Generalized IV Estimation of Nonlinear Rational Expectations Models

Hansen and Singleton (1982)
What is a Rational Expectations (RE) Model?

• An equilibrium of a dynamic model can typically be described by a probability distribution over sequences of data

• RE → Each agent’s subjective belief about the data is a conditional of this equilibrium probability distribution, where the conditioning is on the agent’s information set

• Expectations are thus consistent with outcomes generated by the model

• This means we can use the moment conditions derived from agent’s Euler equations to estimate the model’s parameters using GMM
2 Ways to estimate RE models

Estimation methods for RE models can be categorized by the amount of info they require

1. Full-Information: MLE

2. Limited-Information: GMM
MLE: Full Information

- Goal: Estimate the entire model
- This estimation method is efficient and produces estimates for all the parameters in the model
- Requires econometrician to specify the entire structure of the model, including the distribution of shocks.
- VERY HARD outside of linear models
GMM: Limited Information

• Goal: Estimate only some of the model parameters

• Trade-off: Generally less efficient than MLE, but avoids contaminating the estimation results by model misspecification

• E.g. it avoids having to specify the shock distributions, production technology etc.

• Good for non-linear models
Asset-Pricing Example

• Simple version of Lucas (1978)
  – One consumption good and one asset

• Representative Agent
  – Infinite horizon, CRRA utility
  – Each period chooses between consuming good or investing in asset
The Consumers Problem

\[ \max \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j \frac{C_t^j}{\gamma} \]

s.t. \( C_{t+j} + I_{t+j} \leq r_{t+j} I_{t+j-1} + W_{t+j} \quad j = 0, 1, \ldots, \infty \)

\( C_t \): Consumption
\( I_t \): Investment in (one-period) asset
\( W_t \): Labor Income
\( r_t \): Asset Return
\( \beta \): Discount Factor
\( \gamma \): Preference parameter
Euler Equation

- The agents EE: \( E_t \left[ \beta \left( \frac{C_{t+1}}{C_t} \right)^{\gamma-1} r_{t+1} - 1 \right] = 0 \)

- 1 orthogonality condition for 2 unknowns: \((\gamma, \beta)\)

- Adding instruments \( \zeta_t \), and conditioning down the EE becomes:

\[
E \left[ \beta \left( \frac{C_{t+1}}{C_t} \right)^{\gamma-1} r_{t+1} - 1 \right] \otimes z_t = 0
\]
GMM

- Let \( y_{t+1} = \left[ \frac{C_{t+1}}{C_t}, r_{t+1} \right] \) and \( \theta = [\gamma, \beta] \)

- Define \( f(y_{t+1}, z_t, \theta) = \left[ \beta \left( \frac{C_{t+1}}{C_t} \right)^{\gamma-1} r_{t+1} - 1 \right] z_t \)

\[
g_T(\theta) = \frac{1}{T} \sum_{t=1}^{T} f(y_{t+1}, z_t, \theta)
\]

- GMM: \( \hat{\theta} = \arg \min_{\theta} g_T(\theta) W_T g_T(\theta) \)

where \( W_T \) is the weighting matrix
Optimal Weighting Matrix

• Define the \( j \)th sample autocovariance:

\[
R_t(j) = \frac{1}{T} \sum_{t=1+j}^{T} f(y_{t+1}, z_t, \theta_T) f(y_{t+1}, z_t, \theta_T)^T
\]

• Optimal Weighting Matrix:

\[
W_T^* = \left\{ R_T(0) + \sum_{j=1}^{n-1} [R_T(j) + R_T(j)^T] \right\}^{-1}
\]

where \( \theta_T \) is obtained from the first-step GMM using a suboptimal weighting matrix.
MLE

- Assume that consumption growth and the asset return are conditionally log-normal, so

\[
\mathbb{E}_t \left[ \left( \frac{C_{t+1}}{C_t} \right)^{\gamma^{-1}} r_{t+1} \right] = \exp \left[ \mu_t + \frac{\sigma^2}{2} \right]
\]

- Recall the EE:

\[
\mathbb{E}_t \left[ \left( \frac{C_{t+1}}{C_t} \right)^{\gamma^{-1}} r_{t+1} \right] = \frac{1}{\beta}
\]

- Combining these equations

\[
\mu_t = -\log(\beta) - \frac{\sigma^2}{2}
\]
MLE

• Let $X_t = \log \left( \frac{C_{t+1}}{C_t} \right)$, $R_t = \log(r_t)$, $Y_t = [X_t, R_t]^T$

• Define the error term $V_{t+1}$:

$$V_{t+1} \equiv \log \left( \left( \frac{C_{t+1}}{C_t} \right)^{\gamma^{-1}} r_{t+1} \right) - \mu_t = \alpha X_{t+1} + R_{t+1} + \log \beta + \frac{\sigma^2}{2}$$

• Assume: $\mathbb{E}_t(X_{t+1}) = a(L)^T Y_t + \mu_x$

• System:

$$\begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix} \begin{bmatrix} X_{t+1} \\ R_{t+1} \end{bmatrix} = \begin{bmatrix} a(L)^T \\ 0 \end{bmatrix} \begin{bmatrix} X_t \\ R_t \end{bmatrix} + \begin{bmatrix} \mu_x \\ \log \beta + \frac{\sigma^2}{2} \end{bmatrix} + \begin{bmatrix} W_{t+1} \\ V_{t+1} \end{bmatrix}$$
MLE

• We have

\[
\eta_{t+1} = A_0 Y_{t+1} - A_1(L) Y_t - \mu,
\]

\[
\eta_{t+1} \sim N(\mu, \Sigma)
\]

• Define \( \theta = [\gamma \quad \beta \quad \mu_x \quad a(L) \text{ parameters} \quad \Sigma] \)

• Given \( T \) observations of \( Y_t \) we can estimate \( \theta \) via:

\[
\hat{\theta} = \arg \max_{\theta} -\frac{T}{2} \log |\Sigma| - \frac{1}{2} \sum_{t=0}^{T} \eta_{t+1}^{T} \Sigma^{-1} \eta_{t+1}
\]
Empirical Results

- **GMM:**

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<th>NLAG</th>
<th>$\hat{\alpha}$</th>
<th>$\hat{SE}(\hat{\alpha})$</th>
<th>$\hat{\beta}$</th>
<th>$\hat{SE}(\hat{\beta})$</th>
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- **MLE:**

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Empirical Results

• All of the GMM estimates of gamma are between .03 and .32

• All of the estimates of beta are greater than .99, but less than 1

• Much smaller S.E.’s on beta than gamma → beta estimated more precisely than gamma

• High chi-square p-vals → All estimates reject the stock pricing model with CRRA returns
Hall, Chapter 3

• Iterated GMM estimates:
  \[ \hat{\gamma} = -0.34 \]
  \[ \hat{\beta} = 0.99 \]

• Continuous updating GMM estimates:
  \[ \hat{\gamma} = 0.52 \]
  \[ \hat{\beta} = 0.99 \]