>0.973</td>
<td>0.913</td>
</tr>
<tr>
<td></td>
<td>[.890, .999]</td>
<td>[.861, .999]</td>
<td>[.700, .999]</td>
</tr>
<tr>
<td>$\rho_{US,Ger}$</td>
<td>0.899</td>
<td>-</td>
<td>0.891</td>
</tr>
<tr>
<td></td>
<td>[.874, 1.000]</td>
<td>-</td>
<td>[.700, .997]</td>
</tr>
<tr>
<td>$\rho_{US,Jpn}$</td>
<td>-</td>
<td>0.991</td>
<td>0.905</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>[.877, 1.000]</td>
<td>[.701, 1.000]</td>
</tr>
</tbody>
</table>
Introducing prices

- Consumption information is not enough: we need price information.
- Select 22 moments to match:
  - Consumption: \( \text{Var} \left( \Delta c^h_t \right), \text{cov} \left( \Delta c^h_t, \Delta c^h_{t-1} \right), \text{cov} \left( \Delta c^h_t, \Delta c^h_{t-2} \right), \text{Var} \left( \Delta c^f_t \right), \text{cov} \left( \Delta c^f_t, \Delta c^f_{t-1} \right), \text{cov} \left( \Delta c^f_t, \Delta c^f_{t-2} \right), \text{cov} \left( \Delta c^h_t, \Delta c^f_t \right), \text{cov} \left( \Delta c^h_t, \Delta c^f_{t-1} \right), \text{cov} \left( \Delta c^h_{t-1}, \Delta c^f_t \right) \)
  - Dividend: \( \text{Var} \left( \Delta d^h_t \right), \text{Var} \left( \Delta d^f_t \right), \text{cov} \left( \Delta d^h_t, \Delta d^f_t \right) \)
  - Excess returns: \( \text{Var} \left( r^d,h_t - r^f,h_t \right), \text{cov} \left( r^d,h_t - r^f,h_t, r^d,h_{t-1} - r^f,h_{t-1} \right), \text{Var} \left( r^d,f_t - r^f,f_t \right), \text{cov} \left( r^d,f_t - r^f,f_t, r^d,f_{t-1} - r^f,f_{t-1} \right), \text{cov} \left( r^d,h_t - r^f,h_t, r^d,f_t - r^f,f_t \right) \)
  - Risk free rates: \( \text{Var} \left( r^f,h_t \right), \text{Var} \left( r^f,f_t \right) \)
  - Depreciation rate: \( \text{Var} \left( \frac{e_{t+1}}{e_t} \right) \)
- Use Simulated Method of Moments
## Prices and consumption: results

<table>
<thead>
<tr>
<th></th>
<th>Consumption only</th>
<th>Whole Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Point Estimate</td>
<td>95% CI</td>
</tr>
<tr>
<td>$\rho^h$</td>
<td>0.736 [0.349, 0.996]</td>
<td>0.997 [0.927, 1.000]</td>
</tr>
<tr>
<td>$\rho^f$</td>
<td>0.904 [0.015, 0.997]</td>
<td>0.996 [0.912, 0.999]</td>
</tr>
<tr>
<td>$\varphi^h_e$</td>
<td>1.422 [0.190, 17.318]</td>
<td>0.024 [0.005, 0.213]</td>
</tr>
<tr>
<td>$\varphi^f_e$</td>
<td>0.182 [0.000, 3.502]</td>
<td>0.041 [0.003, 0.157]</td>
</tr>
<tr>
<td>$\sigma^h$</td>
<td>27.629 [4.527, 34.799]</td>
<td>32.792 [28.036, 36.676]</td>
</tr>
<tr>
<td>$\sigma^f$</td>
<td>79.916 [44.407, 87.421]</td>
<td>79.569 [69.767, 88.879]</td>
</tr>
<tr>
<td>$\rho^{hf}$</td>
<td>0.999 [0.353, 1.000]</td>
<td>0.998 [0.853, 1.000]</td>
</tr>
<tr>
<td>$\sigma \left( \frac{e_{t+1}}{e_t} \right)$</td>
<td>-</td>
<td>11.692</td>
</tr>
<tr>
<td>$\rho \left( m^h, m^f \right)$</td>
<td>-</td>
<td>0.922</td>
</tr>
</tbody>
</table>
What’s next?

- Extend the list of moments that can be matched
- Relax assumption of complete home bias in consumption
- Where does $x_t$ come from?
Extending the basic model

- Set $\Psi = 1$ (risk sensitive preferences):

$$U^i_t = \log c^i_t + \frac{\delta}{(1 - \gamma^i)(1 - \delta)} \log E_t \left[ \exp \left\{ (1 - \gamma^i)(1 - \delta)U^i_{t+1} \right\} \right]$$

- Two common factors

  $\Delta c^h_t = \mu^{c,h} + \lambda_1 z^1_{t-1} + \lambda_2 z^2_{t-1} + \varepsilon^{c,h}_t$

  $\Delta c^f_t = \mu^{c,f} + \lambda_2 z^1_{t-1} + \lambda_1 z^2_{t-1} + \varepsilon^{c,f}_t$

  $z^1_{t-1} = \rho_1 z^1_{t-1} + \varepsilon^1_t$

  $z^2_{t-1} = \rho_2 z^2_{t-1} + \varepsilon^2_t$

- If $\lambda_1 = 1$, $\lambda_2 = 0$, $\rho \left( \varepsilon^{c,h}_t, \varepsilon^{c,f}_t \right) = 0.3$, $\rho \left( \varepsilon^1_t, \varepsilon^2_t \right) = 1$, $\rho \left( \varepsilon^{c,h}_t, \varepsilon^1_t \right) = 0$, $\rho \left( \varepsilon^{c,f}_t, \varepsilon^2_t \right) = 0$, then we have the basic model.
Two factors

- One factor: yields are perfectly correlated.
Two factors

- Two factors: low \( \text{corr}(r_f^h, r_f^f) \) and high \( \text{corr}(m^h, m^f) \).
In this model: \( r_t^{fi} = k_0 + k_1 z_{1,t} + k_2 z_{2,t} \)
In this model: $r_t^i = k_0 + k_1 z_{1,t} + k_2 z_{2,t}$
Concluding remarks

Key ingredients
- Separate elasticity of substitution from risk aversion
- Highly persistent predictable component
- Highly correlated predictable components

It is possible to explain
- Low volatility of the depreciation of the US dollar
- High equity premium
- High persistence of the risk free rate
- High correlation of int’l financial markets
- Correlation of bonds
- Low correlation of consumption growths
- Low persistence of consumption growths