RISKS FOR THE LONG RUN: AN EXPLANATION OF
INTERNATIONAL FINANCE PUZZLES

by

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Thomas J. Sargent
DEDICATION

To Anna, my wife, my love, my everything.
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My wife, Anna Bassi, has always been with me and suffered through all the struggle of getting this project completed. This involved long awake nights and an uncountable number of days in which I would not have wanted to be around me. For her patience and the strength that she gave me by never stopping to believe in me, this thesis is as much hers as it is mine.

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This thesis builds on the paper ‘Risks for the long and the real exchange rate’ coauthored with Massimiliano Croce, during my last two years as a graduate student at the Department of Economics at New York University. We got the idea of writing a paper on this topic, while we were officemates and both reading the paper ‘Exchange rates are too smooth or international risk sharing is better than you think’ by Michael Brandt, John Cochrane and Pedro Santa-Clara, as an assignment for Tom Sargent’s macro reading group. It did not take us too long to realize that an alternative to the preferences used by Lucas (1978) was needed in order to account for the high degree of correlation that financial markets seem to display when it comes to discounting future uncertain payoffs. It was like this that we took two off-the-shelf instruments provided by the risk sensitive preferences of Epstein and Zin (1989) and the idea that consumption growth can be modeled as containing a small low frequency component of Bansal and Yaron (2004) and added a third one in the form of the high international correlation of this component to solve, in the matter of a night, what appeared to be an international extension to Mehra and Prescott (1985) equity premium puzzle. The paper was welcomed with excitement by many researcher, but at the same time we kept receiving the request of providing more convincing econometric evidence of the consumption process that we were using.
in the summer and fall of 2005, I started looking into international consumption
and financial data and the results that are reported in Chapter 3 are the outcome
of that effort. I owe Chapter 4 entirely to my job market, as it contains the answers
to many if not all the questions that I received while looking for an academic job in
the United States and in Europe. More recently, I have been trying to extend this
framework to account for other international finance anomalies. Chapter 5 is an
excerpt from my working paper ‘Six anomalies looking for a model’, that I propose
as a unified framework for studying international finance. This is an ambitious and
fun line of research that I hope will produce many more interesting ideas in the
coming years.
ABSTRACT

Brandt, Cochrane, and Santa-Clara (2004) pointed out that the implicit stochastic discount factors computed using prices, on the one hand, and consumption growth, on the other hand, have very different implications for their cross country correlation. They leave this as an unresolved puzzle. We explain it by combining Epstein and Zin (1989) preferences with a model of predictable returns and by positing a very correlated long run component. We also assume that the intertemporal elasticity of substitution is larger than the reciprocal of the coefficient of risk aversion, implying a preference for early resolution of uncertainty. This setup brings the stochastic discount factors computed using prices and quantities close together, by keeping the volatility of the depreciation rate in the order of 12% and the cross country correlation of consumption growth around 30%. We also provide an econometric analysis of the model, by showing that the over-identifying restrictions imposed by financial variables allow us to associate tight confidence intervals to parameters of the consumption process that would otherwise be undetectable. To conclude we generalize the model to provide a unified solution of a number of international finance puzzles including the Backus and Smith (1993) puzzle and the tendency of high interest rate currencies to appreciate.
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Introduction

Ever since the seminal paper of Mehra and Prescott (1985), researchers have tried to reconcile the implications that prices and quantities have for asset returns. The heart of the problem is characterized by Hansen and Jagannathan (1991) that show that in a no arbitrage framework, stochastic discount factors should be at least as volatile as the highest Sharpe ratio of the return of any asset that they are intended to price. Financial data point in the direction of highly volatile stochastic discount factors, while in the context of an economy with a representative consumer with constant relative risk aversion (CRRA) preferences, quantities would imply lowly volatile stochastic discount factors, unless an unreasonably high coefficient of risk aversion is assumed. Brandt, Cochrane, and Santa-Clara (2004) extend the puzzle to a two country economy\footnote{Brandt, Cochrane, and Santa-Clara (2004) construct an index that they regard as a measure of international risk sharing, but we prefer to reformulate their puzzle without invoking the concept of risk sharing.}. They follow Backus, Foresi, and Telmer (1996) in showing
that for any stochastic process for the depreciation rate and returns on domestic and foreign currency denominated assets, there exist stochastic discount factors, whose ratio is equal to the depreciation rate, provided that there are no arbitrage opportunities. This ‘accounting’ relationship allows them to retrieve the correlation of the stochastic discount factors from their volatilities and the standard deviation of the rate of growth of the exchange rate. Using the Hansen and Jagannathan (1991) bound the variance of the pricing kernels in major industrialized countries is in the order of 20% in annualized terms, while the volatility of the depreciation rate between the US and members of the same set of countries is typically between 11% and 15% also in annualized terms\(^2\). The implied correlation of the stochastic discount factors is not less than 0.96, when computed in this way. However data on consumption display a very low cross country correlation, that is generally not higher than 0.3 when the US is one of the two countries; the assumption that agents have CRRA preferences then implies a correlation of the stochastic discount factors of the same magnitude. The implication is that exchange rates will have to move around a lot to prevent arbitrage opportunities across countries: the volatility of the depreciation rate would have to take on values as high as 55%, that is about 5 times what we observe in the actual data. This can be thought as an extension of Mehra and Prescott (1985) equity premium puzzle: in a one country economy,

\(^2\)Source: IMF’s ‘International Financial Statistics’.
consumption growth does not vary enough to keep track of the equity premium, while in a two country economy, consumption growth does not co-vary enough to explain the relative smoothness of the depreciation rate. We want to explain this puzzle.

This thesis is a contribution to a growing literature that goes under the name of ‘risks for the long run’. This class of models has been successful in explaining some long standing puzzles of financial economics, including the afore mentioned high excess return of equities over the risk free rate. Examples of these economies can be found in Bansal and Yaron (2004), Bansal, Gallant, and Tauchen (2002) and Hansen, Heaton, and Li (2004). This paper extends this framework to a two country economy, each of them populated by a representative consumer with Epstein and Zin (1989) recursive preferences. We specify consumption growths as the sum of a slowly moving predictable component and an i.i.d. shock. We assume the former to be relatively smaller than the latter to respect the empirical finding that consumption growth is almost an i.i.d. process. We further assume that the predictable components are highly correlated across countries. This specification makes the puzzle disappear: we manage to obtain highly correlated stochastic discount factors along with a lowly volatile depreciation rate. This result is obtained in combination with consumption growth processes that are as volatile, as correlated and as persistent as observed in the actual data. We also show that the model
can describe the international correlation of financial markets. The success of the model reflects the fact that when agents care about the timing of the resolution of uncertainty, long horizon consumption growth prospects are crucial determinants of stochastic discount factor processes. Having modeled these prospects as being highly correlated we obtain highly correlated stochastic discount factors. Because stochastic discount factors do most of the job of adjusting to avoid international arbitrage opportunities, exchange rates can take a lesser role and be less volatile.

What comes with the territory in these models is the difficulty of identifying the small predictable component of consumption growths in a formal econometric model. Bansal and Yaron (2004) show that the presence of a small predictable component in consumption growth cannot be rejected in the past century of US data and endowing the agents with sure knowledge of its presence helps explaining the behavior of asset returns. Bansal, Gallant, and Tauchen (2002) push the analysis one step forward by using price information to properly identify the properties of the consumption process. Our analysis follows in their footsteps as we demonstrate that consumption data alone contain insufficient evidence for the presence of low frequency components of consumption growths even when a large cross section of countries is employed. However when the econometric system is enriched with restrictions imposed on the model by asset returns and exchange rates, we manage sharply to identify the departure of consumption growths from a purely i.i.d. process.
We also push the analysis one step forward by analyzing the ability of the model to explain a number of international finance puzzles. A well known result in the literature is the so-called Backus and Smith (1993) puzzle, according to which consumption growth differentials should be perfectly correlated with the rate of the depreciation of the domestic currency. This result emerges in the context of a model in which agents’ preferences are time additive. We show that an extension of the model with the features discussed earlier has the ability of producing consumption growth differentials that are as correlated with depreciation rates as they are in the data. This generalized version of the model is also able to account for the tendency of high interest rate currencies to appreciate instead of depreciating. This phenomenon has received a large attention in the literature that dates back to the work of Meese and Rogoff (1983). More recently Backus, Foresi, and Telmer (1996) have shown the properties that stochastic discount factors should have to account for this anomaly. We show that our stochastic discount factors fall in this category.

The thesis is organized as follows. In chapter 1 we describe the determinants of the main puzzle that we want to explain. In regards to the literature of which this is an extension, we denote it as the ‘international equity premium puzzle’. We then write down a simple model that we can solve and calibrate. This provides a useful instrument to show the internal transmission mechanism of the economy. In this context, we show how we can match a large number of statistics in the data.
Then we proceed to extend the model to include in each country a redundant asset that pays dividends, whose process looks like the aggregate dividend paid by the stock market and we explore what our model has to say about the international correlation of financial markets. We also introduce time varying economic uncertainty. The focus of chapter 2 is on the estimation of the model. We study the aforementioned difficulties of providing conclusive evidence of predictable components of consumption growths from quantity data only and show the benefits of introducing prices in the picture. Chapter 3 challenges most of the assumptions that are made in the context of our model and sets the ground for a number of possible extensions that can be explored in this line of the literature. Chapter 4 generalizes the model discussed in chapter 1 by allowing for the presence of two low frequency components in the dynamics of consumption growth and by relaxing the assumptions of the correlation matrix of the shocks to show that the model has the potential to explain a larger number of the international finance puzzles that emerge in the context of the standard CRRA time additive framework. The last chapter concludes the paper, hinting to potential extensions of the model and summarizing the main findings.
Chapter 1

The exchange rate growth volatility puzzle

1.1 The ‘international equity premium puzzle’

We analyze two economies that we denote as home and foreign. Accordingly, we will index variables in the two countries with an $h$ and with an $f$. Assuming that the two countries are characterized by the absence of arbitrage opportunities, the following pricing condition has to be satisfied:

$$E_t \left[ \exp \left\{ m_{t+1}^f \right\} R_{t+1}^f \right] = 1 = E_t \left[ \exp \left\{ m_{t+1}^h \right\} R_{t+1}^h \right]$$
where $m_{t+1}^h$ and $m_{t+1}^f$ are the logarithms of the stochastic discount factors for returns denominated in home and foreign currencies respectively and $R_{t+1}^h$ and $R_{t+1}^f$ are gross returns. If the asset that in the foreign country delivers the return $R_{t+1}^f$ in expectation is also traded in the home country, then a feasible investment strategy consists in converting home prices into foreign prices at the spot exchange rate $e_t$:

$$E_t \left[ \exp \left( m_{t+1}^f \right) R_{t+1}^f \right] = 1 = E_t \left[ \exp \left( m_{t+1}^h \right) \frac{e_{t+1}}{e_t} R_{t+1}^f \right]$$

(1.1)

Equation (1.1) gives us a connection between the rate of depreciation of the home currency and the two stochastic variables $m_{t+1}^h$ and $m_{t+1}^f$. Following Backus, Foresi, and Telmer (1996) and Brandt, Cochrane, and Santa-Clara (2004), we further assume that there is a complete set of markets for currencies and state contingent claims. This uniquely identifies the refinement of condition (1.1) as

$$\pi_{t+1} = m_{t+1}^f - m_{t+1}^h$$

(1.2)

where $\pi_{t+1} = \log \frac{e_{t+1}}{e_t}$. By taking the variance operator on both sides and by denoting $\sigma_{m_i}, \forall i \in \{ h, f \}$ as the standard deviation of the stochastic discount factor in the two countries, $\rho_{m^h, m^f}$ as the correlation of the stochastic discount factors and $\sigma_\pi$ as the volatility of the depreciation rate, we obtain:

$$\rho_{m^h, m^f} = \frac{\sigma_{m^h}^2 + \sigma_{m^f}^2 - \sigma_\pi^2}{2\sigma_{m^h}\sigma_{m^f}}$$

(1.3)

It is useful to restate the puzzle we are after in terms of equation (1.3). The
Hansen and Jagannathan (1991) bound on the volatility of the logarithm of the stochastic discount factor\(^1\) is in the order of 39% in the US and 37% in the United Kingdom. The standard deviation of the log-depreciation rate between the same countries is in the order of 11%. These numbers and equation (1.3) imply a correlation of the stochastic discount factors of approximatively 0.96. In particular, it is shown in the Appendix that the assumption of complete markets along with the Hansen and Jagannathan (1991) imply the following lower bound on the international correlation of stochastic discount factors.

**Proposition 1** Let \( \sigma_{m^h} \) and \( \sigma_{m^f} \) be the lower bounds on the volatilities of the stochastic discount factors in the home and in the foreign country, respectively. If markets are complete both domestically and internationally and the volatility of the depreciation of the home currency is strictly positive, then the following is the lower bound on the correlation of stochastic discount factors:

\[
\rho_{m^h,m^f} = \begin{cases} 
\frac{\sigma_{m^h}^2 - \sigma^2_\pi}{\sigma_{m^h} \sqrt{\sigma_{m^h}^2 - \sigma^2_\pi}} & \text{if } \sigma_{m^f} \leq \sigma_{m^h} - \sigma^2_\pi \\
\frac{\sigma_{m^h}^2 + \sigma_{m^f}^2 - \sigma^2_\pi}{2\sigma_{m^h} \sigma_{m^f}} & \text{if } \sigma_{m^f} \in (\sigma_{m^h} - \sigma^2_\pi, \sigma_{m^h} + \sigma^2_\pi) \\
\frac{\sigma_{m^f}^2 - \sigma^2_\pi}{\sigma_{m^f} \sqrt{\sigma_{m^f}^2 - \sigma^2_\pi}} & \text{if } \sigma_{m^f} \geq \sigma_{m^h} - \sigma^2_\pi
\end{cases}
\]

\(^1\)Since the Hansen and Jagannathan (1991) bound is defined on the levels of the stochastic discount factors, we assume log-normality of the stochastic discount factors to derive the corresponding bound on \( \sigma_{m^h} \) and \( \sigma_{m^f} \).
Proof. See Appendix 2.

This is what the data on prices have to say about the correlation of the stochastic discount factors in the US and in the UK. The puzzle arises when we look at the restrictions imposed by equation (1.3) when quantity data are used. In particular, assuming that the two countries have identical constant relative risk aversion (CRRA) preferences with coefficient of risk aversion $\gamma$, the log-stochastic discount factors are $m_i^t = -\gamma \Delta c_i^t$, $\forall i \in \{h, f\}$ and the correlation $\rho_{m^h, m^f}$ is simply the correlation of consumption growth. This number is about 0.3 that is far below the 0.96 calculated from financial data. Furthermore, consumption growth is notoriously lowly volatile, its standard deviation being 1.37% in the US and 2.83% in the UK. As a consequence, even if we were to set the coefficient of risk aversion to a number as high as 30 to reconcile lowly volatile consumption growths with highly volatile stochastic discount factors, equation (1.3) would point in the direction of an extremely high variance of the depreciation of the home currency: 18.6%. Since this variance is about 15 times what we observe in the data it is natural to wonder whether there is something wrong with the observed exchange rate’s fluctuations of US Dollars versus British Pounds or whether this particular version of consumption based asset pricing model is not suited to describe the data at hand. We can also

\footnote{Brandt, Cochrane, and Santa-Clara (2004) examine market imperfections as the source of...}
see this as a restatement of Mehra and Prescott (1985) equity premium puzzle. In a one country model, consumption growth does not vary enough to explain the excess return over the risk free rate. In a two country model, consumption growth does not co-vary enough to keep track of movements in the exchange rate. This opens up to the rules of the game we want to play. We want to be able to reconcile the implications that both prices and quantities have for equation (1.3), by controlling at the same time for risk aversion, cross country correlation of consumption growth and volatility of the depreciation rate.

In Figure 1.1 we show how the dichotomy prices-quantities extends also to other countries when paired with the US.

1.2 Setup of the economy

1.2.1 Structure of the markets

We study an endowment economy with two countries that we shall denote as home (h) and foreign (f). We assume that there are only two goods in the whole economy and that these goods are country specific. To further simplify the setup, this discrepancy, showing that an unreasonable amount of extra volatility should be added to the stochastic discount factors to reconcile the two measure of correlations.
we impose that preferences are such that there is complete home bias, meaning that each country is willing to consume only the good that it is endowed with. Markets are complete, implying that returns are equalized across countries after accounting for the exchange rate. An equilibrium of this economy exists, in which each country behaves as in autarky both for consumption and asset holdings.
We motivate this extreme structure of the markets in at least three ways. First Brandt, Cochrane, and Santa-Clara (2004) assume this specification, too. As a matter of making our results comparable to theirs, it appears to be justified to follow in their footsteps. Second, it is a well documented fact that there is a marked home bias in consumption movements. Backus, Kehoe, and Kydland (1992) were the first to notice it and Lewis (1999) shows that there is almost a one for one response of consumption growth to domestic output growth after controlling for a world effect and heteroskedasticity. This appears to be true for the US, the UK, Germany, Japan and other G7 countries. Third, we want to use preferences of the Epstein and Zin (1989) type, that are recursive but not time separable and allowing for trades in the consumption goods market would severely complicate our analysis. There exists a literature that examines the dynamics of allocations when international goods markets are open to trade and agents have non time separable preferences. Anderson (2005) and Kan (1995) provide analytical and theoretical tools to study these economies. In our case, the need to also introduce a slowly moving predictable component in the endowment process would make these frameworks difficult to extend. For the sake of explanation, we prefer to study the no-trade limiting case and leave a more realistic structure of the goods market for future extensions.
1.2.2 Preferences and long run risks

We model the two economies as each having a representative consumer with Epstein and Zin (1989) preferences:

\[ U_i^t = \left\{ (1 - \delta)(C_i^t)^{1-\gamma} + \delta \left[ E_t \left[ (U_{t+1}^i)^{1-\gamma} \right] \right]^{\frac{\theta}{1-\gamma}} \right\}^{\frac{1}{\theta}}, \forall i \in \{h, f\} \]

where \( \gamma \) is the coefficient of risk aversion and \( \theta = \frac{1-\gamma}{1-1/\psi} \) implicitly defines the intertemporal elasticity of substitution \( \psi \). The two economies are assumed to be symmetric, having the same preference and transition laws parameters. The implied pricing equation for the \( j^{th} \) asset is

\[ E_t \left[ M_{t+1}^i R_{j,t+1}^i \right] = 1, \forall i \in \{h, f\} \]

where the pricing kernel \( M_{t+1}^i \) is a stochastic process that depends on consumption growth, \( \frac{C_{t+1}^i}{C_t^i} \), on the return on the asset that pays the consumption bundle, \( R_{c,t+1}^i \) and on the preference parameters:

\[ \log M_{t+1}^i = \theta \log \delta - \frac{\theta}{\psi} \log \left( \frac{C_{t+1}^i}{C_t^i} \right) + (\theta - 1) \log R_{c,t+1}^i, \forall i \in \{h, f\} \quad (1.4) \]

In what follows, we will adopt the convention of denoting log-variables in small letters (hence \( m_{t+1}^i = \log M_{t+1}^i \)). The price of an assets that entitles to a stream of country \( i \) consumption bundle costs \( P_{c,t}^i \) and has the following gross return:

\[ R_{c,t+1}^i = \frac{v_{c,t+1}^i + 1}{v_{c,t}^i} \exp \Delta c_{t+1} \]

\[ R_{d,t+1}^i = \frac{v_{d,t+1}^i + 1}{v_{d,t}^i} \exp \Delta d_{t+1} \]

(1.5)

(1.6)
with $v_{c,t+1}^i$ being the price-consumption ratio in each country $i = \{h, f\}$. We complete the system by specifying exogenous laws of motion for consumption as:

$$
\Delta c_t^i = x_{t-1}^i + \varepsilon_{c,t}^i
$$

(1.7)

$$
\Delta d_t^i = \lambda x_{t-1}^i + \varepsilon_{d,t}^i
$$

(1.8)

$$
x_t^i = \rho_x x_{t-1}^i + \varepsilon_{x,t}^i
$$

(1.9)

$\forall i = \{h, f\}$. All shocks are iid normally distributed within each country, but they are allowed to be cross-country correlated according to the covariance matrix $\Sigma$:

$$
\begin{bmatrix}
\varepsilon_{c,t}^h & \varepsilon_{c,t}^f \\
\varepsilon_{x,t}^h & \varepsilon_{x,t}^f
\end{bmatrix} \sim N(0, \Sigma)
$$

(1.10)

where

$$
H_c = \begin{bmatrix}
1 & \rho_{c}^{hf} \\
\rho_{c}^{hf} & 1
\end{bmatrix}
$$

$$
H_x = \begin{bmatrix}
1 & \rho_{x}^{hf} \\
\rho_{x}^{hf} & 1
\end{bmatrix}
$$

In Appendix 1 we show that a first order Taylor expansion of the price-consumption schedule around its steady state $\bar{v}_c^i$ is:

$$
v_{c,t}^i = \bar{v}_c^i \left(1 + \frac{(\psi - 1)x_t^i}{\psi(1 - \rho_x \delta)}\right), \forall i \in \{h, f\}
$$

(1.11)

When the intertemporal elasticity of substitution $\psi$ is larger than one, the coefficient on $x_t$ is positive. Bansal and Yaron (2004) show that this one of the crucial
ingredients that allows high equity premia to be justified by a reasonable coefficient of risk aversion: intertemporal substitution dominates the wealth effect with the consequence of a higher volatility of the return on the consumption asset.

In Appendix 1 we also show how to derive a first order linear approximation of the model. We obtain the following analytical expressions for stochastic discount factors, exchange rates and returns:

\[
m^i_{t+1} = \log\delta - \frac{1}{\psi}x^i_t - \gamma c^i_{c,t+1} + \frac{\delta (1 - \gamma \Psi)}{\psi(1 - \rho_x \delta)} \epsilon^i_{x,t+1}
\]

\[
\frac{e^i_{t+1}}{e^i_t} = m^f_{t+1} - m^h_{t+1}
\]

\[
r^i_{c,t+1} = \bar{r}_c + \frac{1}{\psi}x^i_t + \delta \frac{1 - \frac{1}{\psi}}{1 - \rho_x \delta} \epsilon^i_{x,t+1} + \epsilon^i_{c,t+1}
\]

\[
r^i_{f,t+1} = \bar{r}_f + \frac{1}{\psi}x^i_t, \quad \forall i \in \{h, f\}
\]

where \( r^i_{f,t+1} \) is the log-risk free rate and \( \bar{r}_j \) is the average return on asset \( j \), \( \forall j \in \{c, f\} \). Given the system (1.12), the following two propositions can be stated.

**Proposition 2** For a given choice of parameters and provided that \( \rho^h_{x,f} \geq \rho^h_{c,f} \), the lowest cross country correlation of the stochastic discount factors is achieved for

\[
\psi = \frac{1}{\gamma} \tilde{\delta}, \quad \rho_x = 0, \quad \rho^h_{x,f} = \rho^h_{c,f}
\]

where \( \tilde{\delta} = \frac{1 - 2\rho_x \delta + \delta^2}{\delta^2 (1 - \rho_x^2)} \). Furthermore, if \( \rho^h_{x,f} > \rho^h_{c,f} \), then \((\psi, \rho_x) = \left( \frac{1}{\gamma} \tilde{\delta}, 0 \right)\) is the unique minimizer.

**Proof.** See Appendix 2.
Proposition 3  For a given choice of parameters, the lowest volatility of the depreciation rate is achieved for $\rho_{zf} = 1$.

Proof. See Appendix 2. ■

Proposition 1 and 2 are intended to stress that three crucial ingredients are needed in order to obtain highly correlated stochastic discount factors together with lowly volatile exchange rates. First of all we need to break the tight relationship between risk aversion and intertemporal elasticity of substitution, that constant relative risk aversion preferences would otherwise impose. Indeed, as the subjective discount factor $\delta$ approaches unity, as it is the case when the model is calibrated to a monthly or quarterly decision problem, the minimum correlation of stochastic discount factors is obtained for a value of intertemporal elasticity of substitution that is arbitrarily close to the reciprocal of the coefficient of risk aversion. This is independent of the calibration of the rest of the model. Hence it does not seem implausible to think of the contribution of Brandt, Cochrane, and Santa-Clara (2004) as positing a lower bound on what the correlation of stochastic discount factors in a consumption based asset pricing model would look like. Proposition 1 is silent about agents preferring early or late resolution of uncertainty\(^3\). In the next section we will show that any departure from constant relative risk aversion preferences will

---

\(^3\)Epstein and Zin (1989) argue that an agent prefers early resolution of uncertainty if $\psi > 1/\gamma$, while late resolution is preferred if $\psi < 1/\gamma$. 

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do the job of obtaining highly correlated stochastic discount factors. However we can anticipate that we will not want to make the intertemporal elasticity of substitution too small, as this would result in the counterfactual outcome of infinitely volatile returns, as suggested by the system of equations (1.12).

Proposition 1 stresses the role of the predictable component of consumption growth. The intuition is straightforward. Epstein and Zin (1989) preferences bought us an extra term in the formula for the stochastic discount factors in the two countries, in the form of the return that pays one unit of the consumption bundle at each period. The price of this asset reflects the total expected flow of dividends, in our case consumption, that it entitles to. Since we model consumption as having a predictable component, it is natural to conclude that this term will have an impact on the return of the asset that will be proportional to its persistence. By appropriately raising the correlation of the $x_t$'s we will manage to increase the correlation of the returns on the consumption assets and ultimately the correlation of stochastic discount factors. This is also the way Proposition 2 should be read. As the predictable components of consumption get more and more correlated, the stochastic discount factors behave in the same way, decreasing the role of the exchange rate in eliminating international arbitrage opportunities. In the next section we show how it is only the combination of all the three ingredients put forward by Propositions 1 and 2 that delivers our result.
Proposition 1 argues that a highly persistent $x_t$ component is needed in both countries in order to raise the correlation of the stochastic discount factors. Recent studies by Bansal, Gallant, and Tauchen (2002) and Bansal and Yaron (2004) provide estimates and calibrations of this number to a value very close to unity. Furthermore propositions 1 and 2 combined require a high cross-country correlation of the $x_t$ components as a necessary condition to increase the correlation of the discount factors and to keep the volatility of the depreciation rate at a low level. By staring at equations (1.7) and (1.9) one might be tempted to say that the quasi unit root process of $x_t$ together with the high correlation of $x_t$ across countries is guiding the result, by increasing the cross country correlation of consumption. However an explanation of our result based exclusively on the magnitude effect of $x_t$ on the cross-country correlation of consumption would completely overlook the key feature of the model and we would regard this as a failure of our analysis being the correlation of the consumption processes in the order of 30% in the data. As a consequence we will set the standard deviation of $\varepsilon_x$ to a small number to offset the impact of $\rho_x \approx 1$.

A third ingredient must to be added to the picture. Proposition 2 calls for the need of Epstein-Zin preferences to break the link between risk aversion and intertemporal elasticity of substitution. As $\delta$ and $\rho_x$ approach 1, the timing of the resolution of uncertainty becomes part of the problem and a considerable higher $\rho_{m_h,m_f}$ can be
achieved. As shown in equation (1.4), Epstein-Zin preferences introduce an extra term in the pricing kernel that involves the return on the consumption asset and that is not present with CRRA preferences. As \( x_t \) is a predictable and persistent component of consumption growth, it is going to affect crucially the stream of dividends paid by the consumption asset. Therefore it is going to be key in determining expected value and volatility of this return. Furthermore, the fact that this long run component is highly correlated across countries implies an extremely high correlation of the new term introduced in the pricing kernel by the Epstein and Zin preferences.

In the next section we show how it is only the combination of all the three ingredients put forward by Propositions 1 and 2 that delivers our result.

1.3 Results from a calibrated economy

1.3.1 Choice of parameters

In this section we report the results of a calibrated economy of the type discussed earlier. We assume that the countries share the same calibration, as reported in Table 1.1. As the structure of our two parallel economies mimics those discussed by Bansal and Yaron (2004) and Bansal, Gallant, and Tauchen (2002) most of the coefficients used in our analysis are either estimated or calibrated in those papers.
We choose our model to describe a monthly decision problem and as a consequence we set the subjective discount factor to 0.998. In terms of Proposition 1, this means that the correlation of stochastic discount factors is minimized for values of $\psi \approx 1/\gamma$. We set the coefficient of risk aversion $\gamma$ equal to 4.25, that is a number relatively low compared to what is commonly found in the equity premium puzzle literature and with the number proposed by Brandt, Cochrane, and Santa-Clara (2004) in their extension to an international context. The intertemporal elasticity of substitution, $\psi$ is equal to 2 and is the one estimated by Bansal, Gallant, and Tauchen (2002). As far as the calibration of the parameters of the laws of motion of consumption growth is concerned, our goal is to reproduce the average behavior of consumption growth is the United States and in the United Kingdom. We set $\rho_x = .987$, that is the value estimated by Bansal, Gallant, and Tauchen (2002) and that is slightly higher than the 0.979 calibrated by Bansal and Yaron (2004). The standard error of the idiosyncratic shock to the predictable components is extremely small compared to the one of the idiosyncratic shock to consumption growth, allowing the latter to be the main determinant of the volatility of consumption growth. This is in the spirit of a large part of the literature that models consumption growth as an almost i.i.d. process (see among others Tallarini (2000)). The standard deviation of consumption growth implied by our choice of parameters is approximatively 2.4% in annualized terms, that is in between the average growth of per capita consumption
### Table 1.1

**Baseline Calibration.**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Psi)</td>
<td>Intertemporal elasticity of substitution</td>
<td>2</td>
</tr>
<tr>
<td>(\gamma)</td>
<td>Risk aversion</td>
<td>4.25</td>
</tr>
<tr>
<td>(\delta)</td>
<td>Subjective discount factor</td>
<td>0.998</td>
</tr>
<tr>
<td>(\rho)</td>
<td>Autoregressive coefficient of the long run component (x_t)</td>
<td>0.987</td>
</tr>
<tr>
<td>(\phi_e)</td>
<td>Ratio of long run shock and short run shock volatilities</td>
<td>0.048</td>
</tr>
<tr>
<td>(\sigma)</td>
<td>Standard error of the short run shock to consumption (in %)</td>
<td>(68 \times 10^{-4})</td>
</tr>
<tr>
<td>(\rho_{hf}^x)</td>
<td>Cross country correlation of the long run shock</td>
<td>1.0</td>
</tr>
<tr>
<td>(\rho_{hf}^c)</td>
<td>Cross country correlation of the short run shock to consumption</td>
<td>0.3</td>
</tr>
</tbody>
</table>

**Notes** - The two countries share the same calibration.

of nondurables and services from 1970 to 1998 for the US and the UK.

The choice of the correlations coefficients is driven by the need of matching key features of the data. We set \(\rho_{hf}^x\) to 1 as suggested by Proposition 3 to keep the volatility of the depreciation rate to about 11%. The cross country correlation of the shocks to consumption is chosen so to obtain a correlation of consumption growth in the order of 30%. We will turn to the question of providing evidence for such an extreme correlation coefficient between the shocks to the predictable components of consumption growths in the last section of the paper.

### 1.3.2 Correlation of stochastic discount factors

In the previous section we have discussed how it is the combination of long run risks, Epstein-Zin preferences and cross country highly correlated \(x_t\) to drive the
results. Figure 1.2 details this finding with regard to the correlation of stochastic discount factors. Using equation (1.12), we study how this correlation varies with the coefficient of intertemporal substitution. In each of the two panels, the dark line is drawn according to our baseline calibration. In light of Proposition 1, we are not surprised that the minimum of this graph happens in the vicinity of $\psi \approx 1/\gamma$. By moving away from $1/\gamma$ in either direction, there is a sharp increase in the correlation of stochastic discount factors. However, as stressed in the discussion of the two propositions, increasing the correlation of stochastic discount factors by lowering the intertemporal elasticity of substitution would have the unappealing side effect of pushing the volatilities of returns to infinity. It is also for this reason that in our baseline calibration, we set $\psi = 2$, that leaves us with a correlation that is above 0.9, as price data seem to suggest. Interestingly enough, we do not need to take a stand on the intertemporal elasticity of substitution being greater or smaller than 1, as far as our goal of obtaining highly correlated stochastic discount factors is concerned. Figure 1.2 shows that any $\psi$ larger than 0.5 would do the job of making the stochastic discount factors of the two countries at least 80% correlated, provided that the remaining coefficients are calibrated as in Table 1.1. This comment deserves particular attention, given that the intertemporal elasticity of substitution being greater or smaller than 1 has given rise to a long lasting debate in the literature. Hall (1988) and Lustig and Nieuwerburgh (2005) estimate this number to be below
unity. Guvenen (2005) points out that limited participation in the stock market is able to produce findings consistent with both capital and consumption fluctuations as long as most of the wealth is held by a small fraction of population with a high EIS. Attanasio and Weber (1989) use the *Family Expenditure Survey* to document an intertemporal elasticity of substitution in excess of one in the UK. This paper follows the tradition of the risks for the long run literature in calibrating $\psi = 2$ with the objective of appropriately describing the first two moments of returns, as it will be discussed in the next two sections. In discussing the extensions of this setup we will also argue that a generalization of this model that includes more than one predictable component in the law of motion of consumption growth has the potential of being able to match key moments of returns with a relatively lower intertemporal elasticity of substitution, provided that this coefficient is still higher than the reciprocal of risk aversion.

What happens if we perturb our baseline calibration? The left panel of Figure 1.2 shows that decreasing the correlation of the long run shocks would have the consequence of noticeably decreasing also the correlation of stochastic discount factors. This effect comes through a reduced correlation of the returns on the consumption asset, as we document in the last subsection of our calibration exercise. The right panel of Figure 1.2 shows that reducing the persistence of the predictable components of consumption growths in the two countries would also largely affect the
success of our analysis. Once again, the reduced correlation of stochastic discount factors would come from a reduced correlation in returns. The intuition is that if we reduce the persistence of the predictable part of consumption growth, then the stream of dividends paid by the consumption asset is mainly affected by the idiosyncratic shock to consumption, whose international correlation is quite low. So why do not increase the persistence of the predictable components? Because, by doing this we would increase (at an increasing rate) the share of consumption growth variance that is explained by its predictable component, eventually contradicting the empirical evidence on consumption growth being almost an \(i.i.d\). process. We do not report what happens if we calibrate the coefficient of risk aversion differently, because the outcome is made obvious by Proposition 1: given that the subjective discount factor is close to 1, lowering (increasing) the coefficient of risk aversion would move the minimum of the correlation of stochastic discount factors to the right (left) in Figure 1.2.

1.3.3 Volatility of the depreciation rate

As formulated in section 2, the puzzle has two parts: one is the correlation of stochastic discount factors that we discussed in the previous subsection and the other one is the volatility of the depreciation rate that we analyze here. Figure 1.3 plots
Fig. 1.2 - The role of intertemporal elasticity of substitution. In both panels, the dark line reports the correlation of stochastic discount factors when $\psi$ changes. The grey line on the left panel is drawn for a smaller value of $\rho_x$ everything else equal and the grey line on the right panel is drawn for a lower $\rho_x$.

the volatility of the depreciation rate as obtained from the linear approximation of the model against the coefficient of risk aversion, $\gamma$. The two horizontal dashed lines represent the region in which the volatility of the depreciation rate typically falls for major industrialized countries. In our baseline calibration we set $\gamma = 4.25$ that leaves us with a volatility that is well within the region of interest, as documented by the dark lines in the two panels of Figure 1.3. Increasing the risk aversion of the representative consumers of the two countries to the high levels impelled by the equity premium puzzle literature would push the volatility to counterfactual levels similar to those that motivated Brandt, Cochrane, and Santa-Clara (2004) to postulate the existence of a puzzle. Also in this case, our calibration proves itself crucial. Reducing the correlation of the long run shocks is going to reduce
Fig. 1.3 - The role of risk aversion. In both panels, the dark line reports the volatility of the depreciation rate when $\gamma$ changes. The grey line in the left panel is drawn for a smaller value of $\rho_{hf}$ everything else equal and the grey line in the right panel is drawn for lower $\rho_x$ and $\rho_{hf}$.

the correlation of stochastic discount factors as we documented before and this will open up to a greater role for the exchange rate in avoiding international arbitrage opportunities. The left panel of Figure 1.3 shows that unless agents are made less risk averse, we would be left with too much volatility of the depreciation rate at the calibrated level of risk aversion. This effect could be offset if were also to decrease the persistence of the predictable component, because this would result in the volatility of the stochastic discount factors to fall more than their international correlation. However, as we have shown in Figure 1.2 this is something that we rather avoid and even more so considering the implications that a lowly volatile stochastic discount factor would have for asset returns.
1.3.4 The role of returns

Table 1.2 reports the behavior of other relevant variables as we increase the persistence of the long run component. As already observed, the correlation of the stochastic discount factors is increasing and even more so the closer we get to $\rho_x = 1$. The standard deviation of consumption growth does not change too much, except for the case in which $\rho_x = 0.999$. This is the result of our choice of setting the volatility of the long run term to a tiny number compared to the volatility of $\varepsilon_c$. Indeed the ratio $\sigma_{\varepsilon_c}/\sigma_{\Delta c}$ indicates that the contribution of long run risks to the volatility of consumption growth is always small unless $\rho_x \approx 1$. The two covariance terms that seem to be driving the results are the correlation of the consumption assets and the correlation of consumption growth and consumption asset returns across countries. As the persistence of the $x_t$ component rises, the returns on the consumption asset will mainly reflect the long run perspectives of consumption growth, implying a low correlation with $\Delta c_t$ in any country, that is driven for a large part by the short run shock $\varepsilon_c$, as we have already discussed before. By the same token, returns are going to be increasingly correlated across country as a result of our choice of setting $\rho_x^{hf}$ to one. Since these returns enter the stochastic discount factors of the two countries, a considerably higher correlation of the two can be achieved under our benchmark calibration. Last notice how our choice of setting the correlation of the long run
1.4 Other international moments

In this section we study two generalizations of the model: the introduction of an asset that pays a stream of dividends that mimics the process of US dividends and the introduction of conditional heteroskedasticity. In both cases the focus is on how well our model can match key features of financial markets, given the choice of coefficients that we have described in the previous section and the new ones that we add in this section.
1.4.1 Introducing dividends

We introduce a redundant asset that pays dividends as following a process that is similar to the one that we have used for consumption growth:

\[ \Delta d^i_t = \mu_d + \lambda x_{t-1} + \varepsilon^i_{d,t}, \forall i \in \{h, f\} \]  

(1.13)

with \( \varepsilon^i_{d,t} \) i.i.d. normal with mean zero and variance \((\sigma_d^2)^2, \forall i \in \{h, f\}\). The coefficient \( \lambda \) is referred to as the leverage and is usually set to a number larger than 1. The six shocks of the economy follow a multivariate normal process with covariance matrix \( \tilde{\Sigma} \):

\[
\begin{bmatrix}
\varepsilon^h_{c,t} & \varepsilon^f_{c,t} & \varepsilon^h_{x,t} & \varepsilon^f_{x,t} & \varepsilon^h_{d,t} & \varepsilon^f_{d,t}
\end{bmatrix}' \sim N(0, \tilde{\Sigma})
\]

\( \tilde{\Sigma} = \begin{bmatrix}
\Sigma & 0 \\
0 & \sigma^2_d \varphi^2_{d} H_d
\end{bmatrix} \)  

(1.14)

where

\[ H_d = \begin{bmatrix}
1 & \rho^h_{d} \\
\rho^f_{d} & 1
\end{bmatrix} \]

and \( \Sigma \) is defined as in the previous section. In the Appendix we derive the price-dividend ratio, \( v_{d,t} \) using a first order linear approximation around its steady state. To better describe the nonlinearities in returns we also derive numerical solutions of price-dividend and price-consumption ratios. The Appendix describe in details the numerical algorithm that we employ to minimize the Euler equation errors.
The introduction of the dividends process requires the calibration of four additional parameters. Our guidelines in choosing these coefficients will be to appropriately describe the first two moments of dividend growth, the cross country correlation of dividend growth and the leverage in the US and the UK.

Table 1.3 reports the baseline calibration of $\lambda^i$, $\mu_d^i$, $\varphi_d^i$ and $\rho_{d}^{hf}$, $\forall i \in \{h, f\}$. By construction the variance of dividend growth explained by its predictable component is very small and in the order of 3%. The coefficient $\lambda$ is set in such a way that the ratio $\sigma_{\Delta d}/\sigma_{\Delta c}$ is in the range $(4, 8)$ estimated by Ludvigson, Lettau, and Wachter (2004). In our baseline calibration we set $\lambda = 3.0$ that implies $\sigma_{\Delta d}/\sigma_{\Delta c} = 4.86$. The cross country correlation of the short run shocks to dividends, $\rho_{d}^{hf}$, is set to target the almost null correlation of dividend growths between the US and the UK.

Table 1.4 shows that our baseline calibration is able to match key features of international financial markets. Stochastic discount factors that have the property
### Table 1.4

**Introducing dividends.**

<table>
<thead>
<tr>
<th></th>
<th>US</th>
<th>UK</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho \left( m^h, m^f \right)$</td>
<td>Correlation of SDF</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\sigma \left( \frac{\varepsilon_{t+1}}{\varepsilon_t} \right)$</td>
<td>Volatility of depreciation rate</td>
<td>11.21</td>
<td>11.83</td>
</tr>
<tr>
<td>$E \left( r_d - r_f \right)$</td>
<td>Average excess return</td>
<td>7.02</td>
<td>9.17</td>
</tr>
<tr>
<td>$\sigma \left( r_d - r_f \right)$</td>
<td>Volatility of excess return</td>
<td>17.13</td>
<td>22.83</td>
</tr>
<tr>
<td>$\rho \left( r_d^h - r_d^f, r_d^f - r_f^f \right)$</td>
<td>Correlation of excess returns</td>
<td>0.60</td>
<td>0.58</td>
</tr>
<tr>
<td>$E \left( r_f \right)$</td>
<td>Average risk free rate</td>
<td>1.47</td>
<td>1.62</td>
</tr>
<tr>
<td>$\sigma \left( r_f \right)$</td>
<td>Volatility of risk free rate</td>
<td>1.53</td>
<td>2.92</td>
</tr>
<tr>
<td>$\rho \left( r_f^h, r_f^f \right)$</td>
<td>Correlation of risk free rates</td>
<td>0.65</td>
<td>1.00</td>
</tr>
<tr>
<td>$\sigma \left( r_c \right)$</td>
<td>Volatility of return on cons.</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\rho \left( r_c^h, r_c^f \right)$</td>
<td>Correlation of returns on cons.</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\sigma \left( \Delta c \right)$</td>
<td>Volatility of consumption growth</td>
<td>1.37</td>
<td>2.86</td>
</tr>
<tr>
<td>$\frac{\sigma^2(x)}{\sigma^2(\Delta c)} \times 100$</td>
<td>Share of predictable cons. variance</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\sigma \left( \Delta d \right)$</td>
<td>Volatility of dividend growth</td>
<td>16.85</td>
<td>6.87</td>
</tr>
<tr>
<td>$\rho \left( \Delta c^h, \Delta c^f \right)$</td>
<td>Correlation of consumption growth</td>
<td>0.28</td>
<td>0.35</td>
</tr>
<tr>
<td>$\rho \left( \Delta d^h, \Delta d^f \right)$</td>
<td>Correlation of dividend growth</td>
<td>-0.03</td>
<td>-0.07</td>
</tr>
</tbody>
</table>

Notes - All figures are annualized. All coefficients are set to the numbers reported in Table 1.1 and Table 1.3.

of being highly correlated produce excess returns whose first two unconditional moments are in the range that we observe for the US and the UK. This framework also delivers low average and low volatility excess returns. Bansal and Yaron (2004) showed that this class of models had the potential of appropriately describing this moments in the US. We extend their analysis to show that a similar setup can also explain the same moments in the United Kingdom. We push our analysis one step forward, showing that the model can also deliver the average correlation of excess
returns in the two countries. This is accomplished in spite of the low calibrated correlation of dividend growth. One dimension along which our model fails to match the data is the correlation of risk free rates. Even though we use a numerical solution of the model to construct the moments reported in Table 1.4, a linear approximation of the model can help explaining the result. The Appendix shows that the approximate solution for the risk free rates in this economy is a linear function of the predictable component of consumption growth only and as a consequence the correlation of risk free rates will entirely reflect the perfect correlation of the $x$’s. If on the one hand this can be regarded as a failure of the model, on the other hand this invites us to reflect on the importance of risk free rates and other yields to extent that the econometrician wants to identify the low frequency component of consumption growth. In a later section, we discuss how a generalization of this framework that includes more than one predictable component of consumption growth in each country has the potential of reconciling the correlation of risk free rates that we observe in the data and that we obtain from the model.

1.4.2 Introducing stochastic volatility

To allow for time varying economic uncertainty, we let shocks to consumption, dividends and to their predictable components be conditionally heteroskedastic. In
particular we replace the coefficient $\sigma$ in (1.10) and (1.14) with the time varying coefficient $\sigma_t$, whose square follows the autoregressive process:

\[
(\sigma_t^2)^i = \bar{\sigma}^2 + \nu_1 \left[ (\sigma_{t-1}^2)^i - \bar{\sigma}^2 \right] + \sigma_w \varepsilon_{\sigma,t}^i
\]  

(1.15)

Our guidelines in calibrating the coefficients of equation (1.15) are given by the recognition that a linearized approximation of the model implies the following expression for the conditional variance of returns to the dividend asset:

\[
Var_t (r_{t+1}^d) = (1 - \nu_1)k_0 + \nu_1 Var_{t-1} (r_t^d) + k_1 \sigma_w \varepsilon_{\sigma,t}
\]

with $k_0$ and $k_1$ approximation constants. The literature on conditional variance (see inter alia Bollerslev, Engle, and Nelson (1994)) recommends to choose a highly persistent autoregressive coefficient with a small contribution by the innovation shock. As a consequence, we choose $\nu_1 = 0.96$ and $\sigma_w = 2.3 \times 10^{-6}$. We set $\bar{\sigma} = 68 \times 10^{-4}$, so that the unconditional variance of shocks remains unchanged compared to the previous sections. We also assume that the shocks to the stochastic volatility process are i.i.d. both within and across countries.

Table 1.5 shows the results when stochastic volatility is introduced in the model. The findings of the table favor the statement that our results seem to be robust to the introduction of time varying economic uncertainty. In fact none of the numbers reported seem be dramatically different from the corresponding row of Table 1.4. It
is probably worth noticing that the introduction of stochastic volatility managed to break the perfect correlation of risk free rates. This is due to the the fact that the shocks to the stochastic volatility process are assumed to be uncorrelated. However increasing the contribution of stochastic volatility as a mean of further reduction of the correlation of risk free rates would have the unappealing consequence of increasing the volatility of the depreciation rate and decreasing the correlation of stochastic discount factors, as the first two lines of Table 1.5 seem to suggest.
Chapter 2

Estimation

The previous sections have shown that it is the combination of three crucial ingredients that can address the puzzle raised by Brandt, Cochrane, and Santa-Clara (2004). In particular, we need to disentangle the intertemporal elasticity of substitution from the coefficient of risk aversion and we need to model consumption growth has having a highly persistent and cross country correlated predictable component. Can we justify the assumptions that we made about the consumption processes on the grounds of the data that we have? The literature (Bansal and Yaron (2004) and Hansen, Heaton, and Li (2004)) has typically struggled to find conclusive evidence of the consumption process that we adopt in our model, when the focus is on the US only. We study the implications of adding one more country to the analysis, allowing for shocks to consumption and its predictable component to be cross-country
correlated.

2.1 The data

We use the same data set of Campbell (2003). Data are quarterly observations on private consumption of nondurables and services obtained from NIPA for the US and from IMF’s ‘International Financial Statistics’ (henceforth IFS). These quantities are deflated using the Consumer Price Index provided by CRSP for the United States and by IFS for the other countries\(^1\). The common sample spans from 1970.1 to 1998.4. In the last part of this section we also use price data. In particular, the real US Dollar-British Pound Exchange Rate is derived as the ratio of the price index in UK currency and the MSCI\(^2\) Price Index in dollar terms. The risk free rates are T-bill rates from CRSP for US and the rate at which 91-day T-bills are allotted in the UK. Stock market returns and dividends are for value-weighted portfolios obtained from CRSP for the US and from Morgan Stanley Capital Perspective.

\(^1\)Quarterly series extracted from monthly series by selecting the CPI on the last month of the quarter

\(^2\)Tax credit on dividends, applicable to the UK, is available only to local investors. Since the MSCI calculates returns from the perspective of US investors, it excludes from its indices the tax credits which are available only to local investors. Hence to obtain the gross return to local investors, we add back the tax credits of 25% for the UK.
2.2 Spectral analysis

Since we want to identify $x^h_t$ and $x^f_t$, the low frequency components of consumption growth, we start our empirical investigation by looking at the spectral densities of consumption growth in the US and the UK. Denoting as $Y_t = [\Delta c^h_t \Delta c^f_t]$ the vector of consumption growth in the US and the UK at time $t$, the population spectrum at frequency $\omega$ is

$$S_Y(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{k=\infty} \Gamma_k e^{-ik\omega}$$

(2.1)

where $\Gamma_j = E [Y_t - \bar{Y}] [Y_{t-j} - \bar{Y}]'$ and $\bar{Y} = E[Y_t]$. We follow Hamilton (1994) in estimating $\Gamma_j, \forall j$ with their sample counterparts:

$$\hat{\Gamma}_j = \frac{\sum_{t=j+1}^{T} (Y_t - \bar{Y}) (Y_t - \bar{Y})'}{T}$$

and smoothing (2.1) with a 10 period Bartlett (1964) window

$$\hat{S}_Y(\omega) = \frac{1}{2\pi} \sum_{k=-10}^{k=10} \left(1 - \frac{|k|}{10}\right) \hat{\Gamma}_k e^{-i\omega k}$$

(2.2)

As a measure of the covariance explained at different frequencies, we consider the coherence, defined as:

$$K^2_{\Delta c^h, \Delta c^f}(\omega) = \frac{\left|\frac{\hat{S}_{\Delta c^h, \Delta c^f}(\omega)}{\hat{S}_{\Delta c^h}(\omega) \hat{S}_{\Delta c^f}(\omega)}\right|^2}{\hat{S}_{\Delta c^h}(\omega) \hat{S}_{\Delta c^f}(\omega)}$$

The two panels of Figure 2.1 report the estimated spectra and coherence of US and UK consumption growth (continuous lines) along with the theoretical periodograms and the 95% confidence intervals (dashed lines) under the alternative
assumptions that consumption growth is \( i.i.d. \) (left panel) and that it has a small predictable component (right panel). The latter are computed simulating 1000 independent samples according to model (1.7) with the calibration of Table 1.1 expect for \( \sigma^h_c, \sigma^f_c \) and \( \rho^{hf}_c \), that are set so that the simulated series have the same sample variance and covariance of the actual series. The main finding of Figure 2.1 is that the two alternative assumptions look equally likely to say the least, as the estimated spectra and coherence typically lie within the 95% confidence intervals. This anticipates a recurrent result of the next subsections: quantity data are not enough to identify the low frequency component of consumption growth.

Fig. 2.1 - Sample periodogram. The grey line is the data sample periodogram, while the black line is the theoretical periodogram along with the 95% confidence interval (dashed lines).
2.3 A state space model

The way in which we model consumption growth falls directly in the class of state-space models. In particular, if we denote $Y_t' = \begin{bmatrix} \Delta c_t^h \\
\Delta c_t^f \end{bmatrix}$ as the vector of variables that are observable to the econometrician and $X_t' = \begin{bmatrix} x_t^h \\
x_t^f \end{bmatrix}$ as the vector of variables that are unobservable to the econometrician, we have the following representation:

$$Y_t - \begin{bmatrix} \mu_c^h \\
\mu_f^c \end{bmatrix} = +X_t + \begin{pmatrix} \sigma^h \\
\sigma^f \end{pmatrix}' \begin{bmatrix} 1 & \rho_{c}^{hf} \\
\rho_{c}^{hf} & 1 \end{bmatrix} \begin{bmatrix} \sigma^h \\
\sigma^f \end{bmatrix} \frac{1}{2} \begin{bmatrix} \varepsilon_{t, c}^h \\
\varepsilon_{t, f}^c \end{bmatrix}$$

$$X_{t+1} = \begin{bmatrix} \rho_x^h & 0 \\
0 & \rho_x^f \end{bmatrix} X_t + \begin{pmatrix} \sigma^h & \varphi_{c}^{h} \\
\sigma^f & \varphi_{c}^{f} \end{pmatrix}' \begin{bmatrix} 1 & \rho_{x}^{hf} \\
\rho_{x}^{hf} & 1 \end{bmatrix} \begin{bmatrix} \sigma^h & \varphi_{c}^{h} \\
\sigma^f & \varphi_{c}^{f} \end{bmatrix} \frac{1}{2} \begin{bmatrix} \varepsilon_{t, x}^h \\
\varepsilon_{t, x}^f \end{bmatrix}$$

or, in more compact notation:

$$Y_t - \bar{Y} = AX_t + C \varepsilon_{t, c}$$  \hspace{1cm} (2.3)$$

$$X_{t+1} = BX_t + D \varepsilon_{t, x}$$

Notice that we are now allowing for an asymmetric calibration of the two economies. Aside from one technicality regarding time aggregation of the monthly model into quarterly data on observed consumption growth that we discuss in the Appendix, this model can be directly estimated via maximum likelihood, using the Kalman filter as in Hansen and Sargent (2004). We also impose that the volatility coefficients
Table 2.1

<table>
<thead>
<tr>
<th>Sample size</th>
<th>Real Data</th>
<th>Simulated data</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>T=120</td>
<td>T=120</td>
</tr>
<tr>
<td>$\rho^h$</td>
<td>0.909</td>
<td>0.941</td>
</tr>
<tr>
<td></td>
<td>[0.547, 0.995]</td>
<td>[0.731,1.000]</td>
</tr>
<tr>
<td>$\rho^f$</td>
<td>0.940</td>
<td>0.934</td>
</tr>
<tr>
<td></td>
<td>[0.308, 0.995]</td>
<td>[0.630,1.000]</td>
</tr>
<tr>
<td>$\varphi^h_e$</td>
<td>0.351</td>
<td>0.142</td>
</tr>
<tr>
<td></td>
<td>[0.000, 0.424]</td>
<td>[0.000,0.475]</td>
</tr>
<tr>
<td>$\varphi^f_e$</td>
<td>0.184</td>
<td>0.119</td>
</tr>
<tr>
<td></td>
<td>[0.000, 1.092]</td>
<td>[0.000,1.165]</td>
</tr>
<tr>
<td>$\sigma^h$</td>
<td>38.103</td>
<td>66.083</td>
</tr>
<tr>
<td></td>
<td>[28.562, 38.105]</td>
<td>[51.208,71.617]</td>
</tr>
<tr>
<td>$\sigma^f$</td>
<td>77.944</td>
<td>66.078</td>
</tr>
<tr>
<td></td>
<td>[72.280, 78.009]</td>
<td>[37.388,68.164]</td>
</tr>
<tr>
<td>$\rho_x$</td>
<td>0.897</td>
<td>0.844</td>
</tr>
<tr>
<td></td>
<td>[0.696, 1.000]</td>
<td>[0.467,1.000]</td>
</tr>
<tr>
<td>$\rho_c$</td>
<td>0.208</td>
<td>0.312</td>
</tr>
<tr>
<td></td>
<td>[0.004, 0.406]</td>
<td>[0.066,0.530]</td>
</tr>
</tbody>
</table>

Notes - Estimation of consumption growth processes in the US and in the UK. The second column reports the results of the estimation on real data, while the last two columns report the results of the estimation over 100 independent samples simulated according to the baseline calibration for sample sizes T=120 and T=10000. $\sigma^h_c$, $\sigma^f_c$, $\varphi^h_e$ and $\varphi^f_e$ are nonnegative, that the autoregressive coefficients $\rho^h$ and $\rho^f$ and the correlation coefficients $\rho^hf_x$ and $\rho^hf_c$ are smaller than 1 in absolute value. Due to these constraint, the standard asymptotic theory to compute standard errors of estimated coefficients does not apply. Therefore we follow Runkle (1987) and we construct standard errors by bootstrapping 1000 independent samples from the estimated smoothed residuals.
Table 2.2

ESTIMATING LONG RUN RISKS WITH MORE COUNTRIES.

<table>
<thead>
<tr>
<th></th>
<th>US, UK and Germany</th>
<th>US, UK and Japan</th>
<th>US, UK, Germany and Japan</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_{US}$</td>
<td>0.911</td>
<td>0.906</td>
<td>0.919</td>
</tr>
<tr>
<td></td>
<td>[.530, .990]</td>
<td>[.552, .984]</td>
<td>[.796, .996]</td>
</tr>
<tr>
<td>$\rho_{UK}$</td>
<td>0.928</td>
<td>0.947</td>
<td>0.934</td>
</tr>
<tr>
<td>$\rho_{Ger}$</td>
<td>0.932</td>
<td>-</td>
<td>0.993</td>
</tr>
<tr>
<td></td>
<td>[.274, .985]</td>
<td>-</td>
<td>[.614, .996]</td>
</tr>
<tr>
<td>$\rho_{Jpn}$</td>
<td>-</td>
<td>0.989</td>
<td>0.99</td>
</tr>
<tr>
<td></td>
<td></td>
<td>[.215, .989]</td>
<td>[.504, .998]</td>
</tr>
<tr>
<td>$\phi_{US}$</td>
<td>0.351</td>
<td>0.040</td>
<td>0.177</td>
</tr>
<tr>
<td></td>
<td>[.0001.035]</td>
<td>[.000, .348]</td>
<td>[.000, .206]</td>
</tr>
<tr>
<td>$\phi_{UK}$</td>
<td>0.317</td>
<td>0.182</td>
<td>0.232</td>
</tr>
<tr>
<td></td>
<td>[.0001.484]</td>
<td>[.000,1.557]</td>
<td>[.000, .322]</td>
</tr>
<tr>
<td>$\phi_{Ger}$</td>
<td>0.019</td>
<td>-</td>
<td>0.002</td>
</tr>
<tr>
<td></td>
<td>[.000, .774]</td>
<td>-</td>
<td>[.000, .407]</td>
</tr>
<tr>
<td>$\phi_{Jpn}$</td>
<td>-</td>
<td>0.087</td>
<td>0.090</td>
</tr>
<tr>
<td></td>
<td></td>
<td>[.000,1.558]</td>
<td>[.000,1.312]</td>
</tr>
<tr>
<td>$\sigma_{US}$</td>
<td>38.105</td>
<td>33.622</td>
<td>38.106</td>
</tr>
<tr>
<td></td>
<td>[28.002,38.108]</td>
<td>[27.903,38.104]</td>
<td>[33.409,38.106]</td>
</tr>
<tr>
<td>$\sigma_{UK}$</td>
<td>69.861</td>
<td>69.864</td>
<td>69.863</td>
</tr>
<tr>
<td></td>
<td>[69.798,69.907]</td>
<td>[69.844,69.898]</td>
<td>[69.693,69.956]</td>
</tr>
<tr>
<td>$\sigma_{Ger}$</td>
<td>70.436</td>
<td>-</td>
<td>69.861</td>
</tr>
<tr>
<td></td>
<td>[70.106,70.450]</td>
<td>-</td>
<td>[69.847,69.866]</td>
</tr>
<tr>
<td>$\sigma_{Jpn}$</td>
<td>-</td>
<td>70.452</td>
<td>70.440</td>
</tr>
<tr>
<td></td>
<td></td>
<td>[70.435,70.490]</td>
<td>[70.429,70.452]</td>
</tr>
<tr>
<td>$\rho_{x,UK}$</td>
<td>0.920</td>
<td>0.973</td>
<td>0.913</td>
</tr>
<tr>
<td></td>
<td>[.890, .999]</td>
<td>[.861, .999]</td>
<td>[.700, .999]</td>
</tr>
<tr>
<td>$\rho_{x,Ger}$</td>
<td>0.899</td>
<td>-</td>
<td>0.891</td>
</tr>
<tr>
<td></td>
<td>[.874, 1.000]</td>
<td>-</td>
<td>[.700, .997]</td>
</tr>
<tr>
<td>$\rho_{x,Jpn}$</td>
<td>-</td>
<td>0.991</td>
<td>0.905</td>
</tr>
<tr>
<td></td>
<td></td>
<td>[.877, 1.000]</td>
<td>[.701, 1.000]</td>
</tr>
<tr>
<td>$\rho_{x,UK,Ger}$</td>
<td>0.891</td>
<td>-</td>
<td>0.917</td>
</tr>
<tr>
<td></td>
<td>[.851, 1.000]</td>
<td>-</td>
<td>[.700, .999]</td>
</tr>
<tr>
<td>$\rho_{x,UK,Jpn}$</td>
<td>-</td>
<td>-</td>
<td>0.898</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-</td>
<td>[.648, 1.000]</td>
</tr>
<tr>
<td>$\rho_{x,Ger,Jpn}$</td>
<td>-</td>
<td>-</td>
<td>0.884</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-</td>
<td>[.698, .999]</td>
</tr>
<tr>
<td>$\rho_{c,UK}$</td>
<td>0.142</td>
<td>0.169</td>
<td>0.216</td>
</tr>
<tr>
<td></td>
<td>[-.048, .341]</td>
<td>[-.015, .423]</td>
<td>[.089, .400]</td>
</tr>
<tr>
<td>$\rho_{c,Ger}$</td>
<td>0.111</td>
<td>-</td>
<td>0.151</td>
</tr>
<tr>
<td></td>
<td>[-.088, .285]</td>
<td>-</td>
<td>[-.022, .337]</td>
</tr>
<tr>
<td>$\rho_{c,UK,Jpn}$</td>
<td>-</td>
<td>0.237</td>
<td>0.275</td>
</tr>
<tr>
<td></td>
<td></td>
<td>[.028, .483]</td>
<td>[.082, .446]</td>
</tr>
<tr>
<td>$\rho_{c,UK,Ger}$</td>
<td>0.272</td>
<td>-</td>
<td>0.309</td>
</tr>
<tr>
<td></td>
<td>[.096, .469]</td>
<td>-</td>
<td>[.222, .464]</td>
</tr>
<tr>
<td>$\rho_{c,UK,Jpn}$</td>
<td>-</td>
<td>0.182</td>
<td>0.194</td>
</tr>
<tr>
<td></td>
<td></td>
<td>[-.052, .358]</td>
<td>[.017, .332]</td>
</tr>
<tr>
<td>$\rho_{c,Ger,Jpn}$</td>
<td>-</td>
<td>-</td>
<td>-0.06</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-</td>
<td>[-.219, .202]</td>
</tr>
</tbody>
</table>

Notes - Estimation of consumption growth processes in the US, the UK, Germany and Japan. In squared brackets we report the 95% confidence intervals.
The second column of Table 2.1 reports the estimation results on actual consumption growth data obtained from using Kalman filter and maximum likelihood as detailed in the Appendix. The point estimates of autoregressive coefficients and correlations would not be too far from the values that we calibrated them to in Table 1.1 if it was not for the big 95% confidence intervals associated to them. Furthermore the identification coefficients of the $x_t$ processes, $\varphi^h_x$ and $\varphi^f_x$ are not significantly different from zero. We take these findings as confirming the graphical intuition of Figure 2.1 that we cannot reject neither provide positive evidence of this low frequency component from consumption data only. To check the power of our estimation procedure we also simulated 100 independent samples according to the baseline calibration reported in Table 1.1 for sample sizes $T=120$ (the same of the actual data sample) and $T=10000$ (a much bigger sample). The results for the Monte Carlo experiment of sample size $T = 120$ confirms the difficulty of tightly identifying the coefficients of the model that we experienced on the actual data sample. When we push the sample size to infinity, we find out, not surprisingly, that the estimation procedure is consistent. The punch line seems to be that we simply do not have enough quantity observations in the actual data.
2.4 More quantities, better estimates?

In principle, if the low frequency component of consumption growth is really perfectly correlated across countries, estimating the model including consumption observations from a cross section of countries should help to better identify the coefficients. We explore this idea by including also Germany and Japan in our analysis. The estimation procedure follows the same logic and technique of the previous subsection, while we relegate the construction of the matrices $A$, $B$, $C$ and $D$ in (2.3) to the Appendix. The results as reported in Table 2.2 show that the inclusion of additional countries does not help in tightening the confidence intervals around the estimated coefficients. This is not entirely a negative results, as it would have been a reduction of uncertainty around the wrong values of the parameter. It simply confirms that quantities are not enough to identify the low frequency component of consumption growth.

2.5 More prices, better estimates?

The spirit of this paper and of the literature that this paper belongs to is to try to understand the properties of the consumption process that would allow us to price assets in a no arbitrage framework. Even though these characteristics of consumption cannot be clearly identified from quantity data only, exchange rates
and asset returns should help in this task. We may extend the procedure adopted in the previous subsections to estimate a larger system that also includes exchange rates and asset returns, but we prefer not to do so. The reason is that our model has been shown to provide an excellent description of a number of unconditional moments of international data, but, at least in this basic setup, it is still unsatisfactory along a few dimensions. The unconditional correlation of risk free rates and the unconditional correlation of consumption differentials and exchange rates are certainly hard to get numbers that would require more attention in the research to come. Furthermore the conditional distribution of exchange rates has typically been object of debate as described by Backus and Smith (1993) and more recently by Rey and Gourinchas (2005). Also, our assumptions of complete home bias in consumption and of complete markets are likely to provide an additional source of noise in the exchange rates. For all these reasons, we implement a GMM style estimation, in which we only focus on the set of moments that are the focus of attention of this paper. Alternatively this can be interpreted as a way of providing statistical support to our calibration as described in Tables 1.1 and 1.3. In order to increase the speed of the estimation routine, we restrict our attention to the linearized version of the model\textsuperscript{3}, as summarized by the system of equations (1.7), (1.9), (1.12) and (1.13).

\textsuperscript{3}If we were to use a numerical solution of the model, this would entail re-solving for the equilibrium price-consumption and price-dividend ratios schedules, every time a new vector of parameters is searched for.
### Table 2.3

**Prices and long run risks.**

<table>
<thead>
<tr>
<th></th>
<th>Consumption only</th>
<th>Whole Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Point Estimate</td>
<td>95% CI</td>
</tr>
<tr>
<td>$\rho^h$</td>
<td>0.736</td>
<td>[0.349,0.996]</td>
</tr>
<tr>
<td>$\rho^f$</td>
<td>0.904</td>
<td>[0.015,0.997]</td>
</tr>
<tr>
<td>$\varphi^h_c$</td>
<td>1.422</td>
<td>[0.190,17.318]</td>
</tr>
<tr>
<td>$\varphi^f_c$</td>
<td>0.182</td>
<td>[0.000,3.502]</td>
</tr>
<tr>
<td>$\sigma^h$</td>
<td>27.629</td>
<td>[4.527,34.799]</td>
</tr>
<tr>
<td>$\sigma^f$</td>
<td>79.916</td>
<td>[44.407,87.421]</td>
</tr>
<tr>
<td>$\rho^h_x$</td>
<td>0.999</td>
<td>[0.353,1.000]</td>
</tr>
<tr>
<td>$\rho^h_f$</td>
<td>0.222</td>
<td>[-0.988,0.999]</td>
</tr>
<tr>
<td>$\psi^h$</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\psi^f$</td>
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<td>-</td>
</tr>
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<td>$\gamma^h$</td>
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<td>$\gamma^f$</td>
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<td>-</td>
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<td>$\lambda^h$</td>
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<td>-</td>
</tr>
<tr>
<td>$\lambda^f$</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\varphi^h_d$</td>
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<td>-</td>
</tr>
<tr>
<td>$\varphi^f_d$</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\rho^h_f$</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

#### Unconditional moments (at point estimates)

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma(\Delta c^h)$</td>
<td>1.406</td>
<td>1.588</td>
</tr>
<tr>
<td>$\sigma(\Delta c^f)$</td>
<td>2.819</td>
<td>2.812</td>
</tr>
<tr>
<td>$\rho(\Delta c^h, \Delta c^f)$</td>
<td>0.296</td>
<td>0.332</td>
</tr>
<tr>
<td>$\sigma(\varepsilon_{t+1}/\varepsilon_t)$</td>
<td>-</td>
<td>11.692</td>
</tr>
<tr>
<td>$\rho(m^h, m^f)$</td>
<td>-</td>
<td>0.922</td>
</tr>
<tr>
<td>$\rho(r^h_{d,t}, r^f_{d,t})$</td>
<td>-</td>
<td>0.475</td>
</tr>
</tbody>
</table>

Notes - The second column reports the results of the estimation on consumption data only, while the fourth column reports the results of the estimation of the whole model. In parenthesis we report the 95% confidence intervals. The subjective discount factors $\delta^h$ and $\delta^f$ are calibrated as in Table 1. The bottom panel of the table reports the unconditional moments in annualized terms of the variables of interest using the point estimates.
In particular, we focus on the variances, the first two auto-covariances, the covariance and the first order cross auto-covariance of consumption growths, on the variances and the covariance of dividend growths, on the variance of the depreciation rate, on the variances of risk free rates, on the variances, the first order auto-covariances and the covariance of excess returns and on the average excess returns. There is a total of 22 moments conditions to identify 17 coefficients, as detailed in Table 2.3. Due to the analytical difficulty of computing these moment conditions for quarterly data starting from a model that is specified at a monthly frequency, we follow Lee and Ingram (1991) and Duffie and Singleton (1993) in implementing a Simulated Method of Moments. This amounts to computing the afore mentioned moments from the data set that we have and minimizing the distance between them and their counterparts obtained from simulating the model. Since the model includes consumption growth and returns moment conditions, the use of a Newey and West (1987) weighting matrix would suggest to put a much bigger weight on consumption relative to returns moments. To avoid this unappealing consequence, we use an identity weighting matrix. The estimator is going to be inefficient, but consistency is guaranteed. One more caveat is brought into the problem by the fact that we need to constrain the correlation coefficients to be bounded by 1 in absolute value\(^4\). Although this constraint is not binding in the

\(^4\)We also impose nonnegativity constraints on volatilities and on preference parameters, al-
estimation, Andrews (2002) warns that theoretically when the true parameter is on
the boundary, GMM/SMM estimators are not asymptotically normally distributed.
As a consequence, the asymptotic results of Hansen (1982) cannot be employed to
derive confidence intervals of the estimated coefficients. We derive 95% confidence
intervals using a Monte Carlo procedure. After consistently estimating the param-
eters of the model using a simulated sample of 120,000 monthly observations, we
use these coefficients to generate 1,000 independent samples of length 120 quarters,
the same that we have in the data. We then estimate the coefficients of interest
using each of these simulated samples and construct confidence regions based on
the parameters’ distribution across samples.

Table 2.3 shows that adding price restrictions helps in tightening the point esti-
mates of the consumption laws of motion parameters to the values that we specified
in the theoretical analysis of the model. This procedure is still unable to properly
identify the preference parameters and the coefficients that describe the laws of mo-
ton of dividends. Our explanation of this difficulty relies on the fact that we are
only employing 13 moment conditions that involve these parameters in the form of
exchange rate, risk free rates and stock returns. Future extensions should focus on
generalizing the set of properties of the model, allowing for a wider set of moments
restrictions to be used in the empirical analysis.

though they are never binding.
Chapter 3

Misconceptions and extensions

If the main anomaly addressed in the previous sections is a generalization of Mehra and Prescott (1985) equity premium puzzle, then why not use any of the other solutions that the literature has proposed over the years? What is the meaning of the exchange rate in an economy without trade? What if the predictable components are not observable? Where are these predictable components of consumption growth coming from? These are only few of the questions that will be addressed in this chapter.
3.1 Competing explanations?

Twenty an counting years of literature on the equity premium puzzle have delivered a multitude of competing explanations of this failure of the standard time-separable utility framework. Here we focus only on two of them: the habit model of Campbell and Cochrane (1999) and the limited participation study of Vissing-Jørgensen (2002). Since the lower bound on the international correlation of stochastic discount factors outlined in Proposition 1 is based on the assumption of market completeness, we also explore the possibility of relaxing this assumption.

3.1.1 Alternative solution of the equity premium puzzle

Campbell and Cochrane (1999) assume that a representative consumer has preferences defined around a time varying habit level for consumption:

\[
U_t = \mathbb{E}_t \sum_{t=0}^{\infty} \delta^t \frac{(C_t - X_t)^{1-\gamma} - 1}{1 - \gamma}
\]

where \(\delta\) is the subjective discount factor, \(\gamma\) is the coefficient of relative risk aversion, \(C_t\) is the level of consumption at time \(t\) and \(X_t\) is the afore-mentioned level of habit. After defining the logarithm of the surplus consumption ratio as \(s_t = \log(C_t - X_t) - \log C_t\), the model is closed by specifying exogenous laws of motion for habits and
consumption:

\[
\log \left( \frac{C_{t+1}}{C_t} \right) = \Delta c_{t+1} = \mu_c + v_{t+1}
\]

\[
s_{t+1} = (1 - \phi) \bar{s} + \phi s_t + \lambda (s_t) v_{t+1}
\]

where

\[
v_{t+1} \sim N \left( 0, \sigma^2 \right), \quad \bar{s} = \log \sigma \sqrt{\frac{\gamma}{1 - \phi}} \quad \text{and} \quad \lambda (s_t) = \frac{1}{\exp \bar{s}} \sqrt{1 - s (s_t - \bar{s})} - 1
\]

The logarithm of stochastic discount factors can be computed as:

\[
m_{t+1} = \log \frac{\partial U_{t+1}/\partial C_{t+1}}{\partial U_t/\partial C_t} = \log \delta - \gamma \mu_c - \gamma \left\{ (\phi - 1) (s_t - \bar{s}) + \left[ 1 + \lambda (s_t) \right] v_{t+1} \right\}
\]

Assuming that the home and the foreign country share the same parametrization, allows us to obtain stochastic discount factors for home and foreign currency denominated as that mimic the one reported in equation (3.1):

\[
m_{t+1}^i = \bar{m} - \gamma (\phi - 1) \sum_{j=0}^{+\infty} (\phi L)^j \lambda (s_{t-1}^i) v_{t+1}^i - \gamma \left[ 1 + \lambda (s_t^i) \right] v_{t+1}^i
\]

\[
m_{t+1} = \bar{m} - \gamma \bar{m}_{t+1}, \quad \forall i \in \{h, f\}
\]

where \( s_t^i \) and \( v_t^i \) are international generalization of the variables reported in the setup of the model. It is apparent from the previous equation that the international correlation of stochastic discount factors is driven by the correlation of the shocks \( v_t^h \) and \( v_t^f \). But the correlation of these two shocks cannot be set arbitrarily, because
it also determines the cross-country correlation of consumption growths. Figure 1.1 shows how this correlation is typically very low, when not negative if the United States are the home country. Hence this results in the inability of the model of matching simultaneously the high correlation of stochastic discount factors (and as a by-product the low volatility of the depreciation of the US Dollar) and the low correlation of consumption growth between countries.

Would this finding be changed by the introduction of long run risks? No, because agents are not concerned about them. To see this just modify the setup of the model by introducing a common low frequency component in the dynamics of consumption growth for the two countries

\[
\Delta c_{t+1}^i = \mu_c + x_t + \nu_{i+1}^i, \quad \forall i \in \{h, f\}
\]

\[
x_{t+1} = \rho x_t + w_{t+1}, \quad w_{t+1} \sim N(0, \varphi_e \sigma)
\]

and observe that log-stochastic discount factors are equal to:

\[
m_{t+1}^i = \bar{m} - \gamma (\bar{m}_{t+1}^i + x_t), \quad \forall i \in \{h, f\}
\]

(3.3)

Now, since the volatility of the shock to the long run component has to be small to be consistent with empirical evidence of consumption growth following an almost i.i.d. process, the contribution of \( x_t \) to the international correlation of stochastic discount factors will too small to increase it significantly over the low degree of correlation.
of consumption growths. Adding a long run shock to the model does not affect the result, if agent do not care about the timing of the resolution of uncertainty.

Vissing-Jørgensen (2002) proposes a limited participation approach and using survey data she estimates the preference parameters using the Euler equations of those agents that hold a non-zero position in a given asset. In her exercise she revises the finding of Hall (1988) that the intertemporal elasticity of substitution should be close to zero and extends the results of Attanasio and Weber (1989) for UK survey data. It would be of extreme empirical interest to use the survey data for these countries to compute the international correlation of consumption growth for those agents, whose behavior is appropriately described by the Euler equations in question. Although and educated guess would lead us to think that this correlation should probably be higher than the one that we observe in aggregate data, on the grounds of international risk sharing motive, we remain skeptic that it would be so high to explain on its own the near one correlation of stochastic discount factors suggested by international financial markets. Nevertheless, we regard this as an interesting extension of the current study and we recognize the possibility of a higher correlation of consumption growths as a mean of relaxing our somewhat extreme assumptions on the persistence and cross-country correlation of consumption’s low frequency components.
3.1.2 What if markets are incomplete?

Brandt, Cochrane, and Santa-Clara (2004) explore the possibility of markets being incomplete, concluding that this does not offer an answer to the puzzle that they set forward. Their argument is straightforward. If markets are incomplete, any stochastic discount factor that is obtained as the sum of the complete markets stochastic discount factor and an orthogonal source of zero mean noise is an admissible one. In formulas:

\[
\widetilde{M}_t^{i} = M_t^i + \xi_t^i, \quad \forall i \in \{h, f\}
\]

The relationship between stochastic discount factors and depreciation of the domestic currency must still hold for the minimum variance stochastic discount factors \(M_{t+1}^h\) and \(M_{t+1}^f\):

\[
\frac{M_{t+1}^f}{M_{t+1}^h} = \frac{e_{t+1}}{e_t}
\]

Since the complete markets part of the stochastic discount factors is highly correlated across countries, the only way to lower their correlation is to: 1) let \(\xi_t^h\) and \(\xi_t^f\) be poorly or negatively correlated and 2) let \(\xi_t^h\) and \(\xi_t^f\) be highly volatile. Then the question becomes: how much volatility should we add to the stochastic discount factors to make them as correlated as consumption data would suggest? Brandt, Cochrane, and Santa-Clara (2004) show that variance of the stochastic discount factors should be so high that it would be impossible for any consumption based
asset pricing model to reproduce it with a reasonable calibration of the preference parameters. In this way they rule out market incompleteness as a way to resolve the international equity premium puzzle.

3.2 Justifying the real exchange rate in the absence of trade

What is the meaning of the exchange rate in an economy without trade? If all assets are traded domestically and internationally, there must be Euler equations for holdings of any asset. The Euler equations for holdings of the same asset by a domestic and a foreign investor will then tie down the depreciation of the exchange rate in an arbitrage free environment. This is in the spirit of Lucas (1978) tree’s model: although there is no trade going on in the economy, we can still give a price to any asset by reading it off the first order conditions.

As an alternative, it is possible to think of the current setup of the model as the limiting case of an economy in which there is a consumption aggregator that attaches a weight $\alpha$ to domestic goods and a weight $1 - \alpha$ to foreign goods. The exchange rate is then defined as the relative price of the consumption aggregate in the two countries. We focus on the case in which $\alpha$ tends to zero. With standard
time separable preferences it is trivial to show that this exchange rate exists (see for example Obstfeld and Rogoff (1996)). With the kind of preferences that we are using, the matter is more complicated, but following Anderson (2005) it can be shown to exist.

This also opens up to the question of what would happen if we were to relax the assumption of complete home bias. In principle this is a very hard question to give an answer to, given the degree of difficulty that is involved in solving for the dynamics of risk sensitive allocations\(^1\). However it can be anticipated that things are not going to differ too much from what we previously reported, because a realistic calibration along the lines of what is suggested by Lewis (1999) would have to take into account the remarkable degree of consumption home bias that we observe in the data.

### 3.3 Unobservable predictable components

What if the predictable components \(x^h_t\) and \(x^f_t\) are unobservable? We can address this question by assuming that the representative consumers in the two countries only observe consumption growths and knowing the model through which they

\(^1\)With standard preferences, the solution of the Pareto problem is given by consumption allocations that are functions only of the current endowment and of the Pareto vector. With non-time separable preferences instead, consumption allocations are functions of current endowments, Pareto weights and continuation utility. Anderson (2005) suggests to recast this problem in terms of a time varying Pareto vector.
evolve, the use the Kalman filter to extract the predictable components. In formulas, let the joint dynamics of consumption growth be described by the following state-space representation:

\[
\Delta c_t = CX_t + Dw_t \quad (3.4)
\]

\[
X_{t+1} = AX_t + Bw_t, \quad w_t \sim N(0, I_4)
\]

where

\[
\Delta c_t = \begin{bmatrix} \Delta c^h_t \\ \Delta c^f_t \end{bmatrix}, \quad X_t = \begin{bmatrix} x^h_{t-1} \\ x^f_{t-1} \end{bmatrix}, \quad w_t = \begin{bmatrix} \varepsilon^h_{x,t} & \varepsilon^f_{x,t} & \varepsilon^h_{c,t} & \varepsilon^f_{c,t} \end{bmatrix}'
\]

and

\[
A = \begin{bmatrix} \rho^h_x & 0 \\ 0 & \rho^h_x \end{bmatrix}, \quad B = \sigma_x \begin{bmatrix} 1 & 0 & 0 & 0 \\ \rho^h_x \sqrt{1 - \rho^2_x} & 0 & 0 \end{bmatrix},
\]

\[
C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = \sigma_c \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & \rho_c \sqrt{1 - \rho^2_c} \end{bmatrix}
\]

Following Hansen and Sargent (2004), we can write the innovation representation:

\[
\Delta c_t = CX_t + a_t \quad (3.5)
\]

\[
\hat{X}_{t+1} = A\hat{X}_t + Ka_t
\]

where \( K \) is the solution of the recursion

\[
K = (A\Sigma C' + BD') (C\Sigma C' + DD')^{-1}
\]

\[
\Sigma = A\Sigma A' + BB' - (A\Sigma C' + BD') (C\Sigma C' + DD')^{-1} (A\Sigma C' + BD')'
\]

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and the covariance matrix of $a_t$ is $E a_t a'_t = C \Sigma C' + D D'$.

Fig. 3.1 - Correlation of stochastic discount factors when the predictable components are unobservable. The red line reports the correlation under the assumption of unobservable components, while the dashed-dot line represents the benchmark case of observable predictable components. The calibration of parameters is the same of Table 1.1.

The relevant state variable for time $t$ is now $\hat{X}_{t+1}$, that according to (3.5) depends only on time $t$ information. The relevant vector of innovation is $a_t$. Using the linear approximation of the model that we derive in Appendix 1, we can then write the
stochastic discount factor as a function of the filtered information:

\[
\begin{bmatrix}
    m_{t+1}^h \\
    m_{t+1}^f
\end{bmatrix} = \log \delta - \frac{1}{\Psi} \hat{X}_{t+1} - \gamma a_t + \frac{\delta (1 - \gamma \Psi)}{\Psi (1 - \rho_x \delta)} K a_t
\]  

(3.6)

where \( \rho_x = \rho_x^h = \rho_x^f \). The correlation of stochastic discount factors can be computed directly from equation (3.6).

Using the same calibration of Table 1.1, we can compare the results under the two cases of observable and unobservable \( x_t \)'s. Figure 3.1 depicts the two scenarios. The thing that should be noticed is that for the range of values of interest for the intertemporal elasticity of substitution, the correlation that is obtained by filtering the low frequency components from consumption growths (red line) is almost identical to the one that we obtain in the baseline model where agents can condition on \( x_t^h \) and \( x_t^f \) as part of their information sets. To be more specific, the correlation in the case described in this section appears to be slightly higher than the one analyzed in Chapter 1, whenever \( \Psi \geq 1/\gamma \). For the complementary case in which agents prefer late resolution of uncertainty (i.e. \( \Psi \leq 1/\gamma \)), the situation is reversed with the correlation that is obtained when only consumption growth is observable even taking on negative values. However, since the model delivers counterintuitive results for asset returns (namely their volatility tends to infinity) for this range of values of the preference parameters, we rule them out of our analysis.
3.4 The origins of $x_t$

If this common low frequency component of consumption growths has so much to do with the ability of the model of reproducing a number of moments of international variables, it is natural to ask: where is it coming from? Also, does this mean that consumption growth in major industrialized countries is co-integrated? And then again, what is the meaning of having a cross-country common component in an economy where there is no actual trade going on? We try to address all these questions in this section.

3.4.1 What is $x_t$?

In the paper we let data tell us what consumption growth should reasonably look like and imposing a subset of the equilibrium conditions impelled by the Euler equations we derived asset prices. Now the question becomes: where is the low frequency component of consumption growth coming from? It seems reasonable to think of $x_t$ as being embedded in the technology process and affecting consumption through a fully specified real business cycle model. What is the degree of persistence of technological progress that would justify our consumption process? Our model also requires the predictable component to be perfectly correlated across countries, leading us to further investigate the question of what are the causes of this common
business cycle component. We believe that future research should focus on the deeper economic reasons that link quantities and prices in a fully specified model. To our knowledge there is no paper in the literature that examines the source of common international low frequency components of real business cycle variables.

### 3.4.2 Are consumption growths co-integrated?

After this long discussion about consumption growths having a common long run component, one might ask whether these two variables are co-integrated in the sense of Engle and Granger (1987). We did not explicitly model consumption growths as being co-integrated, but we could have. As a matter of fact, this may be convenient from an estimation point of view as we could first estimate the cointegrating relation between consumption growths and then use the joint sample of consumption growths in the two countries to filter a unique low frequency component. So to speak, we could have a longer sample to estimate it. Future developments of the econometric analysis should focus on this extension.

### 3.4.3 A common component with no trade?

In equilibrium, the two countries that we consider behave as side by side autarky economies. Then one might ask: ‘what is the meaning of having two countries with
no trade that have a common predictable component?" The answer is that two countries can use the same technology to produce different goods, that, in this case, end up being not traded. At the same time, the consumption processes of the representative agents in the two countries are not going to be perfectly correlated, because their decisions are subject to a multitude of country specific shocks, such as monetary or fiscal shocks. There is a sense in which this question relates to the one that calls for the deeper economic meaning of $x_t$: having unveiled that this endowment economy setup is able to explain a large number of phenomena that we observe in finance, the next task of this literature must be concerned with the underlying economy that delivers the consumption processes that we have been using so far.

3.5 Term structure evidence

3.5.1 Other evidence of consumption growth’s low frequency components?

Risk free rates and yields in general should bear a lot of information about the predictable component of consumption growth, as pointed out by the linearized system (1.12). Figure 3.2 reports the spectra and coherence of real risk free rates
in the US and the UK. The sharp increase observed at low frequencies in all the subplots seems to confirm the idea that there is potentially a lot of information to be used to tighten the estimates provide earlier in the paper. Unfortunately the version of the model that we analyze in section 1 prevents us from being able to do this, due to the implied perfect correlation of yields on bonds of any maturity. Is there anything that we can do to better exploit yield curve data? In the next section, we show how to derive risk free rates and stochastic discount factors when consumption growth in the two countries is assumed to be affected by two factors\(^2\). In the context of this generalized version of the model discussed so far, we show that the inclusion of an extra factor can break the perfect correlation of risk free rates and at the same time preserve all the appealing results shown in the previous sections. A unified framework to study consumption, yield curves and long run risks is a priority for future research\(^3\).

### 3.5.2 The (un)covered interest rate parity puzzle

An overwhelming literature has documented the failure of the so called interest rate parity for a broad range of countries and time periods. This is to say that the

---

\(^2\)The analytical results in the Appendix are derived under the assumption that \(\psi = 1\), that is when Epstein and Zin (1989) preferences collapse to Hansen’s risk sensitive preference studied, inter alia, in a related paper by Tallarini (2000).

\(^3\)In a recent paper Piazzesi and Schneider (2006) show that by modeling consumption growth and inflation as both containing low frequency components, it is possible to obtain upward sloping yield curve that are able to reproduce the behavior of the post war US’s bonds market.
Fig. 3.2 - Spectra and coherence of real risk free rates in the United States and in the United Kingdom.

The idea that higher interest rate countries’ currency should appreciate does not seem to accurately approximate the behavior of international markets. Meese and Rogoff (1983) and more recently Elliott and Pesavento (2006) are just a few examples of how a naive regression of exchange rate growth on interest rate differentials produces coefficients that are not only different from one, but also significantly negative.\(^4\)

\(^4\)We report the evidence from our dataset in the next chapter.
Our international no-arbitrage condition implies the uncovered interest parity up to a constant. This is true when we do not have stochastic volatility: in this case expected exchange rate growth is going to depend on a constant, on short term interest rates differential and on the conditional volatility of consumption growth in the two countries (that are time-varying). Fama (1984) and Backus, Foresi, and Telmer (1996) have noted how the introduction of stochastic volatility in an affine term structure model could resolve the uncovered interest rate puzzle and a number of the anomalies that in some form are derived from it. We extend their finding to this particular consumption based asset pricing model in the next chapter.
Chapter 4

Toward a new paradigm for international finance?

In this chapter we generalize the model described in chapter 1 by assuming that the laws of motions of the two countries are characterized by the presence of two instead of only one low frequency components. We also relax the structure of the variance-covariance matrix of the shocks. With these two extensions, we show that the model is fit to explain many of the anomalies that arise in the international finance literature in the context of the standard time-separable preferences framework.

This chapter is organized as follows. In the next section we describe the puzzles that emerge in a two country Mehra and Prescott (1985) asset pricing model. We
will refer to this model as the benchmark model. In the second and last section, we show how the generalizations outline above are able to solve these anomalies, hence providing a unified framework for international finance.

4.1 The benchmark model

We assume that there are two countries, home and foreign and following the notation of chapter 1 we index with an ‘h’ home variables and with an ‘f’ foreign variables. The two countries are each populated by a representative consumer with time additive constant relative risk aversion (henceforth CRRA) preferences defined over the consumption aggregates \( C^h_t \) and \( C^f_t \):

\[
U^i_t = \max_{(C^i_t)_t} \left( 1 - \delta \right) \frac{C^i_{t+1}}{1 - \gamma} + \delta E_t \left[ U^i_{t+1} \right], \quad \forall i \in \{h, f\} \quad \text{and} \quad t \geq 0
\]

Stochastic discount factors are defined as the intertemporal marginal rates of substitution \( M^h_{t+1} \) and \( M^f_{t+1} \):

\[
M^i_{t+1} = \frac{\partial U^i_t / \partial C^i_{t+1}}{\partial U^i_t / \partial C^i_t} = \delta \left( \frac{C^i_{t+1}}{C^i_t} \right)^{-\gamma}, \quad \forall i \in \{h, f\} \quad \text{and} \quad t \geq 0 \quad (4.1)
\]

We assume that the logarithm of consumption follows a unit root process in each of the two countries:

\[
\log C^i_{t+1} - \log C^i_t = \Delta c^i_{t+1} = \mu_c + \lambda^i \varepsilon^i_{t+1}, \quad \forall i \in \{h, f\} \quad \text{and} \quad t \geq 0
\]
where $\lambda^h_t$ and $\lambda^f_t$ are stochastic variance processes with $AR(1)$ dynamics:

$$
\lambda^i_t = \sigma (1 - \rho) + \rho \lambda_{t-1}^i + \varphi \lambda_{t-1}^i \epsilon_{\lambda,t}^i, \quad \forall i \in \{h, f\}
$$

As we argued in the previous chapters, this assumption can be justified on the grounds of the empirical evidence that consumption growth is statistically indistinguishable from an $i.i.d.$ process in large cross section of countries. In terms of economic theory, the assumption of exogenous consumption processes can either be interpreted as the result of an endowment economy in which there exist only two goods and agents’ preferences display complete specialization toward one of the two or as post-trade allocations.

We will now describe the failure of this model to account for many features of international data. We have already discussed extensively the equity premium puzzle and the depreciation rate volatility puzzle in section 1. In this chapter we extend the previous findings by focussing on a larger set of countries: USA, UK, Germany, Japan, France, Canada and Sweden. In calibrating the parameters of the home and foreign countries, our guidelines will be once again to match the almost $i.i.d.$ behavior of consumption and dividend growth and their degree of international correlation. Table 4.1 reports the symmetric calibration for the two countries that is obtained in this spirit. The first two plots of figure 4.1 confirm the inability of the model to account for the international correlation of stochastic discount factors.
Table 4.1

**Benchmark Model: Baseline Calibration.**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma$</td>
<td>Risk aversion</td>
<td>7.5</td>
</tr>
<tr>
<td>$\delta$</td>
<td>Subjective discount factor</td>
<td>.9987</td>
</tr>
<tr>
<td>$\mu_c$</td>
<td>Average consumption growth</td>
<td>.0015</td>
</tr>
<tr>
<td>$\mu_d$</td>
<td>Average dividend growth</td>
<td>.0015</td>
</tr>
<tr>
<td>$\varphi_d$</td>
<td>Dividend to consumption growth standard error ratio</td>
<td>4.0</td>
</tr>
<tr>
<td>$\varphi_\lambda$</td>
<td>Stochastic volatility to consumption growth std error ratio</td>
<td>.001</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>Standard error of the shock to consumption growth</td>
<td>.0115</td>
</tr>
<tr>
<td>$\rho_\lambda$</td>
<td>Persistence of stochastic volatility</td>
<td>.970</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\varepsilon_{ch}$</th>
<th>$\varepsilon_{cf}$</th>
<th>$\varepsilon_{ch}$</th>
<th>$\varepsilon_{cf}$</th>
<th>$\varepsilon_{dh}$</th>
<th>$\varepsilon_{df}$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>1.000</td>
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<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>1.000</td>
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<td>0.000</td>
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<td>-0.500</td>
<td>1.000</td>
<td>0.000</td>
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<td>0.000</td>
<td>0.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
</tbody>
</table>

Notes - The two countries share the same calibration.

and hence for the volatility of the depreciation of the US dollar even when a larger set of countries is considered. The third plot of the same figure shows the existence of a considerable equity premium puzzle for a vast majority of industrialized countries. Figure 4.2 shows that the introduction of a moderate amount of stochastic volatility that enables the model to keep track of the conditional heteroskedasticity of stock markets’ returns does not change our conclusions about the inability of the benchmark model to explain the two anomalies.
FIG. 4.1 - Benchmark model without stochastic volatility letting $\gamma$ vary. The six subplots depict the behavior (from the top left to the right bottom) of the cross country correlation of stochastic discount factors, of the volatility of the depreciation of the domestic currency, of the volatility of the stochastic discount factor, of the correlation of consumption growth differentials and exchange rate depreciation, of the cross country correlation of risk free rates, of the cross country correlation of dividend paying assets and of the slope of the uncovered interest parity regression. For each panel, the solid line represents the median across 250 simulations of length 150, while the dotted lines are 95% confidence intervals and the dashed dotted line is the data measured equivalent.
Fig. 4.2 - Benchmark model letting $\gamma$ vary. The six subplots depict the behavior (from the top left to the right bottom) of the cross country correlation of stochastic discount factors, of the volatility of the depreciation of the domestic currency, of the volatility of the stochastic discount factor, of the correlation of consumption growth differentials and exchange rate depreciation, of the cross country correlation of risk free rates, of the cross country correlation of dividend paying assets and of the slope of the uncovered interest parity regression. For each panel, the solid line represents the median across 250 simulations of length 150, while the dotted lines are 95% confidence intervals and the dashed dotted line is the data measured equivalent.
Backus and Smith (1993) point out that equation (1.2) would suggest that the ratio of stochastic discount factors should be perfectly correlated to the rate of depreciation of the domestic currency. In the benchmark model, this translates into a statement about the correlation between consumption growths differentials and growth of the exchange rate:

\[
corr \left[ \gamma \left( \Delta c^h_{t+1} - \Delta c^f_{t+1} \right), \log \frac{e_{t+1}}{e_t} \right] = 1
\]  

(4.3)

The horizontal dash-dot line in the fourth plot of figure 4.1 shows that this prediction is strongly contradicted by the data that call for a correlation of 0.146. The result appears to be robust to the introduction of a small amount of stochastic volatility, as shown in figure 4.2.

In chapter 1, we pointed out that stock market returns appear to be moderately to extremely correlated across countries despite the fact that fundamentals, namely consumption and dividend growths, are not. In the context of the benchmark model, this can be formalized by postulating a process for the growth of the dividend paid by a value weighted portfolio in the home and in the foreign country and then using the stochastic discount factors (4.1) to give a price to the asset that pays such a dividend.

Assuming for the moment that stochastic volatility is shut down and that divi-
dends in the two countries follow log-unit root processes:

\[ \Delta d^i_{t+1} = \mu_d + \epsilon^i_{d,t+1}, \quad \forall i \in \{h, f\} \]  

(4.4)

Then use the Campbell and Shiller (1988) approximation to write the logarithm of the returns on the asset that pay the dividend processes in (4.4) as

\[ r^i_{t+1} = k_0 + \Delta d^i_{t+1} - k_1 dp^i_{t+1} + dp^i_t, \quad \forall i \in \{h, f\} \]

where \( dp^i_t \) denotes the logarithm of the dividend to price ratio in each country and \( k_0 \) and \( k_1 \) are approximation constants. The Euler equations for holdings of the two assets can be used to derive that the log-dividend to price ratio has to be constant at any point in time. Hence returns must be proportional to the innovation in the dividend process in each country:

\[ r^h_{t+1} = \tilde{r}^h + \epsilon^h_{d,t+1} \]

\[ r^f_{t+1} = \tilde{r}^f + \epsilon^f_{d,t+1} \]  

(4.5)

This implies that in the absence of stochastic volatility the correlation of stock returns is equal to the correlation of the growth of the dividends that they entitle to. The sixth plot of figure 4.1 shows how the former is in the order of 0.5, while the latter is almost zero for the seven countries that we consider in our study. We will refer to this anomaly as the dividends-returns correlation puzzle. In the appendix we show how to generalize the preceding formulas to the case in which dividends and
consumption have time varying economic uncertainty. However figure 4.2 shows that this modification does not affect the inability of the model of producing sufficiently highly correlated returns, unless an unreasonably high coefficient of risk aversion ins assumed.

If the addition of stochastic volatility appears to be of no help with regard to the anomalies that we are after, it comes natural to ask what the use of it is. Besides being needed to account for the degree of persistence of the conditional volatility of asset returns as documented by a long literature that dates back to Engle (1982), stochastic volatility may provide an explanation for the imperfect degree of international correlation of risk free rates. More specifically, in the absence of stochastic volatility, risk free rates are constant over time and hence perfectly correlated across countries, as documented by the fifth plot in figure 4.1. The introduction of imperfectly correlated time varying economic uncertainty is instead able to account for the degree of correlation of risk free rates as shown in figure 4.2.

An empirical evidence that has received a large attention in the literature is the well documented tendency of high interest rate currency to appreciate instead of depreciate, as one might think. This evidence relies on the fact that the coefficient \( \beta_1 \) in the regression

\[
e_{t+1} - e_t = \beta_0 + \beta_1 \left( r_t^{inh} - r_t^{ff} \right) + \text{residual}
\]  

(4.6)
turns out to be significantly negative instead of being equal to unity. In the dataset that we employ, the average estimate of $\beta_1$ is about $-0.5$. Fama (1984) and Backus, Foresi, and Telmer (1996) among others outline the properties that the joint distribution of stochastic discount factors must have in order to account for the anomaly. If use the benchmark model to study regression (4.6), we obtain that $\beta_1$ is equal to:

$$
\hat{\beta}_1 = \frac{\text{cov} \left( e_{t+1} - e_t, r^{fh}_t - r^{ff}_t \right)}{\text{Var} \left( r^{fh}_t - r^{ff}_t \right)}
$$

$$
= \frac{\text{cov} \left( m^f_{t+1} - m^h_{t+1}, E_t \left[ m^f_{t+1} - m^h_{t+1} \right] + 0.5V_t \left( m^f_{t+1} - m^h_{t+1} \right) \right)}{\text{Var} \left( r^{fh}_t - r^{ff}_t \right)}
$$

$$
= \frac{\text{cov} \left( E_t \left[ m^f_{t+1} - m^h_{t+1} \right], E_t \left[ m^f_{t+1} - m^h_{t+1} \right] + 0.5V_t \left( m^f_{t+1} - m^h_{t+1} \right) \right)}{\text{Var} \left( r^{fh}_t - r^{ff}_t \right)}
$$

Notice that

$$
E_t \left[ m^f_{t+1} - m^h_{t+1} \right] = \gamma E_t \left[ \Delta c^h_{t+1} - \Delta c^f_{t+1} \right] = 0
$$

that implies that when $\beta_1$ is estimated from the benchmark model, it is equal to zero\(^1\). The seventh panel of figure 4.2 shows that this implication is at odds with the evidence from international data.

\(^1\)This is certainly true for the case in which there is stochastic volatility that ensures that $\text{Var} \left( r^{fh}_t - r^{ff}_t \right) \neq 0$. When there is no time varying economic uncertainty, $\beta_1$ is undetermined. However we can rule out this case on the grounds of the counterfactual implication that risk free rates ought to be perfectly correlated.
4.2 A unified framework for international finance puzzles

In this section we describe a model that is able to account for the failure of the benchmark model relative to the anomalies discussed in the previous section.

We will retain the basic set of assumptions that were made in the context of the previous model, specifically that markets are complete and that there are two countries with two country-specific goods, each populated by a representative consumer whose preferences display a complete specialization toward the domestic good.

We modify the assumption of time separable preferences, by introducing a preference for the timing of the resolution of uncertainty:

\[
U_t^i = \max_{(c_t^i)_{t=0}^\infty} (1 - \delta) \log C_t^i + \delta \theta \log E_t \exp \left\{ \frac{U_t^{i+1} \theta}{\theta} \right\}, \quad \forall i \in \{h, f\} \tag{4.7}
\]

with \(\theta = \frac{1}{1-\gamma}\). These preferences are known in the literature as Hansen’s risk sensitive preferences and have been used, among others, by Anderson (2005) and Tallarini (2000) in related studies. Hansen and Sargent (2006) show that they can be interpreted as the limiting case of Epstein and Zin (1989) preferences when the intertemporal elasticity of substitution approaches unity. They collapse to the special case of time additive preferences when \(\gamma = 1\).

Given this setup of the economy, stochastic discount factors can be computed
as:

\[ M_{i,t+1}^{i} = \frac{\partial U_{i,t}^{i}}{\partial C_{i,t+1}^{i}} \]

\[ = \delta \left( \frac{C_{i,t+1}^{i}}{C_{i,t}^{i}} \right) E_{t} \exp \left\{ \frac{U_{i,t+1}^{i}}{\theta} \right\}, \quad \forall i \in \{h, f\} \text{ and } \forall t \geq 0 \quad (4.8) \]

The model is closed by specifying the process for the logarithm of consumption growth in each country as:

\[ \Delta c_{i,t+1}^{i} = \mu_{c} + \lambda_{1}^{i} z_{1,t}^{i} + \lambda_{2}^{i} z_{2,t}^{i} + \lambda_{t}^{i/2} \varepsilon_{c,t+1}^{i}, \quad \forall i \in \{h, f\} \text{ and } \forall t \geq 0 \quad (4.9) \]

where

\[ z_{1,t+1} = \rho_{1} z_{1,t} + \varphi_{e\lambda}^{h} \varepsilon_{1,t+1} \]

\[ z_{2,t+1} = \rho_{2} z_{2,t} + \varphi_{e\lambda}^{f} \varepsilon_{2,t+1} \]

\[ \lambda_{t+1}^{i} = \sigma^{2} (1 - \rho_{\lambda}) + \rho_{\lambda} \lambda_{t}^{i} + \varphi_{\lambda}^{h} \lambda_{t}^{i/2} \varepsilon_{\lambda,t+1}^{i}, \quad \forall i \in \{h, f\} \quad (4.10) \]

By defining \( \eta_{t+1} = [\varepsilon_{1,t+1}^{i} \varepsilon_{2,t+1}^{i} \varepsilon_{c,t+1}^{i} \varepsilon_{c,t+1}^{s} \varepsilon_{\lambda,t+1}^{i} \varepsilon_{\lambda,t+1}^{s} ] \), the correlation matrix of the shocks is \( R = E[\eta_{t+1}^{i} \eta_{t+1}^{i}] \).

This model follows in the footsteps of the recent literature on risks for the long run, that consists in including a small predictable component in the dynamics of consumption growth. Bansal and Yaron (2004) show that this process cannot be rejected on the grounds of the past century of observations on US consumption. Bansal, Gallant, and Tauchen (2002) reinforce this finding by providing positive
evidence for the presence of a low frequency component in US consumption growth. Colacito and Croce (2006) extend this evidence to a cross section of countries, also positing a high degree of cross country correlation of consumption growth’s low frequency components.

The relevant vector of state variables is \( s_t = [z_{1,t} \ z_{2,t} \ \lambda^h_t \ \lambda^f_t \ h_t] \), where \( h_t = \lambda^h_t \lambda^f_t^{1/2} \). The Appendix shows that an analytical solution of the model exists in which the logarithm of the stochastic discount factors, \( m^h_{t+1} \) and \( m^f_{t+1} \), take the form:

\[
\begin{align*}
m^h_{t+1} &= -v_{0,m} - v_m s_t^l + \frac{1}{\vartheta} \left( \tilde{v}_1 \lambda^h_t + \tilde{v}_2 \lambda^f_t \right) \eta^l_{t+1} \\
m^f_{t+1} &= -v_{0,m}^* - v_m^* s_t^l + \frac{1}{\vartheta} \left( v_{1}^* \lambda^h_t + v_{2}^* \lambda^f_t \right) \eta^l_{t+1} \tag{4.11}
\end{align*}
\]

with \( v_{0,m}, v_m, v_{0,m}^*, \tilde{v}_1, v_{1}^*, v_{2}^* \) defined in the Appendix.

What does this model have to say about the six anomalies that we discussed in the previous section? To answer this question we calibrate the model according to Table 4.2 by closely replicating the autocorrelation functions of consumption and dividend growths and their international correlations.

When stochastic volatility is shut down, figure 4.3 shows how a moderate value of risk aversion is able to account for many of the failures of the benchmark model. When \( \gamma = 1 \) agents’ preferences specialize to the time separable case discussed earlier with the only novelty being given by the presence of long run risks in the
Table 4.2

**Long Run Risks Model: Baseline Calibration.**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
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<tr>
<td>$\gamma$</td>
<td>Risk aversion</td>
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</tr>
<tr>
<td>$\delta$</td>
<td>Subjective discount factor</td>
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<tr>
<td>$\mu_c$</td>
<td>Average consumption growth</td>
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<td>Home country loading on $z_1$ in the consumption process</td>
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<td>Home country loading on $z_2$ in the consumption process</td>
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<tr>
<td>$\varphi_d$</td>
<td>Dividend to consumption growth standard error ratio</td>
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<tr>
<td>$\varphi_\lambda$</td>
<td>Stochastic volatility to consumption growth std error ratio</td>
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<tr>
<td>$\sigma$</td>
<td>Standard error of the shock to consumption growth</td>
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<td>$\rho_\lambda$</td>
<td>Persistence of stochastic volatility</td>
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<td>$z_1$ to consumption growth standard error ratio</td>
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<tr>
<td>$\varphi_2$</td>
<td>$z_2$ to consumption growth standard error ratio</td>
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Notes - The two countries share the same calibration.
laws of motion of consumption growth. Looking at the first four panels of figure 4.3, this innovation alone is not able to explain the volatility of the depreciation rate puzzle, the equity premium puzzle and the Backus and Smith (1993) puzzle. If we assume that the low frequency components of consumption growth are also affecting the dynamics of dividend growths

\[ \Delta d_{i,t+1} = \mu_d + \lambda_{d_z}^i z_{1,t} + \lambda_{d_z}^i z_{2,t} + \lambda_{d_z}^{i/2} \varphi_d \varepsilon_{d,t+1}, \quad \forall i \in \{h, f\} \text{ and } \forall t \geq 0 \]  

then the fifth and sixth panel of figure 4.3 show that it is possible to reproduce the degree of correlation of risk free rates and stock markets’ returns, no matter the type of preferences that are used. The version of the model without stochastic volatility still falls short to explaining the uncovered interest rate parity anomaly, as documented by the last plot of the same figure.

Figure 4.4 documents that a high persistence of the predictable components of consumption growth is needed in order for the model to account for the anomalies we are after. This picture suggests that the data should provide strong identifying restrictions for the persistence of the low frequency components, since it is only for a narrow range of its values, that the model is able to account for the volatility of the depreciation rate and meet the lower bound on the volatility of the stochastic discount factors impelled by asset returns. The same argument applies to the Backus and Smith (1993) puzzle. As we anticipated from figure 4.3, the ability of the model
of reproducing the degree of comovement of asset returns is entirely due to the presence of the low frequency components of consumption and dividend growth.

As the correlation of long run risks varies, figure 4.5 suggests that the variable that seems to be affected the most is the volatility of the depreciation of the exchange rate. This is a direct consequence of the increasing relationship existing between the correlation of consumption’s low frequency components and the extent at which the representative consumers in the two countries tend to discount future uncertain payoffs in the same way.

Without time varying economic uncertainty the model with long run risks has proven itself unable to explain what we referred to as the uncovered interest rate parity anomaly. The last panel of figure 4.6 shows that the introduction of the same amount of stochastic volatility that we adopted in the benchmark model is able to account for the anomaly, by retaining at the same time the ability of the model to explain all other puzzles.

Hence, introducing two low frequency components in the dynamics of consumption growth in observance of the empirical finding that consumption and dividend growths are almost i.i.d. processes, relaxing the assumptions on the correlation matrix of the shocks and modeling stochastic volatility in a way that is consistent with time varying risk premia allows us to extend the set of moments that the model can explain well beyond the volatilities of the depreciation of the US dollar and
of the stochastic discount factors. This provides a useful instrument for studying international finance, that is worth investing further in the future.

Fig. 4.3 - Long run risks model letting $\gamma$ vary. The six subplots depict the behavior (from the top left to the right bottom) of the cross country correlation of stochastic discount factors, of the volatility of the depreciation of the domestic currency, of the volatility of the stochastic discount factor, of the correlation of consumption growth differentials and exchange rate depreciation, of the cross country correlation of risk free rates, of the cross country correlation of dividend paying assets and of the slope of the uncovered interest parity regression. For each panel, the solid line represents the median across 250 simulations of length 150, while the dotted lines are 95% confidence intervals and the dashed dotted line is the data measured equivalent.
Fig. 4.4 - Long run risks model letting $\rho_1 = \rho_2$ vary. The six subplots depict the behavior (from the top left to the right bottom) of the cross country correlation of stochastic discount factors, of the volatility of the depreciation of the domestic currency, of the volatility of the stochastic discount factor, of the correlation of consumption growth differentials and exchange rate depreciation, of the cross country correlation of risk free rates, of the cross country correlation of dividend paying assets and of the slope of the uncovered interest parity regression. For each panel, the solid line represents the median across 250 simulations of length 150, while the dotted lines are 95% confidence intervals and the dashed dotted line is the data measured equivalent.
Fig. 4.5 - Long run risks model letting $\rho_{1,2}$ vary. The six subplots depict the behavior (from the top left to the right bottom) of the cross country correlation of stochastic discount factors, of the volatility of the depreciation of the domestic currency, of the volatility of the stochastic discount factor, of the correlation of consumption growth differentials and exchange rate depreciation, of the cross country correlation of risk free rates, of the cross country correlation of dividend paying assets and of the slope of the uncovered interest parity regression. For each panel, the solid line represents the median across 250 simulations of length 150, while the dotted lines are 95% confidence intervals and the dashed dotted line is the data measured equivalent.
Fig. 4.6 - Long run risks model letting $\varphi_\lambda$ vary. The six subplots depict the behavior (from the top left to the right bottom) of the cross country correlation of stochastic discount factors, of the volatility of the depreciation of the domestic currency, of the volatility of the stochastic discount factor, of the correlation of consumption growth differentials and exchange rate depreciation, of the cross country correlation of risk free rates, of the cross country correlation of dividend paying assets and of the slope of the uncovered interest parity regression. For each panel, the solid line represents the median across 250 simulations of length 150, while the dotted lines are 95% confidence intervals and the dashed dotted line is the data measured equivalent.
Conclusion

We have shown that allowing for an intertemporal elasticity of substitution larger than the reciprocal of the coefficient of risk aversion and for a persistent and highly cross-correlated forecastable component of consumption growth in the economy described by Brandt, Cochrane, and Santa-Clara (2004) it is possible to reconcile the measure of cross-country correlation of the stochastic discount factors obtained from data on consumption and from data on prices. This result is achieved in combination with a lowly volatile depreciation rate and without requiring a high correlation of the consumption processes and high coefficients of risk aversion. We have also shown how key features of the data can be described by the same parametrization that allows us to meet our primary goal, extending in this way the set of properties of the models that take into account long run risks beyond what pointed out by
Bansal and Yaron (2004).

By challenging our explanation of what we referred to as the ‘international equity premium puzzle’ with other competing explanations of Mehra and Prescott (1985) one-country equity premium puzzle, we have shown that our model seems to be the only one able to account for the anomaly. This should provide over-identifying power to the closed economy studies of Bansal, Gallant, and Tauchen (2002). We have also shown that our conclusions are robust to a number of modifications to the basic set of assumption, ranging from the introduction of stochastic volatility to the unobservability of the predictable components of consumption growths.

The fact that a generalization of the model that allows for the presence of two low frequency components in the dynamics of consumption growth, by remaining consistent with the empirical evidence on consumption growth being almost an \textit{i.i.d.} process in the cross section of countries that we focus on, is able to account for the Backus and Smith (1993) anomaly and for the tendency of high interest rate currencies to appreciate is symptomatic that the model provides solid grounds to analyze a large number of international finance phenomena.

Although many things are left to explain, we are optimistic about the positive findings of this project to hold also in more general setups that have the potential of extending the set of properties of this class of models even further.
Appendix

1 Derivation of moments

We express the relevant moments of the variables of the model as a function of the set Υ of deeper parameters defined as Υ = \{θ, ψ, δ, ρ_x, λ, σ_c, σ_d, ρ_{cf}, ρ_{xf}, ρ_{df}\}.

We assume that:

\[
\Delta c_{i,t+1} = x^i_{t} + \varepsilon^i_{c,t+1}, \quad \varepsilon^i_{c,t} \sim iid \mathcal{N}(0, \sigma^2_c)
\]

\[
\Delta d^i_{t+1} = \lambda x^i_t + \varepsilon^i_{d,t+1}, \quad \varepsilon^i_{d,t} \sim iid \mathcal{N}(0, \sigma^2_c)
\]

\[
x^i_{t+1} = \rho_x x^i_t + \varepsilon^i_{x,t+1}, \quad \varepsilon^i_{x,t} \sim iid \mathcal{N}(0, \sigma^2_x) \quad \forall i \in \{h, f\} \tag{A.13}
\]

The Euler equation for the asset that pays one unit of the consumption bundle at each period is:

\[
1 = E_t [M^i_{t+1} R_{c,t+1}^i] \tag{A.14}
\]
with $M_{t+1} = \delta^\theta \left( \frac{C_{t+1}^i}{C_t^i} \right)^{-\frac{\psi}{\theta}} \left( R_{c,t+1}^i \right)^{(\theta-1)}$. The return on the consumption asset can be expressed in terms of the price-consumption ratio $v_{c,t}$ as $R_{c,t+1}^i = \frac{v_{c,t+1}^i}{v_{c,t}^i} \exp \Delta c_{t+1}^i, \forall i \in \{h, f\}$. This implies that (A.14) can be written as:

$$
(v_{c,t}^i)^\theta = E_t \left[ \delta e^{\Delta c_{t+1}^i (1-\gamma)} (1 + v_{c,t+1}^i)^\theta \right] \tag{A.15}
$$

Linearizing (A.15) around the steady state of $v_{c,t}$ defined as $v_{c,ss}^i = \frac{\bar{v}_i^i}{1-\delta}$, we obtain:

$$
v_{c,t}^i = E_t \left[ \delta + \frac{\delta}{1-\delta} \left( 1 - \frac{1}{\psi} \right) \Delta c_{t+1}^i + \delta v_{c,t+1}^i \right]
$$

that can be solved forward delivering the price-consumption ratio as a function of the state variable $x_t^i$ and of the deeper parameters of the model:

$$
v_{c,t}^i = \alpha_c^i + \beta_c^i x_t^i \tag{A.16}
$$

with $\alpha_c^i = \frac{\delta^\theta_e}{1-\delta}$ and $\beta_c^i = \frac{\delta^\theta_e (1-\frac{1}{\psi})}{1-\rho_\delta \delta^\theta_e}$, $\forall i \in \{h, f\}$. Similarly, we can solve for the price-dividend ratio $v_{d,t}^i$

$$
v_{d,t}^i = \alpha_d^i + \beta_d^i x_t^i \tag{A.17}
$$

with $\alpha_d^i = \frac{\delta^\theta_d}{1-\delta}$ and $\beta_d^i = \frac{\delta^\theta_d (\lambda_i - \frac{1}{\psi})}{1-\rho_\delta \delta^\theta_d}$, $\forall i \in \{h, f\}$.

Log-returns on the assets that pay streams of consumption and stream of dividends follow immediately from (A.13), (A.16) and (A.17):

$$
r_{c,t+1}^i = -\log(\delta) + \left[ \frac{1}{\psi} \right] x_t^i + \left[ \delta \left( 1 - \frac{1}{\psi} \right) \frac{1}{1-\rho_x \delta} \right] \epsilon_{x,t+1}^i + \epsilon_{c,t+1}^i \tag{A.18}
$$

$$
r_{d,t+1}^i = -\log(\delta) + \left[ \frac{1}{\psi} \right] x_t^i + \left[ \delta \left( \lambda - \frac{1}{\psi} \right) \frac{1}{1-\rho_x \delta} \right] \epsilon_{x,t+1}^i + \epsilon_{d,t+1}^i \tag{A.19}
$$
Also the log-stochastic discount factor is:

\[ m_{i,t+1} = \log \delta - \frac{1}{\psi} x_t^i - \gamma c_{i,t+1}^i + \frac{\delta (1-\gamma \Psi)}{\psi (1-\rho_x \delta)} c_{x,t+1}^i \]  

(A.20)

The log-risk free rate \( r_f^i \) is obtained as the solution of:

\[
  r_f^i = -E_t \left[ \delta^\theta \exp \left\{ -\frac{\theta}{\psi} \Delta c_{t+1}^i + (\theta - 1) r_{c,t+1}^i \right\} \right]
\]

\[
  = -\log(\delta) + \frac{1}{2} \left[ \left| \gamma \right|^2 \sigma_c^2 + \left[ \delta \left( 1 - \frac{1}{\psi} \right) \frac{1}{1-\rho_x \delta} \right]^2 \sigma_c^2 \right] + \frac{1}{\psi} x_t^i \]  

(A.21)

The following moments follow directly from the previous formulas and from the assumption that both countries share the same parametrization:

\[ \text{Var}(\pi) = 2 \left[ \left( 1 - \rho_x^{hf} \right) \Gamma_0 \sigma_x^2 + \left( 1 - \rho_c^{hf} \right) \gamma^2 \sigma_c^2 \right] \]  

(A.22)

\[ \text{corr}(m_t^h, m_t^f) = \frac{\Gamma_0 \rho_x^{hf} \sigma_x^2 + \gamma^2 \rho_c^{hf} \sigma_c^2}{\Gamma_0 \sigma_x^2 + \gamma^2 \sigma_c^2} \]

\[ \text{Var}(r_{c,t+1}^i) = \Gamma_0 \sigma_x^2 + \sigma_c^2 \]

\[ \text{Cov}(r_{c,t+1}^h, r_{c,t+1}^f) = \Gamma_0 \rho_x^{hf} \sigma_x^2 + \rho_c^{hf} \sigma_c^2 \]

\[ \text{Cov}(\Delta c_{t+1}^i, r_{c,t+1}^i) = \Gamma_1 + \sigma_c^2 \]

\[ \text{Cov}(\Delta c_{t+1}^h, r_{c,t+1}^f) = \Gamma_1 \rho_x^{hf} + \rho_c^{hf} \sigma_c^2 \]

\[ \text{corr}(r_{d,t}^h - r_f^h, r_{d,t}^f - r_f^f) = \frac{\Gamma_2 \rho_x^{hf} \sigma_x^2 + \rho_d^{hf} \sigma_d^2}{\Gamma_2 \sigma_x^2 + \sigma_d^2} \]

where \( \Gamma_0 = \left( \frac{1}{\psi} \right)^2 \frac{1}{1-\rho_x^2} + \left[ \frac{\delta (1-\gamma \Psi)}{\psi (1-\rho_x \delta)} \right]^2 \), \( \Gamma_1 = \frac{\sigma_x^2}{\psi (1-\rho_x \delta)} \), and \( \Gamma_2 = \left[ \delta \left( \lambda - \frac{1}{\psi} \right) \frac{1}{1-\rho_x \delta} \right]^2 \).
2 Proof of Propositions

Proof of Proposition 1. Assuming complete markets, the correlation $\rho_{m^h,m^f}$ is given by equation (1.3). The problem to solved takes the following form:

\[
\text{choose } \sigma_{m^h}, \sigma_{m^f} \\
\text{to } \min \rho_{m^h,m^f} \\
\text{s.t. } \sigma_{m^h} \geq \sigma_{m^h} \\
\sigma_{m^f} \geq \sigma_{m^f}
\]

where $\sigma_{m^h}$ and $\sigma_{m^f}$ are the Hansen and Jagannathan (1991) bounds. Attaching Lagrange multipliers $\lambda^h$ and $\lambda^f$ to the two constraints, the first order necessary conditions are:

\[
\frac{\partial \rho_{m^h,m^f}}{\partial \sigma_{m^h}} - \lambda^h = 0 \\
\frac{\partial \rho_{m^h,m^f}}{\partial \sigma_{m^f}} - \lambda^f = 0
\]

along with the two constraints. Four cases must be taken into account.

Case 1: $\lambda^h = \lambda^f = 0$. This implies $\sigma^2_\pi = 0$, a contradiction.

Case 2: $\lambda^h = 0$, $\lambda^f > 0$. The system of first order conditions implies:

\[
\begin{cases}
\sigma_{m^f} = \sigma_{m^f} \\
\sigma^2_{m^h} = \sigma^2_{m^f} - \sigma^2_\pi \geq \sigma^2_{m^h} \Rightarrow \frac{\sigma^2_{m^f}}{\sigma^2_{m^h}} \geq \frac{\sigma^2_\pi}{\sigma^2_{m^h}} \\
\frac{\sigma^2_{m^f} - (\sigma^2_{m^f} - \sigma^2_\pi) + \sigma^2_\pi}{2\sigma^2_{m^f} - \sigma^2_\pi} - \lambda^f = 0 \Rightarrow \lambda^f > 0
\end{cases}
\]
that is, if $\sigma^2_{mf} \geq \pi^2 + \sigma^2_{mh}$, the minimum is achieved for $\sigma_{mf} = \sigma_{mf}$ and $\sigma_{mh} = \sqrt{\sigma^2_{mf} - \pi^2}$.

**Case 3:** $\lambda^h > 0, \lambda^f = 0$. This case is symmetric to the previous one.

**Case 4:** $\lambda^h > 0, \lambda^f > 0$. The system of first order conditions implies:

\[
\begin{align*}
\sigma_{mf} &= \sigma_{mf} \\
\sigma_{mh} &= \sigma_{mh} \\
\frac{\sigma^2_{mh} - \sigma^2_{mh} - \sigma^2_{mf}}{2\sigma^2_{mf}} - \lambda^f &= 0 \\
\frac{\sigma^2_{mh} - \sigma^2_{mf} + \sigma^2_{mf}}{2\sigma^2_{mh}} - \lambda^h &= 0
\end{align*}
\]

that is, if $\sigma_{mf} \in (\sigma_{mh} - \sigma^2_{mf}, \sigma_{mh} + \sigma^2_{mf})$ the minimizer is the Hansen and Jagannathan (1991) bound itself.

Combining the four cases, we obtain:

\[
\rho_{mh,mf} = \begin{cases} 
\frac{\sigma^2_{mh} - \sigma^2_{mf}}{\sigma_{mh} \sqrt{\sigma^2_{mh} - \sigma^2_{mf}}} & \text{if } \sigma_{mf} \leq \sigma_{mh} - \sigma^2_{mf} \\
\frac{\sigma^2_{mh} + \sigma^2_{mf} - \sigma^2_{mf}}{2\sigma_{mh} \sigma_{mf}} & \text{if } \sigma_{mf} \in (\sigma_{mh} - \sigma^2_{mf}, \sigma_{mh} + \sigma^2_{mf}) \\
\frac{\sigma^2_{mf} - \sigma^2_{mf}}{\sigma_{mf} \sqrt{\sigma^2_{mf} - \sigma^2_{mf}}} & \text{if } \sigma_{mf} \geq \sigma_{mh} - \sigma^2_{mf}
\end{cases}
\]

That concludes the proof. ■
Proof of Proposition 2. For any choice of the parameters that satisfy

\[ \rho^j_x \neq 1 \quad \text{and} \quad \rho^j_x \delta^j \neq 1 \]

the following three partial derivatives

\[
\frac{\partial \text{corr}(m^h_t, m^f_t)}{\partial \rho_x^{hf}} = \frac{\Gamma_0 \sigma_x^2}{\Gamma_0 \sigma_x^2 + \gamma^2 \sigma_c^2} > 0 \quad (A.23)
\]

\[
\frac{\partial \text{corr}(m^h_t, m^f_t)}{\partial \Psi} = -\frac{2 \Gamma_0 \gamma^2 \left( \rho_x^{hf} - \rho_c^{hf} \right) \sigma_x^2 \sigma_c^2}{\psi \left( \Gamma_0 \sigma_x^2 + \gamma^2 \sigma_c^2 \right)^2} \quad (A.24)
\]

\[
\frac{\partial \text{corr}(m^h_t, m^f_t)}{\partial \rho_x} = \frac{2 \left( \frac{\rho_x}{1-\rho_x^2} + \frac{\delta^j (1-\gamma \psi)^2}{(1-\rho_x \delta^j)^2} \right) \gamma^2 \left( \rho_x^{hf} - \rho_c^{hf} \right) \sigma_x^2 \sigma_c^2}{\psi^2 \left( \Gamma_0 \sigma_x^2 + \gamma^2 \sigma_c^2 \right)^2} \quad (A.25)
\]

exist and are well defined. (A.23) is positive for all the values of the parameters that respect the two conditions, implying that the correlation of the two stochastic discount factors is strictly increasing with respect to \( \rho_x^{hf} \). When \( \rho_x^{hf} = \rho_c^{hf} \) (A.24) is always zero, meaning that changes in \( \Psi, \gamma \) or \( \delta \) do not affect the correlation of the two stochastic discount factors. If \( \rho_x^{hf} \neq \rho_c^{hf} \), this derivative is zero only if \( \psi = \frac{1}{\gamma} \tilde{\delta} \), where

\[
\tilde{\delta} = \frac{1-2 \rho_x \delta + \delta^2}{\delta^2 (1-\rho_x^2)}.
\]

In particular, when \( \rho_x^{hf} > \rho_c^{hf} \) the sign of the derivative is positive for \( \Psi > \frac{1}{\gamma} \tilde{\delta} \) and negative for \( \psi < \frac{1}{\gamma} \tilde{\delta} \). Notice that

\[
\lim_{\delta \to 1^{-}} \frac{1}{\gamma} \tilde{\delta} \geq \frac{1}{\gamma} \quad \text{and} \quad \lim_{\rho_x \to 1^{-}} \lim_{\delta \to 1^{-}} \frac{1}{\gamma} \tilde{\delta} = \frac{1}{\gamma}
\]

So, for a high persistence of the long run component and an individual discount factor close to one, the minimum of the cross correlation of the two discount factors is achieved for \( \psi = \frac{1}{\gamma} \), that is when the Epstein-Zin preferences collapse to the standard CES utility function. (A.25) is always positive when \( \rho_x^{hf} > \rho_c^{hf} \), negative when
\( \rho_{hf} < \rho_{c}^{hf} \) and equal to zero when \( \rho_{x}^{hf} = \rho_{c}^{hf} \). Therefore if \( \rho_{x}^{hf} > \rho_{c}^{hf} \), the minimum is achieved for \( \rho_{x}=0 \). When \( \rho_{x}^{hf} > \rho_{c}^{hf} \), (A.23), (A.24) and (A.25) imply the existence of one unique minimizer at \( \left( \rho_{x} = 0, \rho_{x}^{hf} = 0, \psi = \frac{1}{\gamma} \right) \).

\[ \text{Proof of Proposition 3.} \] The partial derivative of (A.22) with respect to \( \rho_{x}^{hf} \) exists and is well defined provided that \( \rho_{x} \neq 1 \) and \( \rho_{x}\delta \neq 1 \):

\[
\frac{\partial \text{Var}(\pi)}{\partial \rho_{x}^{hf}} = -2 \left[ \left( \frac{1}{\psi} \right)^2 \frac{1}{1 - \rho_{x}^2} + \left[ \frac{\delta (1 - \gamma \psi)}{\psi (1 - \rho_{x} \delta)} \right]^2 \right] \sigma_{x}^2 \leq 0 \quad (A.26)
\]

In particular, this derivative is always negative, implying that the volatility of the log-depreciation rate achieves its minimum when \( \rho_{x}^{hf} = 1 \).

3 Numerical algorithm

We describe the procedure to numerically approximate the price-consumption and price-dividend ratios for the most general case in which stochastic volatility is in the model, too. We discretize the support of \( x \) and \( \sigma \) into \( I_{x} \) and \( I_{\sigma} \) points respectively, to get \( I = I_{x} \cdot I_{\sigma} \) nodes \((x, \sigma)_{i}, \forall i = 1, ..., I \). In what follows, we will refer to \( x_{i} \) and \( \sigma_{i} \) as the first and the second entry of \((\alpha, s)_{i}\) respectively. Specify \( J \) known linearly independent basis functions \( \phi_{j} ((x, \sigma)_{i}), j \in \{1, ..., J\} \). In our solution, we employ a third order polynomial in \( x \) combined with a first order polynomial in \( \sigma \), implying that \( J = 6 \). The goal is to find basis coefficients \( c_{j}, j = 1, ..., J \) that best
approximate the Euler equation

\[ v^i_c = V((x, \sigma)_i) \approx \sum_{j=1}^{J} c_j \phi_j ((x, \sigma)_i) = \sum_{j=1}^{J} c_j \phi_{j,i} \]  

(A.27)

\[ \forall i = 1, \ldots, I \text{ or, in the equivalent matrix notation:} \]

\[ v_c \approx \Phi c \]

where \( v_c \) is the \( I \times 1 \) vector of approximated value functions at each node, \( \Phi \) is the \( I \times J \) collocation matrix and \( c = [c_1, \ldots, c_J]' \) is the vector of approximation coefficients. We also discretize the support of the three shocks in \( K_1, K_2 \) and \( K_3 \) points and denote \( w_{1,k}, w_{2,k} \) and \( w_{3,k} \) the approximated probability masses associated to each of the nodes. The shocks are assumed to independent. Under these assumptions, we get for each node \( i \in \{1, \ldots, I\} \):

\[ v_{c,i} = \delta \left[ \sum_{\bar{\varepsilon}_c \in K_1} \sum_{\varepsilon_x \in K_2} \sum_{\varepsilon_{\sigma} \in K_3} w_{1,\varepsilon_c} w_{2,\varepsilon_x} w_{3,\varepsilon_{\sigma}} \exp \left\{ \frac{\theta}{\psi} \left( 1 - \frac{1}{\psi} \right) \Delta c'_{i,\varepsilon_c} \right\} \right]^{\frac{1}{\theta}} \]

(A.28)

where

\[ \sigma'_{i,\varepsilon_{\sigma}} = \bar{\sigma} + \nu_1 (\sigma_i - \bar{\sigma}) + \sigma_{w,\varepsilon_{\sigma}} \]

\[ \Delta c'_{i,\varepsilon_c} = x_i + \sigma'_{i,\varepsilon_{\sigma}} \varepsilon_c \]

\[ x'_{i,\varepsilon_x} = \rho x_i + \varphi e \sigma'_{i,\varepsilon_{\sigma}} \varepsilon_x \]
We can now use the following algorithm to solve the Euler equation recursively:

1. guess an initial vector of basis coefficients $c^1$

2. for each node $(s, \alpha)_i$ compute the right hand side of equation (A.28) using $c^1$
   and call $v(c^1)$ the outcome

3. solve for $c^2 = (\Phi'\Phi)^{-1}\Phi'v(c^1)$

4. replace $c^1$ with $c^2$ and iterate until convergence.

Having solved for the price-consumption ratio $v_c$, we can solve the Euler equation for the price-dividend ratio in a similar way:

$$v_{d,i} = \delta^\theta \sum_{\varepsilon_c \in K_1} \sum_{\varepsilon_x \in K_2} \sum_{\varepsilon_\sigma \in K_3} \sum_{\varepsilon_d \in K_1} w_{1,\varepsilon_c} w_{2,\varepsilon_x} w_{3,\varepsilon_\sigma} w_{4,\varepsilon_d} \exp\{m'_{i,(\varepsilon_c,\varepsilon_x,\varepsilon_\sigma)}\}$$

$$\left(1 + \sum_{j=1}^J d_j \phi_j(x'_{i,\varepsilon_x}, \sigma'_{i,\varepsilon_\sigma})\right)$$

(A.29)

where

$$m'_{i,(\varepsilon_c,\varepsilon_x,\varepsilon_\sigma)} = \left(\theta - 1 - \frac{\theta}{\psi}\right) \Delta c'_{i,\varepsilon_c} (\theta - 1) \log\left(1 + \frac{v'_{i,(\varepsilon_c,\varepsilon_x,\varepsilon_\sigma)}}{v_i}\right)$$

4 Estimation of state space models

Since we have quarterly data, but we specified the model to describe a monthly decision problem, the first thing that we need to do is to aggregate consumption
data:

\[ \Delta c^i_t = \bar{x}^i_t + \varepsilon^i_{c,t} \]

\[ \bar{x}^i_{t+1} = \left(1 + \rho_x^i + \rho_x^2\right) \left(\rho_x^3 x^i_{1,t-1} + y^i_{1,t} + \rho_x^i y^i_{2,t} + \rho_x^i y^i_{3,t}\right) \]

\[ x^i_{1,t} = \rho_x^3 x^i_{1,t-1} + \left(y^i_{1,t} + \rho_x^i + y^i_{2,t} + \rho_x^i y^i_{3,t}\right) \]

\[ y^i_{j,t+1} = \varepsilon^i_{x_j,t+1}, \forall j \in \{1, 2, 3\} \quad (A.30) \]

\[ (A.31) \]

where \( \varepsilon^i_{c,t} \sim \text{iid} \mathcal{N}\left(0, 3\sigma_c^2\right) \) and \( \varepsilon^i_{x,t} \sim \text{iid} \mathcal{N}\left(0, \varphi_c^2 \sigma_c^2\right) \). We also assume that \( \varepsilon^i_{c,t} \) and \( \varepsilon^i_{x,t} \) are uncorrelated within the same country. Denoting \( Y^i_t = \Delta c^i_t \) and \( X^i_t = \left[ \bar{x}^i_t \; x^i_{1,t-1} \; y^i_{1,t} \; y^i_{2,t} \; y^i_{3,t} \right]' \), \( \forall i \in \{h, f\} \), we obtain the following state space representation within each country:

\[ Y^i_t = A^i X^i_t + C^i v^i_t \]

\[ X^i_{t+1} = B^i X^i_t + D^i w^i_{t+1} \]

where \( A^i, B^i, C^i \) and \( D^i \) can be easily built from (A.30), \( \forall i \in \{h, f\} \). The whole system can now be constructed by stacking the observable and unobservable variables into vectors \( Y_t = \left[ \Delta c^h_t \; \Delta c^f_t \right]' \) and \( X_t = \left[ X^h_t \; X^f_t \right]' \):

\[ Y_t = AX_t + Cv_t \]

\[ X_{t+1} = BX_t + Dw_{t+1} \]

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where

\[
A = \begin{bmatrix}
A^h & 0 \\
0 & A^f
\end{bmatrix},
\quad C = \begin{bmatrix}
\sigma^h_c & 0 \\
\sigma^f_c \rho_c & \sigma^f_c \sqrt{1 - \rho^2_c}
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
B^h & 0 \\
0 & B^f
\end{bmatrix},
\quad D = \begin{bmatrix}
1 & 0 \\
\rho_x \sqrt{1 - \rho^2_x} & \otimes D^h
\end{bmatrix}
\]

and \(v_t\) and \(w_{t+1}\) are bivariate independent vectors distributed as standard normals.

Standard methods as those described in Hansen and Sargent (2004) can be used to obtain the likelihood function:

\[
L(Y_1, ..., Y_T) = \frac{1}{2} \sum_{t=1}^{T} \left\{ p \log(2\pi) + \log |\Omega_t| + a_t^\prime \Omega_t^{-1} a_t \right\}
\]

where \(\Omega_t = A\Sigma_t A^\prime + CC^\prime\), \(\Sigma_t\) solves the matrix Riccati difference equation \(\Sigma_{t+1} = B\Sigma_t B^\prime + DD^\prime - B\Sigma_t A^\prime \left(A\Sigma_t A^\prime + CC^\prime\right)^{-1} A\Sigma_t B^\prime\), \(a_t\) comes from the innovations representation

\[
a_t = Y_t - A\hat{X}_t
\]

\[
\hat{X}_{t+1} = B\hat{X}_t + K_t a_t
\]

and \(K_t = B\Sigma_t A^\prime \left(A\Sigma_t A^\prime + CC^\prime\right)^{-1}\) is the Kalman gain.

When the state space model extends to three and four countries, the aforementioned matrices \(A, B, C, D\) should be modified as follows (the subscript 3 or 4...
refers to the two alternative extensions):

\[
A_3 = \begin{bmatrix} A & 0 \\ 0 & A^g \end{bmatrix} \quad C_3 = \begin{bmatrix} \sigma_c^h & 0 & 0 \\ \sigma_c^f \rho_c^{hf} & \sigma_c^f \sqrt{1 - \rho_c^{hf}} & 0 \\ \sigma_c^g \rho_c^{hg} & \sigma_c^g \rho_c^{g,h} & \sigma_c^g \tau_{1,c} \end{bmatrix}
\]

\[
B_3 = \begin{bmatrix} B & 0 \\ 0 & B^g \end{bmatrix} \quad D_3 = \begin{bmatrix} 1 & 0 & 0 \\ \rho_x^{h} \sqrt{1 - \rho_x^{hf}} & 0 \\ \rho_x^{h} \left(\frac{\rho_c^{g,h} - \rho_x^{g,h}}{\sqrt{1 - \rho_x^{hf}}}\right) & \tau_{1,x} \end{bmatrix} \otimes D^f
\]

and

\[
A_4 = \begin{bmatrix} A_3 & 0 \\ 0 & A^l \end{bmatrix} \quad C_4 = \begin{bmatrix} \sigma_c^h & 0 & 0 & 0 \\ \sigma_c^f \rho_c^{hf} & \sigma_c^f \sqrt{1 - \rho_c^{hf}} & 0 & 0 \\ \sigma_c^g \rho_c^{hg} & \sigma_c^g \rho_c^{g,h} & \sigma_c^g \tau_{1,c} & 0 \\ \sigma_c^l \rho_c^{hl} & \sigma_c^l \left(\frac{\rho_c^{l} - \rho_x^{l}}{\sqrt{1 - \rho_x^{hf}}}\right) & \sigma_c^l \tau_{2,c} & \sigma_c^l \tau_{3,c} \end{bmatrix}
\]

\[
B_4 = \begin{bmatrix} B_3 & 0 \\ 0 & B^l \end{bmatrix} \quad D_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \rho_x^{h} \sqrt{1 - \rho_x^{hf}} & 0 & 0 \\ \rho_x^{h} \left(\frac{\rho_c^{g,h} - \rho_x^{g,h}}{\sqrt{1 - \rho_x^{hf}}}\right) & \tau_{1,x} & 0 \\ \rho_x^{h} \left(\frac{\rho_c^{l} - \rho_x^{l}}{\sqrt{1 - \rho_x^{hf}}}\right) & \tau_{2,x} & \tau_{3,x} \end{bmatrix} \otimes D^l
\]
where:

\[
\tau_{1,j} = \frac{\sqrt{1 - \rho_j^{h_f} - \rho_j^{h_g} - \rho_j^{f_d}}}{\sqrt{1 - \rho_j^{h_f^2}}}
\]

\[
\tau_{2,j} = \frac{\rho_j^l - \rho_j^l \rho_j^{f_d} - \rho_j^f \rho_j^{h_g} + \rho_j^{h_f} \rho_j^{f_d} + \rho_j^{h_g} \rho_j^{f_d} + \rho_j^{h_f} \rho_j^{f_d}}{\sqrt{(1 - \rho_j^{h_f^2}) (1 - \rho_j^{h_g^2} - \rho_j^{f_d^2} + 2 \rho_j^{h_f} \rho_j^{f_d})}}
\]

\[
\tau_{3,j} = \sqrt{\frac{1 - \rho_j^{h_f^2} - \rho_j^{f_d^2} + 2 \rho_j^l \rho_j^{f_d}}{1 - \rho_j^{h_f^2}}} - \frac{(\rho_j^l - \rho_j^l \rho_j^{h_f^2} - \rho_j^{h_g} \rho_j^{f_d} + \rho_j^{h_f} \rho_j^{f_d} + \rho_j^{h_g} \rho_j^{f_d} + \rho_j^{h_f} \rho_j^{f_d})^2}{(1 - \rho_j^{h_f^2}) (1 - \rho_j^{h_g^2} - \rho_j^{f_d^2} + 2 \rho_j^{h_f} \rho_j^{f_d})}
\]

for \( j \in \{c, x\} \).

5 Solution of the benchmark model

The model is reported in chapter 4 with the addition of the following two equations to describe the dynamics of dividends growth in each country:

\[
\Delta d_{t+1}^h = \mu_d + \varphi_d \lambda_t^{h/2} \varphi_{d,t+1}^h
\]

\[
\Delta d_{t+1}^f = \mu_d + \varphi_d \lambda_t^{f/2} \varphi_{d,t+1}^f
\]

Denote the vector of shocks as \( \xi_{t+1} = \begin{bmatrix} \varepsilon_{c,t+1}^h, \varepsilon_{c,t+1}^f, \varepsilon_{\lambda,t+1}^h, \varepsilon_{\lambda,t+1}^f, \varepsilon_{d,t+1}^h, \varepsilon_{d,t+1}^f \end{bmatrix} \).

Stochastic discount factors are (in logarithms):

\[
m_{t+1}^h = \log(\delta) - \gamma \Delta c_{t+1}^h
\]

\[
m_{t+1}^f = \log(\delta) - \gamma \Delta c_{t+1}^f
\]

Use the Campbell and Shiller (1988)

\[
r_{d,t+1}^h = k_0 + \Delta d_{t+1}^h - k_1 dp_{t+1}^h + dp_t^h
\]
and guess that $dp^h_t = \overline{dp} + v_\lambda \lambda^h_t$. Use the Euler equation to solve for $v_\lambda$:

$$0 = \log E_t \left[ \exp \{ m^h_{t+1} + r^h_{d,t+1} \} \right]$$

$$= dp^h_t + \log E_t \left[ \exp \{ \log(\delta) - \gamma \left( \mu_c + \lambda^{h/2}_t \varepsilon_{c,t+1}^h \right) + k_0 + \mu_d + \varphi_d \lambda^h_t \varepsilon_{d,t+1}^h + k_1 \left( \overline{dp} + v_\lambda \left( \sigma^2(1 - \rho_\lambda) + \rho_\lambda \lambda^h_t + \varphi_\lambda \lambda^{h/2}_t \varepsilon_{\lambda,t+1}^h \right) \right) \} \right]$$

Denote:

$$\overline{dp} = - (\log(\delta) - \mu_c \gamma k_0 \mu_d - k_1 \overline{dp} - k_1 v_\lambda \sigma^2(1 - \rho_\lambda))$$

$$g = \left[ -\gamma, \ 0, \ -k_1 \varphi_\lambda v_\lambda, \ 0, \ \varphi_d, \ 0 \right]$$

Hence the Euler equation implies:

$$dp^h_t = \overline{dp} + \left( k_1 v_\lambda \rho_\lambda - \frac{g W g'}{1} \right) \lambda^h_t \quad \text{(A.32)}$$

where $W = E \left[ \varepsilon_{t+1}^h \varepsilon_{t+1}^{h'} \right]$ and $v_\lambda$ is the solution of

$$v_\lambda = \left( k_1 v_\lambda \rho_\lambda - \frac{g W g'}{1} \right)$$

The solution for the foreign country’s log-dividend to price ratio goes in exactly the same way with the exception of using $f$-indexed variables everywhere and replacing $g$ with $g^* = \left[ 0, \ -\gamma, \ 0, \ -k_1 \varphi_\lambda v_\lambda, \ 0, \ \varphi_d \right]$.  

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Risk free rates are:

\[
    r^h_t = - \log E_t \exp \{ m^h_{t+1} \} = - \log E_t \exp \{ \log(\delta) - \gamma \left( \mu_c + \lambda^{h^{1/2}}_{t+1} \zeta^{h}_{c,t+1} \right) \} = \gamma \mu_c - \log(\delta) - \frac{\gamma^2}{2} \lambda^h_t
\] (A.33)

and

\[
    r^f_t = \gamma \mu_c - \log(\delta) - \frac{\gamma^2}{2} \lambda^f_t
\] (A.34)

Ultimately excess returns are:

\[
    r^h_{d,t} - r^h_t = \bar{\pi} + \left( \frac{\gamma^2}{2} - k_1 v \rho_{\lambda} + v_{\lambda} \right) \lambda^h_t + \lambda^{h^{1/2}}_{t+1} \bar{g}_t \bar{\xi}_{t+1}
\] (A.35)

where \( \bar{g} = \begin{bmatrix} 0, & 0, & -k_1 \varphi_{\lambda} v_{\lambda}, & 0, & \varphi_d, & 0 \end{bmatrix} \).

6 Solution of the long run risks model with unit elasticity of intertemporal substitution

The setup of the model is reported in section 4.2. The solution consists of three steps:

1. approximate \( \lambda^{h^{1/2}}_t, \lambda^{f^{1/2}}_t \) and \( h_t \) starting from \( \lambda^h_t \) and \( \lambda^f_t \)

2. solve for the two value functions

3. obtain the stochastic discount factors
6.1 Useful approximations

A first order linear approximation $\lambda_t^{h^{1/2}}$, $\lambda_t^{f^{1/2}}$ and $h_t = \lambda_t^{h^{1/2}} \lambda_t^{f^{1/2}}$ around the steady state $\overline{\lambda}^h = \overline{\lambda}^f = \sigma^2$ delivers:

$$
\lambda_t^{h^{1/2}} \approx \sigma (1 - \rho) + \rho \lambda_{t-1}^{h^{1/2}} + \frac{1}{2} \varphi \lambda \varepsilon_{\lambda,t}^h
$$

$$
\lambda_t^{f^{1/2}} \approx \sigma (1 - \rho) + \rho \lambda_{t-1}^{f^{1/2}} + \frac{1}{2} \varphi \lambda \varepsilon_{\lambda,t}^f
$$

$$
h_t \approx \sigma^2 (1 - \rho)^2 + \sigma (1 - \rho) \rho \lambda \left( \lambda_{t-1}^{h^{1/2}} + \lambda_{t-1}^{f^{1/2}} \right) + \rho^2 h_{t-1} + \\
+ \frac{\varphi \lambda}{2} \left[ \sigma (1 - \rho) + \rho \lambda \lambda_{t-1}^{h^{1/2}} \right] \varepsilon_{\lambda,t}^h + \frac{\varphi \lambda}{2} \left[ \sigma (1 - \rho) + \rho \lambda \lambda_{t-1}^{f^{1/2}} \right] \varepsilon_{\lambda,t}^f
$$

6.2 Solving for the value functions

I shall start by expressing the utility functions in a more convenient form. Subtracting $\log C_t^h$ from both sides of the home country utility function reported in (4.7), it is possible to obtain:

$$
V_t^h = U_t^h - \log C_t^h
$$

$$
= \theta \delta \log E_t \exp \left\{ \frac{U_{t+1}^h - \log C_{t+1}^h + \log C_t^h - \log C_{t+1}^h \log C_t^h}{\theta} \right\}
$$

$$
= \theta \delta \log E_t \exp \left\{ \frac{V_{t+1}^h + \Delta c_t^h}{\theta} \right\}
$$

Similarly for the foreign country:

$$
V_t^f = \theta \delta \log E_t \exp \left\{ \frac{V_{t+1}^f + \Delta c_{t+1}^f}{\theta} \right\}
$$
Guess that the solution of $V^h_t$ and $V^f_t$ is linear:

$$V^h_t = A + B_1 z_{1,t} + B_2 z_{2,t} + D_\lambda \lambda^h_t + D_\lambda \cdot \lambda^f_t + D_h h_t + E_\lambda \lambda^{h/2}_t + E_\lambda \cdot \lambda^{f/2}_t$$

with the solution for $V^f_t$ taking on the same expression, but with the coefficients being indexed by a $^\ast$. It follows that:

$$V^h_t = \theta \delta \log E_t \exp \left\{ \frac{1}{\theta} \left[ \mu + \lambda^h_t z_{1,t} + \lambda^h_t z_{2,t} + \lambda^{h/2}_t \epsilon^h_{e,t+1} + A + 
    B_1 \left( \rho_1 z_{1,t} + \varphi_{e_1} \lambda^h_{1/2} \epsilon^h_{1,t+1} \right) + B_2 \left( \rho_2 z_{2,t} + \varphi_{e_2} \lambda^h_{1/2} \epsilon^h_{2,t+1} \right) + 
    D_\lambda \left( \sigma^2 (1 - \rho_\lambda) + \rho_\lambda \lambda^h_t + \varphi_\lambda \lambda^{h/2}_t \epsilon^h_{\lambda,t+1} \right) + D_\lambda \cdot \left( \sigma^2 (1 - \rho_\lambda) + \rho_\lambda \lambda^f_t + \varphi_\lambda \lambda^{f/2}_t \epsilon^f_{\lambda,t+1} \right) + 
    D_h \left( \sigma^2 (1 - \rho_\lambda)^2 + \sigma (1 - \rho_\lambda) \rho_\lambda \left( \lambda^{h/2}_t + \lambda^{f/2}_t \right) + \rho_\lambda^2 b_t \right. 
    + \left. \frac{\varphi_\lambda}{2} \left[ \sigma (1 - \rho_\lambda) + \rho_\lambda \lambda^{f/2}_t \right] \epsilon^f_{\lambda,t+1} + \frac{\varphi_\lambda}{2} \left[ \rho_\lambda^2 b_t \right] \right) \right\}$$

Defining the two vectors $v_1$ and $v_2$ as

$$v_1 = \begin{bmatrix} B_1 \varphi_{e_1}, & 0, & 1, & 0, & D_\lambda \varphi_\lambda, & D_h \varphi_\lambda \rho_\lambda \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 0, & B_2 \varphi_{e_2}, & 0, & 0, & D_h \varphi_\lambda \rho_\lambda, & D_\lambda \cdot \varphi_\lambda \end{bmatrix}$$

the value function $V^h_t$ can be rewritten as:

$$V^h_t = A + \delta \left( \lambda^h_t + B_1 \rho_1 \right) z_{1,t} + \delta \left( \lambda^h_t + B_2 \rho_2 \right) z_{2,t} + \delta \left( D_\lambda \rho_\lambda + v_1 \frac{R_t}{d} \right) \lambda_t v +$$

$$+ \delta \left( D_\lambda \cdot \rho_\lambda + v_2 \frac{R_t}{d} \right) \lambda^f_t + \delta \left( D_h \rho_\lambda^2 + \frac{v_1 R_t^f}{d} \right) h_t + \delta \left[ D_h \sigma \left( 1 - \rho_\lambda \right) + E_\lambda \rho_\lambda \right] \lambda^{h/2}_t +$$

$$+ \delta \left[ D_h \sigma \left( 1 - \rho_\lambda \right) + E_\lambda \cdot \rho_\lambda \right] \lambda^{f/2}_t$$

The solution for the parameter $A$ is easy to find, but I will omit it, because it is not going to affect the stochastic discount factors. All other parameters can be found
by matching coefficients:

\[ B_1 = \frac{\delta \lambda_1}{1 - \delta \rho_1}, \quad B_2 = \frac{\delta \lambda_2^h}{1 - \delta \rho_2} \]

The three parameters \( D_\lambda, D_{\lambda^*} \) and \( D_h \) are the solution of a second order system:

\[
D_\lambda = \delta \left[ D_\lambda \rho_\lambda + \frac{v_1 R v_1^\prime}{2 \theta} \right], \quad D_{\lambda^*} = \delta \left[ D_{\lambda^*} \rho_\lambda + \frac{v_2 R v_2^\prime}{2 \theta} \right], \quad D_h = \delta \left[ D_h \rho_h^2 + \frac{v_1 R v_2^\prime}{2 \theta} \right]
\]

Out of the 8 possible solutions, I shall select the one with largest coefficients, because that would be the one that maximizes utility, given that \( \lambda^h_t, \lambda^f_t \) and \( h_t \) are always greater than zero by definition. The last two parameters are

\[
E_\lambda = \frac{\delta D_h \sigma \rho_\lambda (1 - \rho_\lambda)}{1 - \delta \rho_\lambda}, \quad E_{\lambda^*} = \frac{\delta D_h \sigma \rho_\lambda (1 - \rho_\lambda)}{1 - \delta \rho_\lambda}
\]

For the foreign country, the procedure is identical with two exceptions:

1. variables indexed by a (*) will be used

2. the vectors \( v_1 \) and \( v_2 \) are replaced by

\[
v_1^* = \begin{bmatrix} B_1^* \varphi_{e_1}, & 0, & 0, & 0, & D_\lambda \varphi_{\lambda}, & D_h^* \varphi_{h_\lambda} \end{bmatrix}
\]

\[
v_2^* = \begin{bmatrix} 0, & B_2^* \varphi_{e_2}, & 0, & 1, & D_h^* \varphi_{h_\lambda}, & D_{\lambda^*} \varphi_{\lambda} \end{bmatrix}
\]

respectively.
6.3 Solving for the stochastic discount factors

Having solve for the two value functions, it is now possible to solve for the stochastic discount factors. I shall start with the home country:

\[
m^h_{t+1} = \log \delta - \Delta c^h + \log \frac{\exp \left\{ \frac{V^h_{t+1} + \Delta c^h_{t+1}}{\theta} \right\}}{E_t \exp \left\{ \frac{V^h_{t+1} + \Delta c^h_{t+1}}{\theta} \right\}}
\]

\[
= \log \delta - \mu_c - \lambda^h_{1} z_{1,t} - \lambda^h_{2} z_{2,t} + \lambda^{h^{1/2}}_{1} \hat{c}^h_{c,t+1} + \frac{1}{\theta} \left( v_1 \lambda^{h^{1/2}}_t + v_2 \lambda^{f^{1/2}}_t \right) \eta_{t+1} + \frac{-1}{2\theta^2} \left( v_1 Rv^f_1 \lambda^h_t + v_2 Rv^f_2 \lambda^f_t + v_1 Rv^f_2 h_t \right)
\]

Define \( v_{0,m}, \tilde{v}_1 \) and \( v_m \) as:

\[
v_{0,m} = \mu_c - \log \delta
\]

\[
\tilde{v}_1 = v_1 + \begin{bmatrix} 0 & 0 & \theta & 0 & 0 \end{bmatrix}
\]

\[
v_m = \begin{bmatrix} \lambda^h_1 & \lambda^h_2 & v_1 Rv^f_1 \sqrt{2}\theta & v_2 Rv^f_2 \sqrt{2}\theta & v_1 Rv^f_2 \sqrt{2}\theta \end{bmatrix}
\]

and rewrite the stochastic discount factor in compact vector notation:

\[
m^h_{t+1} = -v_{0,m} - v_m s'_t + \frac{1}{\theta} \left( \tilde{v}_1 \lambda^{h^{1/2}}_t + v_2 \lambda^{f^{1/2}}_t \right) \eta_{t+1}^{f'} \quad (A.36)
\]

Similarly for the foreign country:

\[
m^f_{t+1} = -v_{0,m} - v_m s'_t + \frac{1}{\theta} \left( v_1^* \lambda^{h^{1/2}}_t + \tilde{v}_2^* \lambda^{f^{1/2}}_t \right) \eta_{t+1}^{f'} \quad (A.37)
\]

where

\[
\tilde{v}_2^* = v_2^* + \begin{bmatrix} 0 & 0 & 0 & \theta & 0 & 0 \end{bmatrix}
\]

\[
v^*_m = \begin{bmatrix} \lambda^f_1 & \lambda^f_2 & v_1^* Rv^f_1 \sqrt{2}\theta & v_2^* Rv^f_2 \sqrt{2}\theta & v_1^* Rv^f_2 \sqrt{2}\theta \end{bmatrix}
\]

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7 Derivation of returns

In this section we shall show how to derive the risk free rates, the real exchange rate and the returns on assets that pay the dividend whose process is specified in section 4.2.

7.1 Risk free rates

Given the processes for the stochastic discount factors in equations (A.36) and (A.37), risk free rates can be computed as:

\[ r^h_t = \log E_t \exp \{ m^h_{t+1} \} \]
\[ = v_{0,m} + v_m s_t' - \frac{1}{2\theta^2} \left( \tilde{v}_1 R_v \lambda^h_1 + v_2 R_v \lambda^f_2 + \tilde{v}_1 R_v' h_t \right) \]
\[ = v_{0,m} + \tilde{v}_m s_t' \quad (A.38) \]

where

\( \tilde{v}_m = \begin{bmatrix} \lambda^h_1, \lambda^h_2, \frac{v_2 R_v - \tilde{v}_2 R_v}{2\theta^2}, 0, \frac{(v_1 - \tilde{v}_1) R_v}{2\theta^2} \end{bmatrix} \)

Similarly for the foreign country:

\[ r^f_t = v_{0,m} + \tilde{v}^*_m s_t' \quad (A.39) \]

where

\( \tilde{v}^*_m = \begin{bmatrix} \lambda^f_1, \lambda^f_2, 0, \frac{v^*_2 R_v' - \tilde{v}^*_2 R_v'}{2\theta^2}, \frac{v^*_1 R(v^*_2 - \tilde{v}^*_2)}{2\theta^2} \end{bmatrix} \)
7.2 Real exchange rate

Assuming that markets are complete, the growth of the real exchange rate must equal the difference between the two stochastic discount factors:

$$\log \frac{e_{t+1}}{e_t} = m_{t+1}^f - m_{t+1}^h$$

$$= (v_m - v_m^*) s_t^* + \frac{1}{\theta} \left[ (v_1^* - \overline{v}_1) \lambda_t^{h^{1/2}} + (\overline{v}_2^* - v_2) \lambda_t^{f^{1/2}} \right] \eta_{t+1}$$ (A.40)

7.3 Asset returns

Campbell and Shiller (1988) show that the logarithm of the return of an asset can be approximated as:

$$r_{d,t+1}^h \approx k_0 + \Delta d_{t+1}^h - k_1 dp_{t+1}^h + dp_t^h$$

where $\Delta d_{t+1}^h$ is growth of the dividend paid by the asset, $dp_{t+1}^h$ is the logarithm of the dividend to price ratio and $k_0$ and $k_1$ are approximation constants defined as

$$k_0 = \log \left( 1 + \exp \left\{ \overline{dp} \right\} \right) - \frac{\exp \left\{ \overline{dp} \right\} \overline{dp}} {1 + \exp \left\{ \overline{dp} \right\}}, \quad k_1 = \frac{1} {1 + \exp \left\{ \overline{dp} \right\}}$$

with $\overline{dp}$ being the steady state of the dividend to price ratio about which the approximation is taken.

I shall solve for the logarithm of the dividend to price ratio. Guess that the solution of $dp_t^h$ is linear in $z_{1,t}, z_{2,t}, \lambda_t^h, \lambda_t^f, h_t, \lambda_t^{h^{1/2}}, \lambda_t^{f^{1/2}}$:

$$dp_t^h = v_{0,p} + v_{1,p} z_{1,t} + v_{2,p} z_{2,t} + v_{\lambda,p} \lambda_t^h + v_{\lambda^*,p} \lambda_t^f + v_{h,p} h_t + v_{e,p} \lambda_t^{h^{1/2}} + v_{e^*,p} \lambda_t^{f^{1/2}}$$
and use the Campbell and Shiller (1988) approximation of returns in the Euler equation

\[ E_t \left[ \exp\{ m^h_{t+1}\} \exp\{ r^h_{t+1}\} \right] = 1 \]

to write

\[
E_t \left[ \exp \left\{ -v_{0,m} - v_m s'_t + \frac{1}{\theta} \left( \tilde{v}_1 \lambda_t^{h_1/2} + v_2 \lambda_t^{1/2} \right) \eta_{t+1}' + k_0 + \mu_d + \lambda_d z_{1,t} + \lambda_d z_{2,t} + \varphi_d \lambda_t^{h_1/2} \right\} \right] = 1
\]

Substitute the guess solution in the Euler equation:

\[
-dp^h_t = \log E_t \exp \left\{ -k_1 v_{0,p} - k_1 v_{1,p} \left( \rho_1 z_{1,t} + \varphi c_v \lambda_t^{h_1/2} \right) \eta_{t+1}' + k_1 v_{2,p} \left( \rho_2 z_{2,t} + \varphi c_v \lambda_t^{h_1/2} \right) \eta_{t+1}' + k_0 + \mu_d - v_{0,m} - v_m s'_t + \frac{1}{\theta} \left( \tilde{v}_1 \lambda_t^{h_1/2} + v_2 \lambda_t^{1/2} \right) \eta_{t+1}' \right\}
\]

For convenience, denote

\[
v_m = \begin{bmatrix} v_{1,m}, v_{2,m}, v_{\lambda,m}, v_{\lambda^*,m}, v_{h,m} \end{bmatrix}
\]

\[
w_{ap} = \begin{bmatrix} 0, 0, 0, 0, \frac{\varphi}{2} \left[ 1 + \sigma (1 - \rho_\lambda) \right], \frac{\varphi}{2} \left[ 1 + \sigma (1 - \rho_\lambda) \right], 0, 0 \end{bmatrix}
\]

\[
b_1 = -k_1 \begin{bmatrix} v_{1,p} \varphi c_v, 0, 0, v_{\lambda,p} \varphi c_v, v_{h,p} \varphi \frac{c_2}{2}, -\frac{c_2}{k_1}, 0 \end{bmatrix} + \frac{1}{\theta} \begin{bmatrix} \tilde{v}_1, 0, 0 \end{bmatrix}
\]

\[
b_2 = -k_1 \begin{bmatrix} 0, v_{2,p} \varphi c_v, 0, 0, v_{h,p} \varphi \frac{c_2}{2}, v_{\lambda^*,p} \varphi c_v, 0, 0 \end{bmatrix} + \frac{1}{\theta} \begin{bmatrix} v_2, 0, 0 \end{bmatrix}
\]

The solution of the dividend to price ratio schedule can be obtained by matching
coefficients. The easiest to compute are the ones that pre-multiply the low frequency components of consumption growth:

\[
\begin{align*}
v_{1,p} &= \frac{v_{1,m} - \lambda d_1}{1 - k_1 \rho_1}, \\
v_{2,p} &= \frac{v_{2,m} - \lambda d_2}{1 - k_1 \rho_2}.
\end{align*}
\]

The coefficients \(v_{\lambda,p}, v_{\lambda^*,p}\) and \(v_{h,p}\) are the solution of a second order system:

\[
\begin{align*}
v_{\lambda,p} &= k_1 v_{\lambda,p} \rho_{\lambda} + v_{\lambda,m} - b_1 \tilde{R} b'_1, \\
v_{\lambda^*,p} &= k_1 v_{\lambda^*,p} \rho_{\lambda} + v_{\lambda^*,m} - b_2 \tilde{R} b'_2, \\
v_{h,p} &= k_1 v_{h,p} \rho_{\lambda}^2 + v_{h,m} - b_1 \tilde{R} b'_2
\end{align*}
\]

Given the calibration that is used in the paper, the solution of this system turns out to be unique. The remainder of the coefficients is equal to:

\[
\begin{align*}
v_{e,p} &= \frac{k_1 v_{h,p} \sigma (1 - \rho_{\lambda}) \rho_{\lambda}}{1 - k_1 \rho_{\lambda}}, \\
v_{e^*,p} &= v_{e,p} \\
v_{0_1} &= \frac{1}{1 - k_1} \left\{ k_1 \left[ \sigma^2 (1 - \rho_{\lambda}) (v_{\lambda,p} + v_{\lambda^*,p}) + \sigma^2 (1 - \rho_{\lambda})^2 v_{h,p} + \sigma (1 - \rho_{\lambda}) (v_{e,p} + v_{e^*,p}) \right] + \\
& \quad -w_ap \tilde{R} w'_a + v_{0,m} - k_0 + \mu_d \right\}
\end{align*}
\]

For the foreign country, the procedure is identical with two exceptions:

1. variables indexed by a (*) will be used

2. the vectors \(v_1\) and \(v_2\) are replaced by

\[
\begin{align*}
b^*_1 &= -k_1 \left[ v_{1,p}^* \varphi_\epsilon, 0, 0, 0, v_{1,p}^* \varphi_\lambda, v_{h,p}^* \frac{\varphi_\lambda \rho_{\lambda}}{2}, 0, 0 \right] + \frac{1}{\theta} \left[ v_1^*, 0, 0 \right] \\
b^*_2 &= -k_1 \left[ 0, v_{2,p}^* \varphi_\epsilon, 0, 0, v_{h,p}^* \frac{\varphi_\lambda \rho_{\lambda}}{2}, v_{\lambda^*,p}^* \varphi_\lambda, 0, -\frac{\varphi_\epsilon}{k_1} \right] + \frac{1}{\theta} \left[ v_2^*, 0, 0 \right]
\end{align*}
\]

respectively.
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