Recitation:

$\boxed{\text{Oct } 10^{th}}$

- Recall $\succsim$ was defined for individual elements of my choice set.

  So for $x, y \in X$ we would write $x \succsim y$ to denote "x is at least as preferred as y"

- now we have another preference, called the induced preference, $\succsim^*$
  It is defined for sets of elements (Subsets of X)
  we Say one set is preferred to another if the "best element" of the set is better than the "best element" of the other set.

Formally:

for $A, B \subset X$

$$A \succsim^* B$$

iff

$$c(A, \succsim) \succsim c(B, \succsim)$$

this is defined since we are assuming $c(A, \succsim)$ and $c(B, \succsim)$ each to be an individual element (a unique maximizer)

Indirect Preferences ($\succsim$ Indirect Utility)

define $V(A) = u(C(A, \succsim))$,

this says for any set A, $V(A)$ is the utility of the most preferred element/bundle, or in other words, $V(A)$ is the maximum utility you can get from something in A.

In terms of our Budget sets, $B(p,w)$, (which by the way Dr. Krishna also denotes (p,w)),

we say

$$V(p,w) = u(X(p,w))$$

this is exactly the same idea as above.

If I am given a Budget set (I am given prices and wealth), then $V(p,w)$ gives me the maximum utility I can obtain from that Budget Set.

Overall Idea: Indirect Utility = Maximum Utility

Duality

In Class, we had 2 differnt problems

1) Primal - our main problem to solve; usually the Consumer's Problem: Finding the bundle/demand/element that gives me the most utility, but which must be affordable

$$\max_x u(x) \quad s.t. \quad x \in B(p,w)$$

$$= \max_x u(x) \quad st. \quad \langle p, x \rangle \leq w$$

$$= \max_x u(x) \quad st. \quad \langle p, x \rangle = w \quad \text{from WALRAS'S Law}$$

$$= \max_x u(x) \quad st. \quad p_1 x_1 + p_2 x_2 + \cdots + p_n x_n = w$$

so I am looking for the best affordable bundle

The solution is called walrasian demand, or sometimes Marshallian demand, and is denoted $x(p,w)$

(Note: It is always in terms of prices and wealth only)

if I plug my optimal bundle $x(p,w)$ into the utility function, I get my maximum utility possible, called the indirect utility

$$V(p,w) = u(x(p,w))$$

2) Dual

This is a "cost minimization" subject to a minimal utility problem. In other words I want to find the cheapest bundle that allows me to attain some minimum utility level u.

Formally

$$\min_{x} \langle p, x \rangle \quad \text{s.t.} \quad u(x) \geq u$$

$$\min_{x} p_1x_1 + p_2x_2 + \cdots + p_nx_n \quad \text{s.t.} \quad u(x) \geq u$$

since $\langle p, x \rangle$ increases with x and I want to minimize it, and since also $u(x)$ typically increases with x, the inequality above will always hold with equality. So I can rewrite it as

$$\min_{x} p_1x_1 + \cdots + p_nx_n \quad \text{s.t.} \quad u(x) = u$$

The optimal bundle is called the Hicksian demand, and is denoted $h(p, u)$

Note: Hicksian demand is in terms of prices and minimum utility level u; you should never have wealth in your answer!

The whole point of duality is that solving the Primal Problem is essentially the same as solving the Dual Problem. If I know the solution for one of the problems, I can very easily find the other using the formulas on pg. 75 of MWG

ex:

$x(p,w) = h(p, v(p,w))$

~~[illegible]~~ $h(p,v) = x(p, e(p,u))$

I would know how to solve these types of problems. They are all simple Calculus Problems, But I would make sure I know how to do them very easily and quickly.

I Recommend problems in MWG:

3.D.1, 3.D.2, 3.D.5*, 3.D.6, 3.E.2*, 3.E.5, 3.E.6, 3.E.8*

* = longer problem

if I plug my hicksian demand into my objective func $\langle p, x \rangle$, I get the "minimum cost" necessary to achieve utility u, or in other words $h(p,u)$ is the cheapest way to achieve utility u. This "minimum cost" is called the expenditure function, and is denote

$$e(p,u)$$

where $e(p,u) = \langle p, h(p,u) \rangle$

$$e(p,u) = p_1 \cdot h_1(p,u) + p_2 \cdot h_2(p,u) + \cdots + p_n h_n(p,u)$$

so clearly $\frac{\partial e(p,u)}{\partial p_i} = h_i(p,u)$

Another helpful identity to know deals with the original (Primal) problem. It is called <u>Roy's Identity</u>.

$$\underbrace{x_\ell(p,w)}_{\text{the } \ell^{th} \text{ term in my bundle}} = - \frac{\frac{\partial v(p,w)}{\partial p_\ell}}{\frac{\partial v(p,w)}{\partial w}}$$

$1 \leq \cdots \ell \cdots \leq n$

Page 75 in MWG gives all the formulas for Primal + Dual Problems

Duality example

find $x(p,w)$ for $u(x) = \sqrt{x_1} + \sqrt{x_2}$

$$\mathcal{L}: \sqrt{x_1} + \sqrt{x_2} + \lambda(w - p_1x_1 - p_2x_2)$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{1}{2\sqrt{x_1}} - \lambda p_1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{1}{2\sqrt{x_2}} - \lambda p_2 = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = w - p_1x_1 - p_2x_2 = 0$$

Solving for λ, equating x_1 in terms of x_2, and plugging into Budget constraint yields
$\Rightarrow$

$$x_1 = \frac{wp_2}{p_2^2 + p_1p_2} \qquad x_2 = \frac{wp_1}{p_1^2 + p_1p_2}$$

so $$x(p,w) = x(p_1,p_2,w) = \underbrace{\left(\frac{wp_2}{p_2^2 + p_1p_2}, \frac{wp_1}{p_1^2 + p_1p_2}\right)}$$

notice only in terms of prices and wealth!

$$v(p,w) = u(x(p,w)) = \sqrt{\frac{wp_2}{p_2^2 + p_1p_2}} + \sqrt{\frac{wp_1}{p_1^2 + p_1p_2}}$$

Now we will solve dual problem:

$$\min_x \langle p, x\rangle \quad \text{s.t.} \quad u(x) \geq u$$

$$\min_x{}' \; p_1 x_1 + p_2 x_2 \quad \text{s.t.} \quad \sqrt{x_1} + \sqrt{x_2} = u$$

$$\mathcal{L}: \; p_1 x_1 + p_2 x_2 + \lambda(u - \sqrt{x_1} - \sqrt{x_2})$$

$$\left.\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} &= p_1 - \frac{\lambda}{2\sqrt{x_1}} = 0 \\ \frac{\partial \mathcal{L}}{\partial x_2} &= p_2 - \frac{\lambda}{2\sqrt{x_2}} = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= u - \sqrt{x_1} - \sqrt{x_2} = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x_1 &= \frac{p_2 u^2}{p_1 + p_2} \\ x_2 &= \frac{p_1 u^2}{p_1 + p_2} \end{aligned}$$

so $$h(p, u) = \underbrace{\left(\frac{p_2 \cdot u^2}{p_1 + p_2}, \; \frac{p_1 \cdot u^2}{p_1 + p_2} \right)}$$

Notice only in terms of prices and the minimal utility

$$e(p, u) = \langle p, h(p, u)\rangle = \frac{p_1 p_2 u^2}{p_1 + p_2} + \frac{p_1 p_2 u^2}{p_1 + p_2} = \frac{2 p_1 p_2 u^2}{p_1 + p_2}$$

Test 2 Question $\underline{IV}$

$X = \mathbb{R}^2_+$, $\succsim$ is a preference relation on X.

When is $\succsim$ homothetic?

$\succsim$ homothetic if $x \succsim y$ then $\alpha x \succsim \alpha y$ $\forall \alpha > 0$

When is $\succsim$ strongly convex?

$\succsim$ s-convex if $x \succsim z$, $y \succsim z$, $x \neq y$ then $\lambda x + (1-\lambda) y \succ z$
$\lambda \in (0,1)$

Is the lexicographic order homothetic? Yes!

let $x \succsim y$

then either

① $x_1 = y_1$
$x_2 \geq y_2$

or ② $x_1 > y_1$

Case ①

$\alpha x = (\alpha x_1, \alpha x_2)$
$\alpha y = (\alpha y_1, \alpha y_2)$

$x_1 = y_1 \iff \alpha x_1 = \alpha y_1$
$x_2 \geq y_2 \iff \alpha x_2 \geq \alpha y_2$ since $\alpha > 0$

Hence $\alpha x \succsim \alpha y$

Case ②

$x_1 > y_1 \iff \alpha x_1 > \alpha y_1$ since $\alpha > 0$

hence $\alpha x \succ \alpha y \Rightarrow \alpha x \succsim \alpha y$

Is lexicographic strongly convex?

let $x \succeq z$, $y \succeq z$ $x \neq y$ $\lambda \in (0,1)$

~~Either~~

we have different cases

① $x_1 = y_1 = z_1$

$\underbrace{x_2 \geq z_2 \quad y_2 \geq z_2}$

both of these cannot hold with equality, since that would imply $x = y$, so

WLOG assume $x_2 > z_2$ $y_2 \geq z_2$

since $\lambda > 0$, $(1-\lambda) > 0$

$\Rightarrow$ $\lambda x_2 > \lambda z_2$ $\lambda x_1 = \lambda z_1$

$(1-\lambda) y_2 \geq (1-\lambda) z_2$ $(1-\lambda) y_1 = (1-\lambda) z_1$

adding up inequalities yields

$\lambda x_1 + (1-\lambda) y_1 = \lambda z_1 + (1-\lambda) z_1 = z_1$

$\lambda x_2 + (1-\lambda) y_2 > \lambda z_2 + (1-\lambda) z_2 = z_2$

hence $\lambda x + (1-\lambda) y \succ z$

② $\underbrace{x_1 \geq z_1 \quad y_1 \geq z_1}$

for same argument one must hold with strict inequality, WLOG let $x_1 > z_1$
$y_1 \geq z_1$

then since $\lambda > 0$, $(1-\lambda) > 0$

$\Rightarrow$ $\lambda x_1 > \lambda z_1$ and $(1-\lambda) y_1 \geq (1-\lambda) z_1$

adding up inequalities we have

$\lambda x_1 + (1-\lambda) y_1 > \lambda z_1 + (1-\lambda) z_1 = z_1$

hence $\lambda x + (1-\lambda) y \succ z$

I have used the fact that if $a > b$, $c \geq d$, then $a + c > b + d$.