International Asset Pricing and Risk Sharing with Recursive Preferences

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Prepared for Tom Sargent’s PhD class
(Part 2)
Roadmap

Yesterday
- International asset pricing (exchange rates, co-movements of int’l stock market returns, ...)
- Agents have recursive preferences
- No equilibrium trade

Today
- Agents consume bundles of domestic and foreign goods
- Trade arises as an equilibrium outcome
- Efficient risk-sharing with recursive preferences
Battle plan

1. A benchmark model: Anderson, JET 2005
   ▶ one good economy
   ▶ set the stage for risk sharing with recursive preferences
Battle plan

1. A benchmark model: Anderson, JET 2005
   - one good economy
   - set the stage for risk sharing with recursive preferences

2. Leads the way to Colacito and Croce, 2010
   - two goods economy
Agents have risk-sensitive preferences

\[ U_{i,t} = (1 - \delta) \log C_{i,t} + \delta \theta_i \log E_t \exp \left\{ \frac{U_{i,t+1}}{\theta_i} \right\}, \quad \forall i \in \{h, f\} \]

where \( \theta_i = 1 / (1 - \gamma_i) \).
Agents have risk-sensitive preferences

\[ U_{i,t} = (1 - \delta) \log C_{i,t} + \delta E_t[U_{i,t+1}], \quad \forall i \in \{h, f\} \]

where \( \theta_i = 1 / (1 - \gamma_i) \). If \( \theta_i \to -\infty \): time additive case.
Agents have risk-sensitive preferences

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where \( \theta_i = 1/(1 - \gamma_i) \).
A benchmark model: Anderson, JET 2005

Agents have risk-sensitive preferences

\[ U_{i,t} \approx (1 - \delta) \log C_{i,t} + \delta E_t [U_{i,t+1}] + \frac{\delta}{2 \theta_i} V_t [U_{i,t+1}], \quad \forall i \in \{h, f\} \]

where \( \theta_i = 1 / (1 - \gamma_i) \). Conditional Variance matters.
A benchmark model: Anderson, JET 2005

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One good economy

\[ C_{h,t} + C_{f,t} = Z_t \]
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\[ C_{h,t} + C_{f,t} = Z_t \]

- Supply of \( Z_t \) is i.i.d. homoscedastic.

- Complete markets.
Pareto problem

Efficient allocations are the solution to the planner’s problem

choose \( \{C_{h,t}, C_{f,t}\}_{t=0}^{+\infty} \)

to max \( Q = \mu_h U_{h,0} + \mu_f U_{f,0} \)

s.t. \( C_{h,t} + C_{f,t} = Z_t, \quad \forall t \geq 0 \)

where
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s.t. \( C_{h,t} + C_{f,t} = Z_t, \quad \forall t \geq 0 \)

where

- \( \mu_h \) and \( \mu_f \) are the Pareto weights.
- Notation: \( S = \mu_h / \mu_f \).
Risk Sharing with Recursive Preferences: why is this hard?

Take first order conditions of planner’s problem:
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$$\mu_h \frac{1}{C_{h,0}} = \frac{1}{Z_0 - C_{h,0}} \mu_f$$
Take first order conditions of planner’s problem:

\[
\begin{align*}
\mu_h \frac{1}{C_{h,0}} &= \frac{1}{Z_0 - C_{h,0}} \mu_f \\
\exp \left\{ \frac{U_{h,1}}{\theta_h} \right\} \mu_h \frac{1}{C_{h,1}} &= \frac{1}{Z_1 - C_{h,1}} \mu_f \exp \left\{ \frac{U_{f,1}}{\theta_f} \right\} \\
E_0 \exp \left\{ \frac{U_{h,1}}{\theta_h} \right\} &= \frac{E_0 \exp \left\{ \frac{U_{f,1}}{\theta_f} \right\}}{Z_1 - C_{h,1}}
\end{align*}
\]
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Take first order conditions of planner’s problem:

\[ \mu_h \frac{1}{C_{h,0}} = \frac{1}{Z_0 - C_{h,0}} \mu_f \]

\[ \exp \left\{ \frac{U_{h,1}}{\theta_h} \right\} \mu_h \frac{1}{C_{h,1}} = \frac{1}{Z_1 - C_{h,1}} \mu_f \frac{\exp \left\{ \frac{U_{f,1}}{\theta_f} \right\}}{E_0 \exp \left\{ \frac{U_{f,1}}{\theta_f} \right\}} \]

\[ \vdots \]

\[ \exp \left\{ \frac{U_{h,t}}{\theta_h} \right\} \frac{1}{E_{t-1} \exp \left\{ \frac{U_{h,t}}{\theta_h} \right\}} \exp \left\{ \frac{U_{h,1}}{\theta_h} \right\} \mu_h \frac{1}{C_{h,t}} = \frac{1}{Z_t - C_{h,t}} \mu_f \exp \left\{ \frac{U_{f,1}}{\theta_f} \right\} \frac{\exp \left\{ \frac{U_{f,1}}{\theta_f} \right\}}{E_0 \exp \left\{ \frac{U_{f,1}}{\theta_f} \right\}} \frac{\exp \left\{ \frac{U_{f,t}}{\theta_f} \right\}}{E_{t-1} \exp \left\{ \frac{U_{f,t}}{\theta_f} \right\}} \]

We introduce an endogenous state variable and make the problem recursive.

\[
\frac{\mu_{h,t}}{C_{h,t}} = \frac{1}{Z_t - C_{h,t}} \frac{\mu_{f,t}}{\mu_{h,t} - 1} \exp \left\{ \frac{U_{h,t}}{\theta_h} \right\} E_{t-1} \exp \left\{ \frac{U_{f,t}}{\theta_f} \right\}
\]

and \( \frac{\mu_{h,0}}{\mu_{f,0}} = \frac{\mu_h}{\mu_f} \).
Recursive formulation

Lucas and Stokey (JET, 1983) and Kan (JET, 1995):

\[ Q(Z, \mu) = \max_{\{C_i, U_{i}^{j'}, i \in \{h, f\}\}} \left( \sum_{i \in \{h, f\}} \mu_i \left( (1 - \delta) \log C_i(Z, \mu_i) + \delta \theta_i \log \sum_j \pi_j \exp \left\{ U_{i}^{j''}/\theta_i \right\} \right) \right) \]

subject to

\[ \min_{\mu'} Q(Z^{j'}, \mu^{j''}) - \sum_{i \in \{h, f\}} \mu^{j''} U_{i}^{j'} \geq 0 \]
\[ C_h + C_f \leq Z \]
Recursive formulation

Lucas and Stokey (JET, 1983) and Kan (JET, 1995):

\[
Q^{k+1}(Z, \mu) = \max_{\{C_i, U'_i\}} \sum_{i \in \{h, f\}} \mu_i \left[ (1 - \delta) \log C_i(Z, \mu_i) + \delta \theta_i \log \sum_j \pi^j \exp \left\{ U''_i / \theta_i \right\} \right]
\]

subject to

\[
\min_{\mu'} Q^k(Z', \mu''') - \sum_{i \in \{h, f\}} \mu'' U'_i \geq 0
\]

\[
C_h + C_f \leq Z
\]
Allocations in Anderson’s economy

Special case: \( \theta_h \to -\infty, \theta_f \to -\infty \)

\[
C_{h,t} = \mu_h \cdot Z_t = \frac{S}{1+S} Z_t
\]

\[
C_{f,t} = \mu_f \cdot Z_t = \frac{1}{1+S} Z_t
\]

where

\[
S = \frac{\mu_h}{\mu_f}
\]
Allocations in Anderson’s economy

Risk Sensitive Preferences (general case)

\[ C_{h,t} = \mu_{h,t} \cdot Z_t = \frac{S_t}{1 + S_t} Z_t \]
\[ C_{f,t} = \mu_{f,t} \cdot Z_t = \frac{1}{1 + S_t} Z_t \]

where

\[ S_t = S_{t-1} \cdot \frac{\delta \exp \{ U_{h,t}/\theta \}}{E_{t-1} \exp \{ U_{h,t}/\theta \}} \bigg/ \frac{\delta \exp \{ U_{f,t}/\theta \}}{E_{t-1} \exp \{ U_{f,t}/\theta \}} \]
Dynamics and Stationarity

Questions

1. Does $S_t$ move over time?
2. Is the dynamics stationary?

Answers

1. It depends.
2. Typically not...

We shall see two examples of Anderson’s economy...
Example #1: identical risk-sensitive preferences

- Let $\theta_h = \theta_f = \theta$. 

Guess and verify that solution to expected utility control problem also solve recursive risk sharing problem.

With two identical risk-sensitive agents:

$S_t = S_0$, $\forall t$. 

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Example #2: one expected utility agent

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1. Show that $\mu_{h,t}$ is a supermartingale
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- With two non-identical risk sensitive agents: $S_\infty \rightarrow \{0, +\infty\}$. 
Pareto weight of Expected Utility Agent

Fig. 2. Evolution with logarithmic rewards. An example of the evolution of the Pareto weight on agent two when there are two agents who have logarithmic reward functions. The discount factor for both agents is $\gamma = 0.95$. Agent two has time-additive preferences so that $\gamma_2 = 0$ and agent one has risk-sensitive preferences with $\gamma_1 = -1$. The probabilities of the state are given by specification $B$. The initial Pareto weight on agent two is 0.5. The Pareto weight is graphed for $1 \times 10^7$ time periods. (On the graph one point is plotted for every 10,000 periods so that 1000 points are plotted. This masks the local variability in the Pareto weights.)

Consider an economy in which agents have distorted beliefs and time-additive preferences. Let agent $i$ believe that the time zero probability that the history $x_t$ will be realized at time $t$ is $M_{it}(x_t)$. These are well formed beliefs since $M_{it} \geq 0$ and for any $t$, $\sum x_t(M_{it}(x_t)) = 1$. The summation over $x_t$ indicates summation over all histories that can be realized at time $t$. We assume the agent takes these probabilities as being exogenous, even though they will depend upon the Pareto optimal allocation. Let agent $i$'s lifetime utility function be

$$\sum_{t=0}^{\infty} \left( \sum x_t \frac{u_i(x_t)}{\gamma_t} M_{it}(x_t) \right)$$

where $M_{it}$ is treated as exogenous.

- Only the Expected Utility Agent “survives”.

▶
Lessons from Anderson’s economy

- Risk Sharing Scheme with risk sensitive preferences in a one good economy features a tension between:
  1. non-trivial dynamics (i.e. consumption shares move around)
  2. non-degenerate equilibrium (i.e. all agents “survive” in the long-run)
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  1. heterogeneous δ’s?

Colacito and Croce: multiple goods and consumption home bias.
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- Colacito and Croce: multiple goods and consumption home bias.
Setup of Colacito and Croce economy

- Agents have risk-sensitive preferences

\[ U_{i,t} = (1 - \delta) \log C_{i,t} + \delta \theta \log E_t \exp \left\{ \frac{U_{i,t+1}}{\theta} \right\}, \quad \forall i \in \{h, f\} \]

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- Preferences are defined over the consumption aggregate

\[ C_{h,t} = (x_{h,t})^\alpha (y_{h,t})^{1-\alpha} \quad \text{and} \quad C_{f,t} = (x_{f,t})^{1-\alpha} (y_{f,t})^\alpha \]
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- Complete markets.
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- Consumption bias: \( \alpha > 1/2 \).

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- Endowments are i.i.d. homoscedastic

  - \( HL = \{X = 103, Y = 100\} \) and \( LH = \{X = 100, Y = 103\} \)
Allocations

- Find efficient allocations by solving Pareto problem

- Let $k = \frac{\alpha}{1-\alpha}$:

\[
x^h_t = \frac{kS_t}{1 + kS_t} X_t, \quad x^f_t = \frac{1}{1 + kS_t} X_t
\]

\[
y^h_t = \frac{S_t}{k + S_t} Y_t, \quad y^f_t = \frac{k}{k + S_t} Y_t
\]

where

\[
S_t = S_{t-1} \cdot \frac{\delta \exp \{U_{h,t}/\theta\}}{E_{t-1} \exp \{U_{h,t}/\theta\}} \Bigg/ \frac{\delta \exp \{U_{f,t}/\theta\}}{E_{t-1} \exp \{U_{f,t}/\theta\}}
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Questions

1. **Does** $S_t$ **move over time?**
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1. **Does** $S_t$ move over time?
2. **How** does $S_t$ move over time?
3. **Does** $S_t$ have a non-degenerate long-run distribution?
$S_t$ moves around over time

- **Key result**: the current share of consumption goes
  
  1. up in "bad times"
  2. down in "good times"

  This means that $S_t$ is decreasing in $X_t/Y_t$. 
$S_t$ moves around over time

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How does $S_t$ move?

![Graphs showing $\Delta \mu_h^{HL}$ and $\Delta \mu_h^{LH}$]
How does $S_t$ move?

$\rightarrow$ Abundant $X$, scarce $Y$: 
How does $S_t$ move?

→ Abundant $X$, scarce $Y$:

→ Good news for home
How does $S_t$ move?

→ **Abundant** $X$, scarce $Y$:  
→ Good news for home  
→ Home Pareto weight ↓
How does $S_t$ move?

$HL$ and $LH$ graphs showing the movement of $\Delta \mu_h$. "$\rightarrow$ Scarce $X$, abundant $Y$:"

- Bad news for home
- Home Pareto weight ↑
Does $S_t$ have a non-degenerate long-run distribution?
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1. Note that

$$E_t [S_{t+1}] = S_t - \frac{\text{cov}_t \left[ \exp \left\{ \frac{U_{t,t+1}}{\theta} \right\}, S_{t+1} \right]}{E_t \left[ \exp \left\{ \frac{U_{t,t+1}}{\theta} \right\} \right]}$$
Does $S_t$ have a non-degenerate long-run distribution?

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$$E_t[S_{t+1}] = S_t - \frac{\text{cov}_t \left[ \exp \left\{ \frac{U_{f,t+1}}{\theta} \right\}, S_{t+1} \right]}{E_t \left[ \exp \left\{ \frac{U_{f,t+1}}{\theta} \right\} \right]}$$

2. There exists a $S^*_t < 1$ such that

- $U_{f,t+1}^{HL}(S^*_t) = U_{f,t+1}^{LH}(S^*_t)$;
- $U_{f,t+1}^{HL}(S_t) < U_{f,t+1}^{LH}(S_t)$, if $S_t < S^*_t$;
- $U_{f,t+1}^{HL}(S_t) > U_{f,t+1}^{LH}(S_t)$, if $S_t > S^*_t$;
Does $S_t$ have a non-degenerate long-run distribution?

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2. There exists a $S_t^* < 1$ such that

- $U_{f,t+1}^{HL}(S_t^*) = U_{f,t+1}^{LH}(S_t^*)$;
- $U_{f,t+1}^{HL}(S_t) < U_{f,t+1}^{LH}(S_t)$, if $S_t < S_t^*$;
- $U_{f,t+1}^{HL}(S_t) > U_{f,t+1}^{LH}(S_t)$, if $S_t > S_t^*$;

3. This means that the covariance changes sign before and after $S_t^*$. Why?
Difference of continuation utilities

\[
\frac{U_{1,HL} - U_{1,LH}}{U_{2,LH} - U_{2,HL}}
\]
Where is $S_t$ going in the long-run?

- This means that

1. $E_t [S_{t+1}] \geq S_t$, if $S_t \leq S^*_t$;
Where is $S_t$ going in the long-run?

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2. $E_t [S_{t+1}] < S_t$, if $S_t > S^*_t$;

This prevents $\mu_h$, $t$ from converging to zero!
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- This prevents $\mu_{h,t}$ from converging to zero!
Pareto weights: expected growth and distribution

$E(\mu' - \mu)$

→ Mean Reversion
Pareto weights: expected growth and distribution

\[ \mathbb{E}(\mu' - \mu) \]

Expected Pareto weight change

\[ \pi(\mu) \]

Asymptotic Distribution

→ Mean Reversion

→ Symmetric Distribution
Why does $S_t$ move?
Why does $S_t$ move?

Reducing Expected Utility

Reducing Volatility

$E[U_{h,t+1}(s_{t+1}|s_t)]$

$\sigma[U_{h,t+1}(s_{t+1}|s_t)]$

$\gamma = 25$

$\gamma = 1$ (Time Additive Case − No Tradeoff)
Why does $S_t$ move?

Reducing Expected Utility

Reducing Volatility

$\gamma = 25$
Why does $S_t$ move?

Reducing Expected Utility

Reducing Volatility

Trade-off between Expected Utility and Utility Variance
Why does $S_t$ move?

\[
E[U_{h,t+1}(s_{t+1}|s_t)] \quad \sigma[U_{h,t+1}(s_{t+1}|s_t)]
\]

$\gamma = 1$ (Time Additive Case – No Tradeoff)

$\gamma = 25$
Conditional Volatilities

- X/Y
- $\mu_h$
- $V_t(U_{h,t+1})$

Periods
Conditional Volatilities

(periods)
Conditional Volatilities

- $X/Y$
- $\mu_h$
- $V_t(U_{h,t+1})$

Periods: 0 to 100
Utilities’ correlations

→ **Blue curves**: utilities when supply of good $X$ is high

→ **Red curves**: utilities when supply of good $Y$ is high
Utilities’ correlations

→ **Time additive** preferences:

→ Home utility is high (low) when foreign utility is low (high)
Risk-sensitive preferences:

Must take into account international redistribution of wealth
Continuation utilities including redistribution of wealth

\( U_{t+1} \) as a function of \( \mu_t \)
If wealth is similar:

→ Home utility is high (low) when foreign utility is low (high)
Utilities’ correlations

What if one country is more wealthy than the other?
Utilities’ correlations

→ If wealth is not similar:
  → Home utility is high (low) when foreign utility is high (low)
Correlation of utilities increases with wealth inequality
How good is the approximation of $E_t[\Delta s_{t+1}]$?
How good is the approximation of $E_t[\Delta s_{t+1}]$?
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How good is the approximation of $E_t[\Delta s_{t+1}]$?
How good is the approximation of $E_t[\Delta s_{t+1}]$?
How good is the approximation of $V_t[\Delta s_{t+1}]$?
## Introducing Rare Events

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Introducing Rare Events

Four equally likely no-disaster events

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Five equally likely disaster events
Stochastic Discount Factors

▶ International Stochastic Discount Factors:

\[
\log M_{i,t+1} = \log \delta + \log \frac{C_{i,t}}{C_{i,t+1}} + \log \frac{\exp \{ U_{i,t+1}/\theta \}}{E_t \exp \{ U_{i,t+1}/\theta \}}, \quad \forall i \in \{h, f\}
\]
Stochastic Discount Factors

- International Stochastic Discount Factors:

\[ \log M_{i,t+1} = \log \delta + \log \frac{C_{i,t}}{C_{i,t+1}} + \log \frac{\exp \{ U_{i,t+1}/\theta \}}{E_t \exp \{ U_{i,t+1}/\theta \}} , \quad \forall i \in \{ h, f \} \]

- Properties:

1. Volatility is high
2. Volatility is time-varying
3. Correlation is high
4. Correlation is time-varying
Stochastic Discount Factors

- **International Stochastic Discount Factors:**

\[
\log M_{i,t+1} = \log \delta + \log \frac{C_{i,t}}{C_{i,t+1}} + \log \frac{\exp \left\{ U_{i,t+1}/\theta \right\}}{E_t \exp \left\{ U_{i,t+1}/\theta \right\}}, \quad \forall i \in \{h, f\}
\]

- **Properties:**
  1. Volatility is high \(\Rightarrow\) Equity Sharpe ratios are high
  2. Volatility is time-varying \(\Rightarrow\) Equity risk-premia are time-varying
  3. Correlation is high \(\Rightarrow\) Volatility of FX growth is “low”
  4. Correlation is time-varying \(\Rightarrow\) Volatility of FX growth is time-varying
Conditional volatility of SDF

$\sigma_t(M_{h,t+1})/E_t(M_{h,t+1}) \rightarrow \text{Average Volatility} \approx 30\%$
Conditional volatility of SDF

\[
\sigma_t \left( \frac{M_{h,t+1}}{E_t(M_{h,t+1})} \right)
\]

→ Average Volatility ≈ 30%

→ Equity risk-premia

\[
E_t \left[ r_{h,t+1}^c - r_{h,t}^f \right] = -\rho_t \left( \Delta c_{h,t+1}, M_{h,t+1} \right) \sigma_t \left( \Delta c_{h,t+1} \right) \frac{\sigma_t(M_{h,t+1})}{E_t(M_{h,t+1})}
\]
Average Volatility $\approx 30\%$

Equity risk-premia

\[
E_t \left[ r_{h,t+1}^c - r_{h,t}^f \right] = -\rho_t (\Delta c_{h,t+1}, M_{h,t+1}) \sigma_t (\Delta c_{h,t+1}) \frac{\sigma_t (M_{h,t+1})}{E_t (M_{h,t+1})}
\]

are time varying and counter-cyclical.
Conditional Correlations

\[ \text{corr}(\Delta c_t^{f}, \Delta c_t^{h}) \]

\[ \mu \text{ corr}(m_{t+1}^{f}, m_{t+1}^{h}) \]
Conditional Correlations

→ Low, time-varying correlation of consumption
Conditional Correlations

- Low, time-varying correlation of consumption
- High, time-varying correlation of marginal utilities
Conditional volatility of FX growth

\[ \Delta e_{t+1} = m_{f,t+1} - m_{h,t+1} \]
Conditional volatility of FX growth

\[ V_t[\Delta e_{t+1}] = V_t[m_{f,t+1} - m_{h,t+1}] \]
Conditional volatility of FX growth

\[ V_t[\Delta e_{t+1}] = V_t[m_{f,t+1}] + V_t[m_{h,t+1}] - 2\rho_t \cdot \sqrt{V_t[m_{f,t+1}]} \cdot \sqrt{V_t[m_{h,t+1}]} \]
Conditional volatility of FX growth

\[ V_t[\Delta e_{t+1}] = V_t[m_{f,t+1}] + V_t[m_{h,t+1}] - 2\rho \cdot \sqrt{V_t[m_{f,t+1}]} \cdot \sqrt{V_t[m_{h,t+1}]} \]

\[ \sigma_t(\Delta e_{t+1}) \]

\[ \mu_h \]

\[ \rightarrow \text{ Average Volatility } \approx 14\% \]
Conditional volatility of FX growth

$$V_t[\Delta e_{t+1}] = V_t[m_{f,t+1}] + V_t[m_{h,t+1}] - 2\rho_t \cdot \sqrt{V_t[m_{f,t+1}]} \cdot \sqrt{V_t[m_{h,t+1}]}$$

→ Average Volatility ≈ 14%

→ Time-varying exchange rate volatility
Qualitative implications

1. Inverse relationship between
   ▶ Volatility of exchange rate
   ▶ Absolute level of savings
1. Inverse relationship between
   ▶ Volatility of exchange rate
   ▶ Absolute level of savings
Qualitative implications

1. Inverse relationship between
   ▶ Volatility of exchange rate
   ▶ Absolute level of savings

2. Positive relationship between
   ▶ Volatility of consumption
   ▶ Level of savings
Qualitative implications

1. Inverse relationship between
   - Volatility of exchange rate
   - Absolute level of savings

2. Positive relationship between
   - Volatility of consumption
   - Level of savings
Qualitative implications

1. Inverse relationship between
   - Volatility of exchange rate
   - Absolute level of savings

2. Positive relationship between
   - Volatility of consumption
   - Level of savings
Qualitative implications (cont’d)

3. Inverse relationship between
Qualitative implications (cont’d)

3. Inverse relationship between
   - Volatility of exchange rate

\[ \mu_h \sigma_t (\Delta e_{t+1}) \]

\[ corr_t(r_{h,t+1}, r_{f,t+1}) \]
Qualitative implications (cont’d)

3. Inverse relationship between
   ▶ Volatility of exchange rate
   ▶ Correlation of returns
3. Inverse relationship between
   - Volatility of exchange rate
   - Correlation of returns
Concluding remarks

A two-countries model with:

- complete markets
- two goods
- i.i.d. endowments
- risk-sensitive preferences

1. generates
   - dynamic risk-sharing scheme
   - endogenously time varying second moments

2. replicates a number of international finance facts

3. introduce frictions and investments