Exercise 3.2

let O_λ be open, $\lambda \in A$, A is any index set.
(Note that A could be finite, countable, or even not-countable)

Show $\bigcup_{\lambda \in A} O_\lambda$ is open.

I need to show for any $x \in \left(\bigcup_{\lambda \in A} O_\lambda\right)$, that
x is an interior point.

let $x \in \left(\bigcup_{\lambda} O_\lambda\right)$. then $x \in O_{\lambda'}$ for some λ'

since $O_{\lambda'}$ open by assumption, all of its points must be interior points, so in particular x is an interior point.

Q.E.D

(Quite Easily Done)

Let O_1, O_2 be open sets.

Show $O_1 \cap O_2$ open.

Again, I want to show for an arbitrary $x \in (O_1 \cap O_2)$, that x is an interior point

Let $x \in (O_1 \cap O_2)$. Then $x \in O_1$ and $x \in O_2$.

Since O_1 open, $\exists r' > 0$ s.t. $B(x, r') \subset O_1$

since O_2 open, $\exists r'' > 0$ s.t. $B(x, r'') \subset O_2$

choose $r = \frac{\min\{r', r''\}}{2}$

Note that $B(x, r) \subset O_1$ since $r < r'$

and that $B(x, r) \subset O_2$ since $r < r''$

so $B(x, r) \subset (O_1 \cap O_2)$, Hence

$\exists r > 0$ s.t. $B(x, r) \subset (O_1 \cap O_2)$

so x is an interior point.

* (Picture Not that helpful)

Exercise 3.3

$O_n = (-\frac{1}{n}, \frac{1}{n})$

show O_n open.

Key Idea: $(-\frac{1}{n}, \frac{1}{n})$ is a fixed interval for any particular n.

To show O_n open, I need to show every point is an interior point.

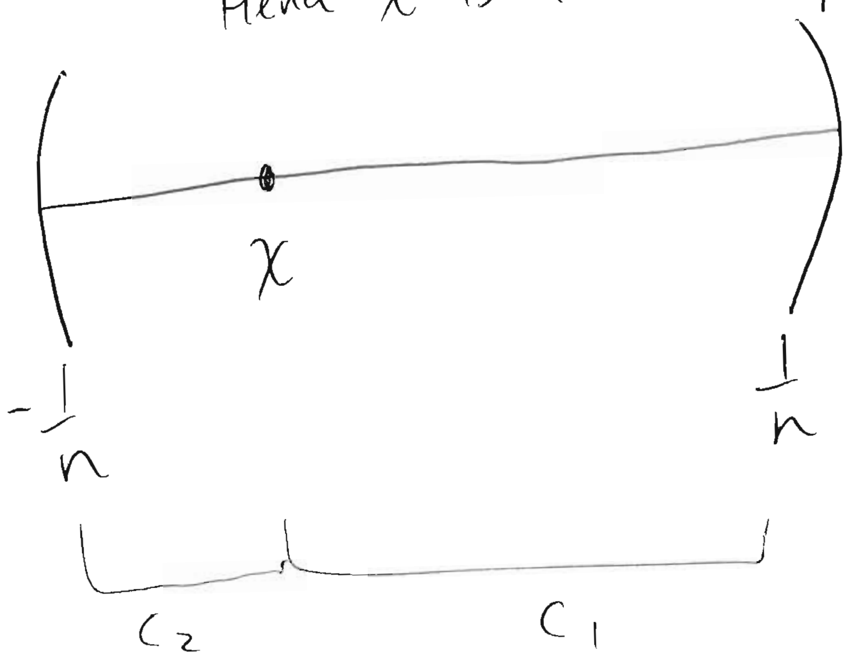
Take $x \in O_n$. (Now x fixed as well)

$$\left.\begin{aligned} d(x, \tfrac{1}{n}) &= c_1 \\ d(x, -\tfrac{1}{n}) &= c_2 \end{aligned}\right\} \text{constants}$$

choose $r < \min\{c_1, c_2\}$

then $\exists\, B(x, r) \in O_n$

Hence x is an interior point



~~$\bigcap_n O_n \ni \{0\}$ why true? clearly $-\frac{1}{n} < 0 < \frac{1}{n}$ $\forall n$ so $0 \in \bigcap_n O_n$ assume $\exists\, x \neq 0$ s.t. $x \in \bigcap_n O_n$.~~

Exercise 3.6

a) given $C \subset D$,

let $y \in \overline{C}$

· by definition $\forall r > 0$, $B(y, r) \cap C \neq \emptyset$

if $y \notin \overline{D}$ $\exists r^* > 0$ s.t.

$B(y, r^*) \cap D = \emptyset$

since $C \subset D \Rightarrow B(y, r^*) \cap C = \emptyset$

(Note I used if $A \subset B$ and $C \cap B$ is empty, then $C \cap A$ must be empty. the proof is easy if you want to prove it yourself. suppose that $C \cap B$ empty but $C \cap A$ non-empty. You will find a contradiction)

b) I will not prove this. TRy to show $\overline{C}$ is a closed set. (show $(\overline{C})^c$ is an open set might be one way to do it)

c) I will not prove this either. Sorry.

d) Since $\emptyset$ and X can also be regarded as closed, $\overline{\emptyset} = \emptyset$ and $\overline{X} = X$ by paragraph above question

Exercise 3.8

Show $F_n = [-1+\frac{1}{n}, 1-\frac{1}{n}]$ is closed in $\mathbb{R}$

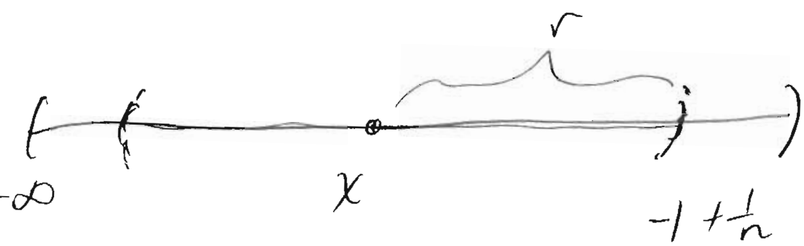
note $F_n^c = \underline{(-\infty, -1+\frac{1}{n})} \cup \underline{(1-\frac{1}{n}, \infty)}$

pick $x \in (-\infty, -1+\frac{1}{n})$. Note x, n fixed now,

so $d(x, -1+\frac{1}{n}) = \text{constant}$

choose $r < d(x, -1+\frac{1}{n})$

Then $B(x,r) = \{y \in \mathbb{R} : d(x,y) < r\} \subset (-\infty, -1+\frac{1}{n})$



Clearly the Ball is ok on left side (by ok I mean the boundary of the ball stops before $-\infty$) and we chose our r to make right side ok.

Hence $\exists\, r > 0$ s.t.

$B(x,r) \subset (-\infty, -1+\frac{1}{n})$ so x interior point, hence $(-\infty, -1+\frac{1}{n})$

Same trick for $(1-\frac{1}{n}, \infty)$. (TRY to do on your own if you get a chance)

so F_n^c = Union of two open sets, which we proved earlier is an open set (Union of open sets is open)

$\bigcup_n F_n = (-1,1)$

Exercise 3.9

let F_α be closed, $\alpha \in A$ index set.

Show $\bigcap_\alpha F_\alpha$ closed.

Note complement of $(\bigcap_\alpha F_\alpha)$ is

$$\left(\bigcap_\alpha F_\alpha\right)^c = \bigcup_\alpha (F_\alpha{}^c)$$ by De Morgan's Law

(Really helpful trick!)

since F_α closed, $F_\alpha{}^c$ open (compliments of closed sets are open)

and we know unions of open sets are open, so

$\bigcup_\alpha (F_\alpha{}^c)$ is open, so $\bigcap_\alpha F_\alpha$ must be closed

Exercise 3.1D

I will not prove this. Sorry. Argue by contradiction looks like the best way. If you really want to know this one, stop by my office and we can talk about it.

Inclusion is not always strict. use Hint

Exercise 3.11.

let $\varepsilon > 0$

assume $x_n \to x$, $x_n \to y$, $x \neq y$.

since $x_n \to x$, $\exists\, N' \in \mathbb{N}$ s.t. for $n > N'$ $d(x_n, x) < \frac{\varepsilon}{2}$

since $x_n \to y$, $\exists\, N'' \in \mathbb{N}$ s.t. for $n > N''$ $d(x_n, y) < \frac{\varepsilon}{2}$

$$d(x,y) \leq d(x, x_n) + d(x_n, y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Since ε can be any positive number, and the distance between x and y is always smaller, we can conclude it must be the case that

$d(x,y) = 0$, which means $x = y$ from our properties of metric spaces.

3.13

I will not prove theorem. See Rudin Principles of Mathematical Analysis pg. 48 for a similar proof. Note he uses "limit points" (a.k.a. cluster points) instead of contact points, so technically is not the same proof. I feel it is close enough to where you could figure it out if you really wanted to.

Exercise 3.14

Let X_n be a convergent subsequence. Show it is Cauchy.

By definition of convergence, $X_n \to X$

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. for all $n > N$

$d(X_n, X) < \varepsilon$

so we can always find an N' s.t. $d(X_n, X) < \frac{\varepsilon}{2}$ for $n, m > N'$

$$d(X_m, X_n) \le d(X_m, X) + d(X_n, X) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

So Cauchy by definition

Exercise 3.16

☆ See Rudin, Principles of Mathematical Analysis Pg. 53 for proof $\mathbb{R}^n$ complete ☆

Let X be discrete metric space.

Let X_n be Cauchy in X. (for $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $n, m > N$ $d(X_n, X_m) < \varepsilon$)

Since can take any $\varepsilon > 0$, take $\varepsilon = \frac{1}{2}$. Note distances can only be zero or one in this metric space.

so for $n, m > N'$, and $\varepsilon = \frac{1}{2}$ if $d(X_n, X_m) < \frac{1}{2}$, $d(X_n, X_m) = 0$

hence $X_n = X_m$ $\forall n, m > N'$. let X be the value of X_n for $n > N'$

the sequence must converge to X.

$d(X, X_n) = 0 < \varepsilon$, $\forall n > N'$, so convergent.