International Asset Pricing with Risk-Sensitive Agents

Riccardo Colacito and Mariano Max Croce

University of Minnesota, 12/7/2010
Motivation

Established empirical facts

- Volatility of exchange rates is time-varying
- Correlation of int’l stock market returns is time-varying
- Equity risk premia are time-varying
- Volatility of consumption growth is time-varying
- ...

...
Established empirical facts

- Volatility of exchange rates is time-varying
- Correlation of int’l stock market returns is time-varying
- Equity risk premia are time-varying
- Volatility of consumption growth is time-varying
- ...

Established theoretical fact: a model with

- complete markets
- time-additive preferences
- \( i.i.d. \) homoscedastic endowments

fails to replicate empirical findings...
Motivation

Established empirical facts

- Volatility of exchange rates is time-varying
- Correlation of int’l stock market returns is time-varying
- Equity risk premia are time-varying
- Volatility of consumption growth is time-varying
- ...

Established theoretical fact: a model with

- complete markets ✓
- time-additive preferences
- \textit{i.i.d.} homoscedastic endowments ✓
Motivation

Established empirical facts

- Volatility of exchange rates is time-varying
- Correlation of int’l stock market returns is time-varying
- Equity risk premia are time-varying
- Volatility of consumption growth is time-varying
- ...

Established theoretical fact: a model with

- complete markets ✓
- time-additive preferences recursive preferences
- \( i.i.d. \) homoscedastic endowments ✓

can replicate empirical findings...
Endogenous heteroscedasticity and predictability

\[ \sigma_t(\Delta e_{t+1}) \]

\[ E_t(r_{t+1}c - r_t) \]

\[ \sigma_t(\Delta c_{t+1}) \]

\[ \rho_t(\Delta c_{t+1}h, \Delta c_{t+1}f) \]

\[ 500 \, 1000 \, 1500 \, 2000 \, 2500 \, 3000 \, 3500 \, 4000 \, 4500 \, 5000 \]

periods
Roadmap

- Setup of the economy
Roadmap

- Setup of the economy
  1. Two country - complete markets economy
  2. Two goods: endowments are \textit{i.i.d.} homoscedastic
Roadmap

Setup of the economy

1. Two country - complete markets economy
2. Two goods: endowments are $i.i.d.$ homoscedastic
3. Home-bias toward the consumption of domestic good
Roadmap

Setup of the economy

1. Two country - complete markets economy
2. Two goods: endowments are \( i.i.d. \) homoscedastic
3. Home-bias toward the consumption of domestic good
4. Agents have risk-sensitive preferences
Roadmap

Setup of the economy

1. Two country - complete markets economy
2. Two goods: endowments are *i.i.d*.
   homoscedastic
3. Home-bias toward the consumption of domestic good
4. Agents have risk-sensitive preferences

Risk Sharing Scheme

- Social planner’s problem: *allocations* and time-varying Pareto weights
Roadmap

Setup of the economy

1. Two country - complete markets economy
2. Two goods: endowments are \( i.i.d. \) homoscedastic
3. Home-bias toward the consumption of domestic good
4. Agents have risk-sensitive preferences

Risk Sharing Scheme

- Social planner’s problem: allocations and time-varying Pareto weights
- Decentralized equilibrium and state-prices
Roadmap

Setup of the economy

1. Two country - complete markets economy
2. Two goods: endowments are \( i.i.d. \) homoscedastic
3. Home-bias toward the consumption of domestic good
4. Agents have risk-sensitive preferences

Risk Sharing Scheme

- Social planner’s problem: \textit{allocations} and time-varying Pareto weights
- Decentralized equilibrium and state-\textit{prices}
- \textbf{Endogenous} time-varying second moments
Roadmap

Setup of the economy

1. Two country - complete markets economy
2. Two goods: endowments are \textit{i.i.d.} homoscedastic
3. Home-bias toward the consumption of domestic good
4. Agents have risk-sensitive preferences

Risk Sharing Scheme

- Social planner’s problem: \textit{allocations} and time-varying Pareto weights
- Decentralized equilibrium and state-\textit{prices}
- \textbf{Endogenous} time-varying second moments

International Asset Pricing

5. Small probability of large endowment drop
Roadmap

Setup of the economy

1. Two country - complete markets economy
2. Two goods: endowments are \( i.i.d. \) homoscedastic
3. Home-bias toward the consumption of domestic good
4. Agents have risk-sensitive preferences

Risk Sharing Scheme

- Social planner’s problem: \textit{allocations} and time-varying Pareto weights
- Decentralized equilibrium and state-\textit{prices}
- \textbf{Endogenous} time-varying second moments

International Asset Pricing

5. Small probability of large endowment drop

Qualitative implications of the model
Setup of the Economy
Agents have risk-sensitive preferences

\[ U_{i,t} = (1 - \delta) \log C_{i,t} + \delta \theta \log E_t \exp \left\{ \frac{U_{i,t+1}}{\theta} \right\}, \quad \forall i \in \{h, f\} \]

where \( \theta = 1 / (1 - \gamma) \).
Setup of the economy

- Agents have risk-sensitive preferences

\[ U_{i,t} = (1 - \delta) \log C_{i,t} + \delta E_t[U_{i,t+1}], \quad \forall i \in \{h, f\} \]

where \( \theta = 1 / (1 - \gamma) \). If \( \theta \to -\infty \): time additive case.
Agents have risk-sensitive preferences

\[ U_{i,t} = (1 - \delta) \log C_{i,t} + \delta \theta \log E_t \exp \left\{ \frac{U_{i,t+1}}{\theta} \right\}, \quad \forall i \in \{h, f\} \]

where \( \theta = 1 / (1 - \gamma) \).
Agents have risk-sensitive preferences

\[ U_{i,t} \approx (1 - \delta) \log C_{i,t} + \delta E_t[U_{i,t+1}] + \frac{\delta}{2\theta} V_t[U_{i,t+1}], \quad \forall i \in \{h, f\} \]

where \( \theta = 1 / (1 - \gamma) \). Conditional Variance matters.
Setup of the economy

- Agents have risk-sensitive preferences

\[ U_{i,t} = (1 - \delta) \log C_{i,t} + \delta \theta \log E_t \exp \left\{ \frac{U_{i,t+1}}{\theta} \right\}, \quad \forall i \in \{h, f\} \]

where \( \theta = 1 / (1 - \gamma) \).
Setup of the economy

- Agents have risk-sensitive preferences

\[ U_{i,t} = (1 - \delta) \log C_{i,t} + \delta \theta \log E_t \exp \left\{ \frac{U_{i,t+1}}{\theta} \right\}, \quad \forall i \in \{h, f\} \]

where \( \theta = 1 / (1 - \gamma) \).

- Preferences are defined over the consumption aggregate

\[
C_{h,t} = (x_{h,t})^\alpha (y_{h,t})^{1-\alpha} \quad \text{and} \quad C_{f,t} = (x_{f,t})^{1-\alpha} (y_{f,t})^\alpha
\]
Agents have risk-sensitive preferences

\[ U_{i,t} = (1 - \delta) \log C_{i,t} + \delta \theta \log E_t \exp \left( \frac{U_{i,t+1}}{\theta} \right), \quad \forall i \in \{h, f\} \]

where \( \theta = 1 / (1 - \gamma) \).

Preferences are defined over the consumption aggregate

\[ C_{h,t} = (x_{h,t})^\alpha (y_{h,t})^{1-\alpha} \quad \text{and} \quad C_{f,t} = (x_{f,t})^{1-\alpha} (y_{f,t})^\alpha \]

Consumption bias: \( \alpha > 1/2 \).
Setup of the economy

- Agents have risk-sensitive preferences

\[ U_{i,t} = (1 - \delta) \log C_{i,t} + \delta \theta \log E_t \exp \left\{ \frac{U_{i,t+1}}{\theta} \right\}, \quad \forall i \in \{h, f\} \]

where \( \theta = 1 / (1 - \gamma) \).

- Preferences are defined over the consumption aggregate

\[ C_{h,t} = (x_{h,t})^\alpha (y_{h,t})^{1-\alpha} \quad \text{and} \quad C_{f,t} = (x_{f,t})^{1-\alpha} (y_{f,t})^\alpha \]

- Consumption bias: \( \alpha > 1/2 \).

- Complete markets.
Setup of the economy

- Agents have risk-sensitive preferences

\[ U_{i,t} = (1 - \delta) \log C_{i,t} + \delta \log E_t \exp \left\{ \frac{U_{i,t+1}}{\theta} \right\}, \quad \forall i \in \{h, f\} \]

where \( \theta = 1 / (1 - \gamma) \).

- Preferences are defined over the consumption aggregate

\[ C_{h,t} = (x_{h,t})^\alpha (y_{h,t})^{1-\alpha} \quad \text{and} \quad C_{f,t} = (x_{f,t})^{1-\alpha} (y_{f,t})^\alpha \]

- Consumption bias: \( \alpha > 1/2 \).

- Complete markets.

- Endowments are \( i.i.d. \) homoscedastic.
Setup of the economy

- Agents have risk-sensitive preferences

\[ U_{i,t} = (1 - \delta) \log C_{i,t} + \delta \theta \log E_t \exp \left\{ \frac{U_{i,t+1}}{\theta} \right\}, \quad \forall i \in \{h, f\} \]

where \( \theta = 1 / (1 - \gamma) \).

- Preferences are defined over the consumption aggregate

\[ C_{h,t} = (x_{h,t})^\alpha (y_{h,t})^{1-\alpha} \quad \text{and} \quad C_{f,t} = (x_{f,t})^{1-\alpha} (y_{f,t})^\alpha \]

- Consumption bias: \( \alpha > 1/2 \).

- Complete markets.

- Endowments are i.i.d. homoscedastic

\( \text{Two states: } HL = \{X = 103, Y = 100\} \text{ and } LH = \{X = 100, Y = 103\} \)
Setup of the economy

- Agents have risk-sensitive preferences

\[ U_{i,t} = (1 - \delta) \log C_{i,t} + \delta \theta \log E_t \exp \left\{ \frac{U_{i,t+1}}{\theta} \right\}, \quad \forall i \in \{h, f\} \]

where \( \theta = 1 / (1 - \gamma) \).

- Preferences are defined over the consumption aggregate

\[ C_{h,t} = (x_{h,t})^\alpha (y_{h,t})^{1-\alpha} \quad \text{and} \quad C_{f,t} = (x_{f,t})^{1-\alpha} (y_{f,t})^\alpha \]

- Consumption bias: \( \alpha > 1/2 \).

- Complete markets.

- Endowments are *i.i.d.* homoscedastic

1. Two states: \( HL = \{X = 103, Y = 100\} \) and \( LH = \{X = 100, Y = 103\} \)
2. Introduce rare events
Risk Sharing Scheme:
Allocations
Planner’s problem

Efficient allocations are the solution to the planner’s problem

choose $\{x_{h,t}, x_{f,t}, y_{h,t}, y_{f,t}\}_{t=0}^{+\infty}$

to max $Q = \mu_h U_{h,0} + \mu_f U_{f,0}$

s.t. $x_{h,t} + x_{f,t} = X_t$

$y_{h,t} + y_{f,t} = Y_t, \quad \forall t \geq 0$
Planner’s problem

Efficient allocations are the solution to the planner’s problem

\[ \begin{align*}
\text{choose} & \quad \{x_{h,t}, x_{f,t}, y_{h,t}, y_{f,t}\}_{t=0}^{+\infty} \\
\text{to max} & \quad Q = \mu_h U_{h,0} + \mu_f U_{f,0} \\
\text{s.t.} & \quad x_{h,t} + x_{f,t} = X_t \\
& \quad y_{h,t} + y_{f,t} = Y_t, \quad \forall t \geq 0
\end{align*} \]

- \( \mu_h \) and \( \mu_f \) correspond to an initial distribution of assets.
- Notation: \( S = \mu_h / \mu_f \).
What to look for...

\[
\mu_h \cdot \frac{\alpha}{x_{h,t}^j} = \frac{1 - \alpha}{x_{f,t}^j} \cdot \mu_f
\]
What to look for...

\[
\frac{\exp \left\{ U_{h,t}^i / \theta \right\}}{E_{t-1} \left[ \exp \left\{ U_{h,t} / \theta \right\} \right]} \cdot \mu_h \cdot \frac{\alpha}{x_h,t} = \frac{1 - \alpha}{x_h^i} \cdot \mu_f \cdot \frac{\exp \left\{ U_{f,t}^j / \theta \right\}}{E_{t-1} \left[ \exp \left\{ U_{f,t} / \theta \right\} \right]}
\]
What to look for...

\[
\frac{\exp\left\{ U_{h,t} / \theta \right\}}{E_{t-1} \left[ \exp\left\{ U_{h,t} / \theta \right\} \right]} \cdot \mu_h \cdot \frac{\alpha}{x_{h,t}} = \frac{1 - \alpha}{x_{f,t}} \cdot \mu_f \cdot \frac{\exp\left\{ U_{f,t} / \theta \right\}}{E_{t-1} \left[ \exp\left\{ U_{f,t} / \theta \right\} \right]}
\]
What to look for...

\[
\frac{\exp \left\{ \frac{U_{h,t}}{\theta} \right\}}{E_{t-1} \left[ \exp \left\{ \frac{U_{h,t}}{\theta} \right\} \right]} \cdot \mu_h \cdot \frac{\alpha}{x_{h,t}^j} = \frac{1 - \alpha}{x_{f,t}^j} \cdot \mu_f \cdot \frac{\exp \left\{ \frac{U_{f,t}}{\theta} \right\}}{E_{t-1} \left[ \exp \left\{ \frac{U_{f,t}}{\theta} \right\} \right]}
\]

$\Rightarrow \ U_{h,t}^j > U_{f,t}^j \cdot \frac{\mu_{h,t}}{\mu_{f,t}}$
What to look for...

\[
\frac{\exp\left\{ \frac{U_{h,t}}{\theta} \right\}}{E_{t-1}\left[ \exp\left\{ \frac{U_{h,t}}{\theta} \right\} \right]} \cdot \mu_h \cdot \frac{\alpha}{x_h,t} = \frac{1 - \alpha}{x_f,t} \cdot \frac{\exp\left\{ \frac{U_{f,t}}{\theta} \right\}}{E_{t-1}\left[ \exp\left\{ \frac{U_{f,t}}{\theta} \right\} \right]} \cdot \mu_f
\]

\[
\mu_{h,t} \quad \mu_{f,t}
\]

\[\rightarrow \quad U_{h,t} > U_{f,t} \cdot \frac{\mu_{h,t}}{\mu_{f,t}} \downarrow \]

\[\rightarrow \quad U_{h,t} < U_{f,t} \cdot \frac{\mu_{h,t}}{\mu_{f,t}} \uparrow\]
What to look for...

\[
\frac{x_{h,t}^j}{x_{h,t}^k} = \frac{x_{f,t}^j}{x_{f,t}^k}
\]
What to look for...

\[
\exp \left\{ \frac{U_{h,t}^j - U_{h,t}^k}{\theta} \right\} \cdot \frac{x_{h,t}^j}{x_{h,t}^k} = \frac{x_{f,t}^j}{x_{f,t}^k} \cdot \exp \left\{ \frac{U_{f,t}^j - U_{f,t}^k}{\theta} \right\}
\]
What to look for...

\[
\exp\left\{\frac{U_{j,h,t} - U_{k,h,t}}{\theta}\right\} \cdot \frac{x_{j,h,t}}{x_{k,h,t}} = \frac{x_{j,f,t}}{x_{k,f,t}} \cdot \exp\left\{\frac{U_{j,f,t} - U_{k,f,t}}{\theta}\right\}
\]
What to look for...

\[
\exp \left\{ \frac{U_{h,t}^j - U_{h,t}^k}{\theta} \right\} \cdot \frac{x_{h,t}^j}{x_{h,t}^k} = \frac{x_{f,t}^j}{x_{f,t}^k} \cdot \exp \left\{ \frac{U_{f,t}^j - U_{f,t}^k}{\theta} \right\}
\]

Time Additive Allocations

\[
\begin{bmatrix}
\left( \frac{x_j^l}{x_k^l} \right)^\alpha \left( \frac{y_j^l}{y_k^l} \right)^{(1-\alpha)} \end{bmatrix}^{(1-\delta)/\theta}
\]

Time Additive Allocations

\[
\begin{bmatrix}
\left( \frac{x_j^f}{x_k^f} \right)^{(1-\alpha)} \left( \frac{y_j^f}{y_k^f} \right)^\alpha \end{bmatrix}^{(1-\delta)/\theta}
\]
What to look for...

\[ \exp \left\{ \frac{U_{h,t}^j - U_{h,t}^k}{\theta} \right\} \cdot \frac{x_{h,t}^j}{x_{h,t}^k} = \frac{x_{f,t}^j}{x_{f,t}^k} \cdot \exp \left\{ \frac{U_{f,t}^j - U_{f,t}^k}{\theta} \right\} \]

Time Additive Allocations

\[
\left[ \left( \frac{x_{f,t}^j}{x_{f,t}^k} \right)^{\alpha} \left( \frac{y_{f,t}^j}{y_{f,t}^k} \right)^{(1-\alpha)} \right]^{(1-\delta)/\theta}
\]

\[ \alpha = \frac{1}{2} \]

Time Additive Allocations

\[
\left[ \left( \frac{x_{h,t}^j}{x_{h,t}^k} \right)^{(1-\alpha)} \left( \frac{y_{h,t}^j}{y_{h,t}^k} \right)^{\alpha} \right]^{(1-\delta)/\theta}
\]
What to look for...

\[
\exp \left\{ \frac{U_{h,t}^j - U_{h,t}^k}{\theta} \right\} \cdot \frac{x_{h,t}^j}{x_{h,t}^k} = \frac{x_{f,t}^j}{x_{f,t}^k} \cdot \exp \left\{ \frac{U_{f,t}^j - U_{f,t}^k}{\theta} \right\}
\]

Time Additive Allocations

\[
\left[ \left( \frac{x_{t}^j}{x_{t}^k} \right)^{1/2} \left( \frac{y_{t}^j}{y_{t}^k} \right)^{1/2} \right]^{(1-\delta)/\theta}
\]

\[\alpha = \frac{1}{2}\]

Time Additive Allocations

\[
\left[ \left( \frac{x_{t}^j}{x_{t}^k} \right)^{1/2} \left( \frac{y_{t}^j}{y_{t}^k} \right)^{1/2} \right]^{(1-\delta)/\theta}
\]
What to look for...

\[
\exp \left\{ \frac{U^j - U^k}{\theta} \right\} \cdot \frac{x^j_{h,t}}{x^k_{h,t}} = \frac{x^j_{f,t}}{x^k_{f,t}} \cdot \exp \left\{ \frac{U^j - U^k}{\theta} \right\}
\]

Time Additive Allocations

\[
\left[ \left( \frac{x^j}{x^k} \right)^{1/2} \left( \frac{y^j}{y^k} \right)^{1/2} \right]^{(1-\delta)/\theta}
\]

\[\alpha = \frac{1}{2}\]

Time Additive Allocations

\[
\left[ \left( \frac{x^j}{x^k} \right)^{1/2} \left( \frac{y^j}{y^k} \right)^{1/2} \right]^{(1-\delta)/\theta}
\]
What to look for...

\[ \exp \left\{ \frac{U^j_{h,t} - U^{k}_{h,t}}{\theta} \right\} \cdot \frac{x^j_{h,t}}{x^k_{h,t}} \neq \frac{x^j_{f,t}}{x^k_{f,t}} \cdot \exp \left\{ \frac{U^j_{f,t} - U^{k}_{f,t}}{\theta} \right\} \]

Time Additive Allocations

\[
\left[ \left( \frac{x^j_t}{x^k_t} \right)^\alpha \left( \frac{y^j_t}{y^k_t} \right)^{(1-\alpha)} \right]^{(1-\delta)/\theta}
\]

\(\alpha > \frac{1}{2}\)

Time Additive Allocations

\[
\left[ \left( \frac{x^j_t}{x^k_t} \right)^{(1-\alpha)} \left( \frac{y^j_t}{y^k_t} \right)^\alpha \right]^{(1-\delta)/\theta}\]
What to look for...

\[
\exp \left\{ \frac{U_{h,t}^{LH} - U_{h,t}^{HL}}{\theta} \right\} \cdot \frac{x_{h,t}^{LH}}{x_{h,t}^{HL}} \neq \frac{x_{f,t}^{LH}}{x_{f,t}^{HL}} \cdot \exp \left\{ \frac{U_{f,t}^{LH} - U_{f,t}^{HL}}{\theta} \right\}
\]

Time Additive Allocations

\[
\left( \frac{103}{100} \right)^{(2\alpha-1)(\delta-1)/\theta}
\]

\[
\alpha > \frac{1}{2}
\]

Time Additive Allocations

\[
\left( \frac{100}{103} \right)^{(2\alpha-1)(\delta-1)/\theta}
\]

\[ \rightarrow \text{Eg 2 states: } HL = \{ X = 103, Y = 100 \}, LH = \{ X = 100, Y = 103 \} \]
What to look for...

\[ \exp \left\{ \frac{U_{h,t}^{LU} - U_{h,t}^{HL}}{\theta} \right\} \cdot \frac{x_{h,t}^{LU}}{x_{f,t}^{HL}} > \frac{x_{f,t}^{LU}}{x_{f,t}^{HL}} \cdot \exp \left\{ \frac{U_{f,t}^{LU} - U_{f,t}^{HL}}{\theta} \right\} \]

Time Additive Allocations

\[
\left( \frac{103}{100} \right)^{(2\alpha - 1)(\delta - 1)/\theta}
\]

\[
\alpha > \frac{1}{2}
\]

Time Additive Allocations

\[
\left( \frac{100}{103} \right)^{(2\alpha - 1)(\delta - 1)/\theta}
\]

→ Eg 2 states: \( HL = \{ X = 103, Y = 100 \} \), \( LH = \{ X = 100, Y = 103 \} \)
What to look for...

\[ \exp \left\{ \frac{U_{h,t}^{LH} - U_{h,t}^{HL}}{\theta} \right\} \cdot \frac{x_{h,t}^{LH}}{x_{h,t}^{HL}} > \frac{x_{f,t}^{LH}}{x_{f,t}^{HL}} \cdot \exp \left\{ \frac{U_{f,t}^{LH} - U_{f,t}^{HL}}{\theta} \right\} \]

\[ U_{h,t}^{LH} \uparrow, \ U_{h,t}^{HL} \downarrow \]

\[ \alpha > \frac{1}{2} \]

\[ U_{f,t}^{HL} \uparrow, \ U_{f,t}^{LH} \downarrow \]
What to look for...

\[
\exp \left\{ \frac{U_{h,t}^{\text{LH}} - U_{h,t}^{\text{HL}}}{\theta} \right\} \cdot \frac{x_{h,t}^{\text{LH}}}{x_{h,t}^{\text{HL}}} > \frac{x_{f,t}^{\text{LH}}}{x_{f,t}^{\text{HL}}} \cdot \exp \left\{ \frac{U_{f,t}^{\text{LH}} - U_{f,t}^{\text{HL}}}{\theta} \right\}
\]

\( U_{h,t}^{\text{LH}} \uparrow, U_{h,t}^{\text{HL}} \downarrow \)

\( \alpha > \frac{1}{2} \)

\( U_{f,t}^{\text{HL}} \uparrow, U_{f,t}^{\text{LH}} \downarrow \)

→ Consumption home bias matters
What to look for...

\[ \exp \left\{ \frac{U_{h,t}^{LH} - U_{h,t}^{HL}}{\theta} \right\} \cdot \frac{x_{h,t}^{LH}}{x_{h,t}^{HL}} > \frac{x_{f,t}^{LH}}{x_{f,t}^{HL}} \cdot \exp \left\{ \frac{U_{f,t}^{LH} - U_{f,t}^{HL}}{\theta} \right\} \]

- Consumption home bias matters
- Utility smoothing matters

\( U_{h,t}^{LH} \uparrow, U_{h,t}^{HL} \downarrow \)
\( U_{f,t}^{HL} \uparrow, U_{f,t}^{LH} \downarrow \)
Allocations

Time Additive Preferences

Let $k = \frac{\alpha}{1-\alpha}$:

\[
\begin{align*}
    x_t^h &= \frac{kS}{1 + kS} X_t, & x_t^f &= \frac{1}{1 + kS} X_t \\
    y_t^h &= \frac{S}{k + S} Y_t, & y_t^f &= \frac{k}{k + S} Y_t
\end{align*}
\]

where

\[
S = \frac{\mu_h}{\mu_f}
\]
Let $k = \frac{\alpha}{1-\alpha}$:

$$x_t^h = \frac{k S_t}{1 + k S_t} X_t, \quad x_t^f = \frac{1}{1 + k S_t} X_t$$

$$y_t^h = \frac{S_t}{k + S_t} Y_t, \quad y_t^f = \frac{k}{k + S_t} Y_t$$

where

$$S_t = S_{t-1} \cdot \frac{\delta \exp \{U_{h,t} / \theta\}}{E_{t-1} \exp \{U_{h,t} / \theta\}} \Bigg/ \frac{\delta \exp \{U_{f,t} / \theta\}}{E_{t-1} \exp \{U_{f,t} / \theta\}}$$

Risk Sensitive Preferences
1. **How** does $S_t$ evolve over time?
Questions

1. **How** does $S_t$ evolve over time?

2. **Why** does $S_t$ move?
Questions

1. **How** does $S_t$ evolve over time?

2. **Why** does $S_t$ move?

3. **What** is $S_t$?
   
   → Decentralized Economy.
How does $S_t$ move?
How does $S_t$ move?

$HL$

$LH$

→ **Abundant** $X$, **scarce** $Y$: 
How does $S_t$ move?

$HL$

$LH$

$\rightarrow$ Abundant $X$, scarce $Y$:

$\rightarrow$ Good news for home
How does $S_t$ move?

→ Abundant $X$, scarce $Y$:
  → Good news for home
  → Home Pareto weight ↓
How does $S_t$ move?

$HL$ and $LH$

$\Delta \mu_h^{HL}$ and $\Delta \mu_h^{LH}$

$X$ scarce, $Y$ abundant:

→ Bad news for home

→ Home Pareto weight ↑
Why does $S_t$ move?
Why does $S_t$ move?

![Graph showing the relationship between $\sigma[U_{h,t+1}(s_{t+1}|s_t)]$ and $E[U_{h,t+1}(s_{t+1}|s_t)]$ for different values of $\gamma$. The graph indicates that reducing expected utility increases volatility. For $\gamma = 25$, the expected utility decreases, leading to an increase in volatility.](image)

The graph illustrates the tradeoff between expected utility and volatility, highlighting how risk-sharing schemes can affect market dynamics. 

---

**Why does $S_t$ move?**

The movement of $S_t$ is influenced by the risk-sharing scheme and international pricing mechanisms. Qualitative implications of these interactions are discussed, emphasizing the importance of understanding market dynamics for effective economic planning.
Why does $S_t$ move?

Reducing Expected Utility

Reducing Volatility

$\gamma = 1$ (Time Additive Case − No Tradeoff)

$\gamma = 25$
Why does $S_t$ move?

Trade-off between Expected Utility and Utility Variance
Why does $S_t$ move?

\[ E[U_{h,t+1}(s_{t+1}|s_t)] \]

\[ \sigma[U_{h,t+1}(s_{t+1}|s_t)] \]

$\gamma = 1$ (Time Additive Case – No Tradeoff)

$\gamma = 25$
Pareto weights: expected growth and distribution

→ Mean Reversion
Pareto weights: expected growth and distribution

→ Mean Reversion

→ Symmetric Distribution
Risk Sharing Scheme:
State Prices
Decentralized equilibrium

Home is endowed with good $X$; Foreign is endowed with good $Y$;

Complete set of one period ahead state contingent securities;

Budget constraints:

$$x_h, t + p_t y_h, t + \sum_{s = t + 1}^{Q_t + 1} (s + 1) A_{t + 1} (s + 1) \leq X_{t + 1}$$

Use welfare theorems to find state prices and asset holdings.
Decentralized equilibrium

- *Home* is endowed with good $X$;
- *Foreign* is endowed with good $Y$;
- Complete set of one period ahead state contingent securities;

Budget constraints:

\[
\begin{align*}
    x_{h,t} + p_t y_{h,t} + \sum_{s_{t+1}} Q_{t+1}(s_{t+1}) A_{t+1}(s_{t+1}) &\leq X_t + A_t \quad [Home] \\
    x_{f,t} + p_t y_{f,t} - \sum_{s_{t+1}} Q_{t+1}(s_{t+1}) A_{t+1}(s_{t+1}) &\leq p_t Y_t - p_t A_t \quad [Foreign]
\end{align*}
\]

- Use welfare theorems to find state prices and asset holdings.
Contingent securities

\[ a_t (\%) \]

\[ \mu_h \]

\[ \rightarrow \] Savings are increasing in Pareto weight
State Prices

\[ \frac{Q_{t+1}^{LH}}{Q_{t+1}^{HL}} = \frac{x_{h,t+1}^{HL}}{x_{h,t+1}^{LH}} \exp \left\{ \frac{U_{h,t+1}^{LH} - U_{h,t+1}^{HL}}{\theta} \right\} \]

→ State prices reflect continuation utilities:
State Prices

State prices reflect continuation utilities:

\[
\frac{Q_{t+1}^{LH}}{Q_{t+1}^{HL}} = \frac{\chi_{h,t+1}^{HL}}{\chi_{h,t+1}^{LH}} \exp \left\{ \frac{U_{h,t+1}^{LH} - U_{h,t+1}^{HL}}{\theta} \right\}
\]

→ Time varying volatility of utilities
Risk Sharing Scheme:
Endogenously time-varying volatilities and correlations
Conditional Volatilities

X/Y

\[ \frac{X}{Y} \]

 periods

\[ \mu_h \]

\[ V_t(U_{h,t+1}) \]

\[ x 10^{-8} \]

HL

LH

Introduction

The Economy

Risk-Sharing Scheme

International Pricing

Qualitative implications

Conclusion
Conditional Volatilities

- Conditional Volatilities
- Introduction
- The Economy
- Risk-Sharing Scheme
- International Pricing
- Qualitative implications
- Conclusion

- Periods

- $X/Y$
- $\mu_h$
- $V_t(U_{h,t+1})$

- HL
- LH

- $x 10^{-8}$
Conditional Volatilities

- Conditional Volatilities
- $\frac{X}{Y}$
- $\mu_h$
- $V_t(U_{h,t+1})$
Conditional Volatilities

\[ \frac{X}{Y} \]

\[ \mu_h \]

\[ V_t(U_{h,t+1}) \times 10^{-8} \]

periods
Conditional Volatilities

Graphs showing the behavior of $X/Y$, $\mu_h$, and $V_t(U_{h,t+1})$ over periods.
Utilities’ correlations

→ Blue curves: utilities when supply of good $X$ is high
→ Red curves: utilities when supply of good $Y$ is high
Utilities’ correlations

→ **Time additive** preferences:
  → Home utility is high (low) when foreign utility is low (high)
Risk-sensitive preferences:

→ Must take into account international redistribution of wealth
Continuation utilities including redistribution of wealth

(i.e. $U_{t+1}$ as a function of $\mu_t$)
If wealth is similar:

→ Home utility is high (low) when foreign utility is low (high)
What if one country is more wealthy than the other?
Utilities’ correlations

→ If wealth is not similar:

→ Home utility is high (low) when foreign utility is high (low)
Correlation of utilities increases with wealth inequality
International Asset Pricing
## Introducing Rare Events

<table>
<thead>
<tr>
<th>$X$</th>
<th>$Y$</th>
<th>$\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>103</td>
<td>103</td>
<td>0.2375</td>
</tr>
<tr>
<td>103</td>
<td>100</td>
<td>0.2375</td>
</tr>
<tr>
<td>100</td>
<td>103</td>
<td>0.2375</td>
</tr>
<tr>
<td>100</td>
<td>100</td>
<td>0.2375</td>
</tr>
<tr>
<td>103</td>
<td>60</td>
<td>0.0100</td>
</tr>
<tr>
<td>100</td>
<td>60</td>
<td>0.0100</td>
</tr>
<tr>
<td>60</td>
<td>60</td>
<td>0.0100</td>
</tr>
<tr>
<td>60</td>
<td>103</td>
<td>0.0100</td>
</tr>
<tr>
<td>60</td>
<td>100</td>
<td>0.0100</td>
</tr>
</tbody>
</table>
## Introducing Rare Events

<table>
<thead>
<tr>
<th>$X$</th>
<th>$Y$</th>
<th>$\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>103</td>
<td>103</td>
<td>0.2375</td>
</tr>
<tr>
<td>103</td>
<td>100</td>
<td>0.2375</td>
</tr>
<tr>
<td>100</td>
<td>103</td>
<td>0.2375</td>
</tr>
<tr>
<td>100</td>
<td>100</td>
<td>0.2375</td>
</tr>
<tr>
<td>103</td>
<td>60</td>
<td>0.0100</td>
</tr>
<tr>
<td>100</td>
<td>60</td>
<td>0.0100</td>
</tr>
<tr>
<td>60</td>
<td>60</td>
<td>0.0100</td>
</tr>
<tr>
<td>60</td>
<td>103</td>
<td>0.0100</td>
</tr>
<tr>
<td>60</td>
<td>100</td>
<td>0.0100</td>
</tr>
</tbody>
</table>

Four equally likely no-disaster events
### Introducing Rare Events

Five equally likely disaster events:

<table>
<thead>
<tr>
<th>$X$</th>
<th>$Y$</th>
<th>$\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>103</td>
<td>103</td>
<td>0.2375</td>
</tr>
<tr>
<td>103</td>
<td>100</td>
<td>0.2375</td>
</tr>
<tr>
<td>100</td>
<td>103</td>
<td>0.2375</td>
</tr>
<tr>
<td>100</td>
<td>100</td>
<td>0.2375</td>
</tr>
<tr>
<td>103</td>
<td>60</td>
<td>0.0100</td>
</tr>
<tr>
<td>100</td>
<td>60</td>
<td>0.0100</td>
</tr>
<tr>
<td>60</td>
<td>60</td>
<td>0.0100</td>
</tr>
<tr>
<td>60</td>
<td>103</td>
<td>0.0100</td>
</tr>
<tr>
<td>60</td>
<td>100</td>
<td>0.0100</td>
</tr>
</tbody>
</table>
Stochastic Discount Factors

International Stochastic Discount Factors:

\[ \log M_{i,t+1} = \log \delta + \log \frac{C_{i,t}}{C_{i,t+1}} + \log \frac{\exp \{ U_{i,t+1}/\theta \}}{E_t \exp \{ U_{i,t+1}/\theta \}}, \quad \forall i \in \{h, f\} \]
Stochastic Discount Factors

- International Stochastic Discount Factors:

\[
\log M_{i,t+1} = \log \delta + \log \frac{C_{i,t}}{C_{i,t+1}} + \log \frac{\exp \left\{ U_{i,t+1}/\theta \right\}}{E_t \exp \left\{ U_{i,t+1}/\theta \right\}}, \quad \forall i \in \{h, f\}
\]

- Properties:
  1. Volatility is high
  2. Volatility is time-varying
  3. Correlation is high
  4. Correlation is time-varying
Stochastic Discount Factors

International Stochastic Discount Factors:

\[
\log M_{i,t+1} = \log \delta + \log \frac{C_{i,t}}{C_{i,t+1}} + \log \frac{\exp\left\{ U_{i,t+1}/\theta \right\}}{E_t \exp\left\{ U_{i,t+1}/\theta \right\}}, \quad \forall i \in \{h, f\}
\]

Properties:

1. Volatility is high ⇒ Equity Sharpe ratios are high
2. Volatility is time-varying ⇒ Equity risk-premia are time-varying
3. Correlation is high ⇒ Volatility of FX growth is “low”
4. Correlation is time-varying ⇒ Volatility of FX growth is time-varying
Conditional volatility of SDF

\[ \sigma_t(M_{h,t+1})/E_t(M_{h,t+1}) \]

\[ \mu_h \]

→ Average Volatility ≈ 30%
→ Average Volatility ≈ 30%
→ Equity risk-premia

\[ E_t \left[ r_{h,t+1}^c - r_{h,t}^f \right] = -\rho_t (\Delta c_{h,t+1}, M_{h,t+1}) \sigma_t (\Delta c_{h,t+1}) \frac{\sigma_t (M_{h,t+1})}{E_t (M_{h,t+1})} \]
Conditional volatility of SDF

\[ E_t \left( r_{h,t+1}^c - r_{h,t}^f \right) = -\rho_t (\Delta c_{h,t+1}, M_{h,t+1}) \sigma_t (\Delta c_{h,t+1}) \frac{\sigma_t (M_{h,t+1})}{E_t (M_{h,t+1})} \]

are time varying and counter-cyclical.

→ Average Volatility ≈ 30%
→ Equity risk-premia

\[ \rightarrow \text{Average Volatility} \approx 30\% \]
\[ \rightarrow \text{Equity risk-premia} \]
Conditional Correlations

\[
corr(\Delta c_{t+1}^h, \Delta c_{t+1}^f)
\]

\[
corr(m_{t+1}^h, m_{t+1}^f)
\]
Conditional Correlations

\[ \text{corr}_t(\Delta c_{t+1}^h, \Delta c_{t+1}^f) \]

\[ \mu \text{corr}_t(m_{t+1}^h, m_{t+1}^f) \]

→ Low, time-varying correlation of consumption
Conditional Correlations

\[ \text{corr}(\Delta c_{t+1}^h, \Delta c_{t+1}^f) \]

\[ \text{corr}(m_{t+1}^h, m_{t+1}^f) \]

→ Low, time-varying correlation of consumption
→ High, time-varying correlation of marginal utilities
Conditional volatility of FX growth

\[ \Delta e_{t+1} = m_{f,t+1} - m_{h,t+1} \]
Conditional volatility of FX growth

\[ V_t[\Delta e_{t+1}] = V_t[m_{f,t+1} - m_{h,t+1}] \]
Conditional volatility of FX growth

\[ V_t[\Delta e_{t+1}] = V_t[m_{f,t+1}] + V_t[m_{h,t+1}] - 2\rho_t \cdot \sqrt{V_t[m_{f,t+1}]} \cdot \sqrt{V_t[m_{h,t+1}]} \]
Conditional volatility of FX growth

\[ V_t[\Delta e_{t+1}] = V_t[m_{f,t+1}] + V_t[m_{h,t+1}] - 2\rho \cdot \sqrt{V_t[m_{f,t+1}]} \cdot \sqrt{V_t[m_{h,t+1}]} \]

→ Average Volatility ≈ 14%
Conditional volatility of FX growth

\[ V_t[\Delta e_{t+1}] = V_t[m_{f,t+1}] + V_t[m_{h,t+1}] - 2\rho_t \cdot \sqrt{V_t[m_{f,t+1}] \cdot V_t[m_{h,t+1}]} \]

\[ \sigma_t(\Delta e_{t+1}) \]

\[ \mu_h \]

\rightarrow \text{Average Volatility } \approx 14\% \\
\rightarrow \text{Time-varying exchange rate volatility}
Qualitative implications
Qualitative implications

1. Inverse relationship between
   - Volatility of exchange rate
   - Absolute level of savings

![Graph showing the relationship between volatility of exchange rate and absolute level of savings.]

\[ \sigma_{t+1}(\Delta e_t) \]
Qualitative implications

1 Inverse relationship between
   - Volatility of exchange rate
   - Absolute level of savings
Qualitative implications

1. Inverse relationship between
   - Volatility of exchange rate
   - Absolute level of savings

2. Positive relationship between
   - Volatility of consumption
   - Level of savings
1. Inverse relationship between
   - Volatility of exchange rate
   - Absolute level of savings

2. Positive relationship between
   - Volatility of consumption
   - Level of savings
Qualitative implications

1. Inverse relationship between
   - Volatility of exchange rate
   - Absolute level of savings

2. Positive relationship between
   - Volatility of consumption
   - Level of savings
Inverse relationship between
Inverse relationship between

- Volatility of exchange rate
In Qualitative implications (cont’d)

3 Inverse relationship between
   - Volatility of exchange rate
   - Correlation of returns
Inverse relationship between

- Volatility of exchange rate
- Correlation of returns
Concluding remarks

A two-countries model with:

- complete markets
- two goods
- i.i.d. endowments
- risk-sensitive preferences

1. generates
   - dynamic risk-sharing scheme
   - endogenously time varying second moments

2. replicates a number of international finance facts

3. introduce frictions and investments