# Recitation Friday Oct. 24th

Announcements
- Problems from latest homework set from Mas-colell are easy to solve and recommend you do all of them. Very do-able, and can check your work afterwards. As always do not try to reverse engineer solutions
- Any questions want to go over from the homework?

- Now talking about Monotone comparative statics. How does a function change when one of it's variables change?

Ex: $f(x) = x^2$ if raise $X$ by $\varepsilon$, $(x+\varepsilon)$ how does $f$ change?
with a one variable setting it is really easy: can just plug into equation.
$f(x+\varepsilon) = (x+\varepsilon)^2 = x^2 + 2x\varepsilon + \varepsilon^2$ change is

Consider a harder problem: How will $f(x,y)$ change when increase $x$? changing $x$ might change choice of $y$ (In equilibrium or optimum solution)

ex: $f(x,y) = x^2y - y^2 - y$
maximize function
to solve for optimal value of $f$, take FOC

$\frac{\partial f}{\partial x}$: $2xy = 0 \Rightarrow x=0$ or $y=0$

$\frac{\partial f}{\partial y}$: $x^2 - 2y - 1 = 0$ $\quad x^2 = 2y+1$ $\quad x=0 \Rightarrow y = -\frac{1}{2}$ $\quad y=0 \Rightarrow x = \pm 1$

$f^*(0, -\frac{1}{2}) = \boxed{\frac{1}{4}}$

$f^*(\pm 1, 0) = 0$ if increase $x$ by $\varepsilon$, how does solution change?
$2(x+\varepsilon)y = 0$ $\quad x = -\varepsilon$ or $y = 0$

$x^2 = 2y+1 \Rightarrow (-\varepsilon, \frac{\varepsilon-1}{2})$ or $(\pm 1, 0)$

$f(-\varepsilon, \frac{\varepsilon-1}{2}) = \varepsilon^2\frac{(\varepsilon-1)}{2} - (\frac{\varepsilon-1}{2})^2 - \frac{(\varepsilon-1)}{2} = (\varepsilon-1)[\frac{\varepsilon^2}{2} - \frac{\varepsilon-1}{2} - \frac{1}{2}] = (\varepsilon-1)[\frac{1}{2}\varepsilon^2 - \frac{\varepsilon}{2}] = \varepsilon\frac{(\varepsilon-1)(\varepsilon-1)}{2} = \frac{(\varepsilon-1)^2}{2}\varepsilon$

$f(\pm 1, 0) = \boxed{0}$

Notice value of $y$ changes

Notice both y value changed + optimal value changed. Want a procedure or formula for coming up with the change of the overall function due to change in X only. To do this can calculate how x changes y, and then treat like one-dimensional case

$\boxed{\text{Need Implicit Function Thm}}$

$f: \mathbb{R}^{mn} \to \mathbb{R}^m$ cts. diff $f: (x_1, \ldots, x_n, y_1, \ldots, y_m) \to (f_1(x_1), \ldots, f_m(x_n))$

$\left[ (x,y) \to \underset{(y(x))}{y} \right]$ ← want to be able to come up with this function

if $f(a,b) = c$ and $\left[\frac{\partial f}{\partial y}\right]$ matrix is invertible, then $\exists$ open sets $U, V$ with $a \in U$ $b \in V$ and $\exists\, g: U \to V$ s.t. $f(x,y) = f(x, g(x))$ $\forall x \in U, \forall y \in V$

$$\left[\frac{\partial f}{\partial y}\right] = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_m} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial y_1} & \cdots & \frac{\partial f_m}{\partial y_m} \end{bmatrix}$$ invertible, non singular

Note: This theorem allows me to have any finite number of parameters I want $(x_1, \ldots, x_n)$ and as many finite number of endogenous variables as I want $(y_1, \ldots, y_m)$.

This is a very general framework.

ex $X =$ (hours worked, height, age) $y =$ (wage, weight, [illegible], car worth)

so this would allow me to not only tell me
in what direction $f(x,y)$ changes (increases or decreases) with $x$,
but it tells me exactly how much it changes by

So it gives me ordinal information (in which direction changes) and cardinal information (how much exactly is the change.)

Problem is that I need to assume differentiability of my objective function $f(x,y)$, and that it only tells me information for small changes in $x$ + (in small neighborhoods of $t_{a,b}$) as in previous example)

This leads to Increasing Differences.

- Tells how objective function changes with change in parameters
- Pro's: need only one assumption that is very weak and easier to satisfy than IFT. You do not need differentiability. Later we will talk about why this is important
- Drawbacks: only tells me direction of change, not amount of change (gives only ordinal information; no cardinal information)

Def: $u(x,y)$ has increasing differences in $(x,y)$ if given that $x_h > x_e$ + $y_h > y_e$,
then

$$u(x_h, y_h) - u(x_h, y_e) \geq u(x_e, y_h) - u(x_e, y_e)$$

called strict increasing differences if strict inequality above

$\boxed{\text{Def}}$

$$\phi(x) = \underset{y}{\text{argmax}}\ u(x,y)$$

$\phi(x)$ = "optimal $y$ in terms of $x$" - actually set of optimal $y$'s

Note: $\phi(x)$ could be multivalued (why a set)

$\phi(y)$ defined similarly: $\phi(y) = \underset{x}{\text{argmax}}\ u(x,y)$

$\boxed{\text{Thm}}$ if $u$ has increasing differences in $(x,y)$ and $y_h > y_\ell$, $x_h \in \phi(y_h)$ $x_\ell \in \phi(y_\ell)$

then either:

(1) $x_h \geq x_\ell$

or

(2) $x_h \in \phi(y_\ell)$ and $x_\ell \in \phi(y_h)$
(both $x$'s are max's for both $y$'s)

(if $u$ has strictly increasing differences in $(x,y)$ then only (1) holds.)

Proof

if $x_h \geq x_\ell$, then (1) is satisfied.
so we will assume $x_\ell > x_h$ and show (2) must be satisfied

if $x_\ell > x_h$, $y_h > y_\ell$ since $u$ has increasing differences

(*) $u(x_\ell, y_h) - u(x_\ell, y_\ell) \geq u(x_h, y_h) - u(x_h, y_\ell)$

Since $x_h \in \phi(y_h)$, $u(x_h, y_h) \geq u(x, y_h)$ $\forall x$
take $x = x_\ell$ and get:

$$u(x_h, y_h) \geq u(x_\ell, y_h)$$

similarly $u(x_\ell, y_\ell) \geq u(x_h, y_\ell)$

rewrite (*) as

$$\underbrace{u(x_\ell, y_h) - u(x_h, y_h)}_{\leq 0} \geq \underbrace{u(x_\ell, y_\ell) - u(x_h, y_\ell)}_{\geq 0}$$

so must be zero on both sides:

hence $u(x_\ell, y_h) = u(x_h, y_h)$ and hence $x_\ell \in \phi(y_\ell)$
also $u(x_h, y_\ell) = u(x_\ell, y_\ell)$ and $x_h \in \phi(y_h)$

so $x_\ell$ and $x_h$ are solutions to both problems

Q.E.D.

Example:
Now we will use this new math idea and do some economics.

"What happens to (optimal) profit if the quantity produced changes?"

(Can assume optimal profit since firms are always acting in best interest to max profits. Objective of firm is to max profits.)

$\pi(p,q) = p \cdot q - c(q)$

$c(q)$ is cost function. Do not know anything about it. Definitely cannot assume differentiability

Think about situations where output can only be a natural number
<u>Ex</u>: produce 57 cars. Cannot produce 57.3 cars.

Then cannot assume differentiability. Cannot define derivative on discrete set! (if f diff $\Rightarrow$ f cts, if f not cts $\Rightarrow$ f not diff. f not cts in q if $q \in \mathbb{N}$)
does $\pi(p,q)$ have increasing differences?

let $p_h > p_l$ and $q_h > q_l$

$$\pi(p_h, q_h) - \pi(p_l, q_h) \overset{?}{\geq} \pi(p_h, q_l) - \pi(p_l, q_l)$$

$$p_h q_h - p_l q_h \overset{?}{\geq} p_h q_l - p_l q_l$$

$$q_h(\cancel{p_h - p_l}) \overset{?}{\geq} q_l(\cancel{p_h - p_l})$$

true. even holds with strict inequality

so $\Pi(p, y)$ has strictly increasing differences.

Can apply earlier theorem and say
if $P_h \geq P_l$
and $q_h \in \phi(P_h)$ and $q_l \in \phi(P_l)$

then

so in optimum, (firms always maximizing profits)

if increase the price charged to consumer, then will want to at least as much, never less.

similarly from another perspective

if $p_h \in \phi(q_h)$ and $p_l \in \phi(q_l)$, $q_h > q_l$

then $P_h \geq p_l$.
so if produce more output, will want to charge a higher price to consumers.

Math Stuff Talked about in class

correspondence: $\Psi : \mathbb{R}^m \to \mathbb{R}^n$ (multivalued)

function $f : \mathbb{R}^m \to \mathbb{R}$ (single valued)

can compare single numbers, but how compare sets of numbers?
Then we can compare sets of optimums.

Define Strong Set order

for $S, T \in \mathbb{R}$,

$S \geq_s T$ if $x \in S, y \in T$ $[(x,y) \in S \times T]$,
and $y \geq x$ then $x \in T$ and $y \in S$ $[x, y \in S \cap T]$

$S \; ( \overset{x}{\cdot} \quad\quad )$

$T \; ( \quad\quad \underset{y}{\cdot} )$

ex: $(0,2) \geq_s (0,1)$

$(1,3) \geq_s (0,2)$
$(5,6) \geq_s (1,2)$

$\left\{ \begin{array}{l} x = 1\tfrac{1}{2} \\ y = 2\tfrac{1}{2} \end{array} \right.$ breaks definition

ask: $\boxed{\underset{S}{(1,4)} \; ?\geq_s \; \underset{T}{(2,3)}}$

No!!

$S \; ( \overset{x}{\cdot} \;\; \overset{}{1} \quad 2 \quad 3 \quad 4 )$

$T \; ( \underset{2}{} \; \underset{y}{\cdot} \; \underset{3}{} )$

Whole point of that is now can state:

Thm
if $u(x,y)$ has strictly increasing differences in $(x,y)$, and $x_h > x_l$

then $\phi(x_h) \geq_s \phi(x_l)$

Three Random other math definitions

1) Poset - Partially Ordered Set

$(X, \geq)$ where $X$ is an arbitrary set and $\geq$ is a Binary relation on $X$ that is
- reflexive: $a \geq a \quad \forall a \in X$
- transitive: $a \geq b$ and $b \geq c \Rightarrow a \geq c$
- antisymmetric: $a \geq b$ and $b \geq a \Rightarrow a = b$

Join/supremum "Least Upper bound"
$a \vee_X b = \inf \{ c \in X : c \geq a, c \geq b \}$

Meet/infimum "Greatest Lower Bound"
$a \wedge_X b = \sup \{ c \in X : a \geq c, b \geq c \}$

in $\mathbb{R}$ $a \vee_{\mathbb{R}} b = \max \{a, b\}$

$a \wedge_{\mathbb{R}} b = \min \{a, b\}$

In General, Join and meet may not exist
If they always exist, we have a lattice

a Poset is called a <u>lattice</u> if

$a \vee_X b \in X$ and $a \wedge_X b \in X$