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Characterizing Concave Functions

Recall the following definitions from previous handouts. A set $X \subset \mathbb{R}^n$ is *convex* if $x, y \in X$ implies $\lambda x + (1-\lambda)\, y \in X$ for all $\lambda \in (0,1)$.

A function $f : X \to \mathbb{R}$ is *(strictly) concave* if, for all $x \neq y$,

$$f(\lambda y + (1-\lambda)\, x) \geq (>) \lambda f(y) + (1-\lambda)\, f(x)$$

for all $\lambda \in (0,1)$. In these notes, we characterize a concave function in terms of its first and second derivatives.

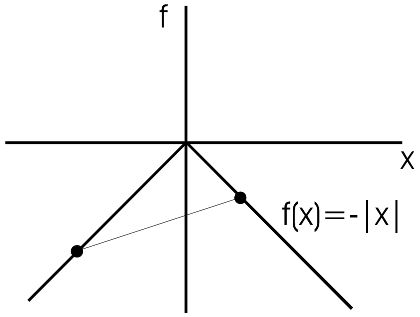
Characterizations of Univariate Concave Functions

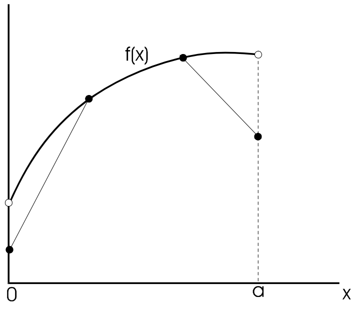
In this section we characterize concave functions of a single variable in terms of their first and second derivatives. In this case the domain of a concave function is a convex subset of $\mathbb{R}$, which is just an interval I of $\mathbb{R}$.

Continuity and Differentiablity of Concave Functions

A concave function need not be differentiable everywhere. For example $f(x) = -|x|$ is a concave function that is not differentiable at 0. However, right and left hand derivatives must exist on the interior of I.

- We prove in the Appendix that right and left hand derivatives always exist on the interior of the domain and that $f^-(x) \geq f^+(x)$. For example, if $f(x) = -|x|$, then $f^-(0) = 1$ and $f^+(0) = -1$.
- Since both righthand and lefthand derivatives exist on the interior, it follows from our earlier results on differentiable functions that f is both right and left continuous and therefore continuous. However, concave functions need not be continuous at the boundary as illustrated in the right hand example below.





The First Derivative of Concave Functions

We now turn attention to functions that are differentiable.

Theorem 1: Suppose $f : I \to \mathbb{R}$ is differentiable. (a) f is concave if and only if for all $x, y \in I$ with $x \neq y$, we have

$$f(y) - f(x) \leq f'(x) \cdot (y - x)\,. \tag{1}$$

(b) f is strictly concave if and only if the inequality is always strict.

Proof. We will prove Part (a). Part (b) is proved in the Appendix.

(only if) Suppose f is concave. Then for all $\lambda \in (0, 1)$, we have

$$f(\lambda y + (1 - \lambda)\,x) \geq \lambda f(y) + (1 - \lambda)\,f(x)$$

which is equivalent to

$$f(\lambda\,(y - x) + x) \geq \lambda\,(f(y) - f(x)) + f(x).$$

Then rearranging terms and dividing by λ,we obtain

$$\frac{f(\lambda\,(y - x) + x) - f(x)}{\lambda} \geq f(y) - f(x).$$

Now define $\Delta x \equiv \lambda(y - x)$. Then the relation above may be written as

$$\frac{f(\Delta x + x) - f(x)}{\Delta x}\,(y - x) \geq f(y) - f(x) \quad \text{for all } \Delta x \in (0, y - x)\,.$$

Then letting $\Delta x \to 0$, we have

$$f'(x)\,(y - x) = \lim_{\Delta x \downarrow 0} \frac{f(\Delta x + x) - f(x)}{\Delta x}\,(y - x) \geq f(y) - f(x).$$

(if) Let $x, y \in I$ and suppose $x \neq y$. Now consider any $\lambda \in [0, 1]$. Let $z = \lambda y + (1 - \lambda)\,x = \lambda\,(y - x) + x$. We need to show that relation (1) implies

$$f(z) = f(\lambda y + (1 - \lambda)\,x) \geq \lambda f(y) + (1 - \lambda)\,f(x)$$

Letting z play the role of x in relation (1), we have

$$\begin{aligned} f(x) - f(z) &\leq f'(z)\,(x - z) \\ f(y) - f(z) &\leq f'(z)\,(y - z) \end{aligned}$$

Observe that $z - x = \lambda(y - x)$ and $y - z = (1 - \lambda)\,(y - x)$. So if we multiply the top relation by $1 - \lambda$ and the bottom equation by λ, and substitute for $z - x$ and $y - z$, we have

$$\begin{aligned} (1 - \lambda)\,(f(x) - f(z)) &\leq -\lambda\,(1 - \lambda)\,f'(z)\,(y - x) \\ \lambda\,(f(y) - f(z)) &\leq \lambda\,(1 - \lambda)\,f'(z)\,(y - x) \end{aligned}$$

Adding the two relations and rearranging terms then yields

$$(1 - \lambda)\,f(x) + \lambda f(y) \leq f(z).$$

■

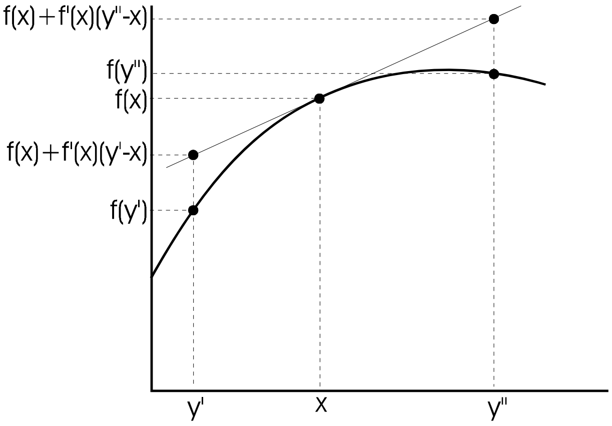
Corollary 1: Suppose $f : I \to \mathbb{R}$ is differentiable. (a) f is concave if and only if for all $x, y \in I$ with $x \neq y$, we have

$$f(y) - f(x) \geq f'(y)\,(y - x)\,.$$

(b) f is strictly concave if and only if the inequality is always strict.

Proof. Reversing the roles of x and y, relation (1) is equivalent to $f(x) - f(y) \leq f'(y)\,(x - y)$. Multiplying each side by -1 then yields the desired result. ■

Theorem1 is illustrated below.



Notice that the inequality holds both for the case where $y > x$ and $y < x$.

- We show in the Appendix that even if a function is not differentiable everywhere, it is concave if and only if for each $x \in int(I)$, there is an $a \in \mathbb{R}$ such that $f(y) - f(x) \leq a(y - x)$ for all $y \in I$. This is an example of a *supporting hyperplane* (for one dimension) that is an important tool in economic analysis.

The Second Derivative of a Concave Function

Theorem 2: Suppose $f : I \to \mathbb{R}$ is twice continuously differentiable. (a) f is concave if and only if $f'' \leq 0$. (b) If $f'' < 0$, then f is strictly concave.

Proof. We prove first that f concave implies $f'' \leq 0$. Consider any $x, y \in I$, with $x \neq y$. Then Theorem 1 and Corollary 1 imply that

$$\begin{aligned} f'(y)(y - x) &\leq f(y) - f(x) \\ f'(x)\,(y - x) &\geq f(y) - f(x). \end{aligned}$$

Subtracting the second equation from the first then yields

$$\big(f'(y) - f'(x)\big)\,(y - x) \leq 0.$$

Dividing by $(y - x)^2$ then yields

$$\frac{f'(y) - f'(x)}{y - x} \leq 0.$$

Letting $y \to x$, we then obtain

$$f''(x) = \lim_{y \to x} \frac{f'(y) - f'(x)}{y - x} \leq 0.$$

We show next that $f'' \leq 0$ implies relation (1) and therefore that f is concave. Suppose $y > x$. By the mean value theorem, there is a $z \in (x, y)$ such that

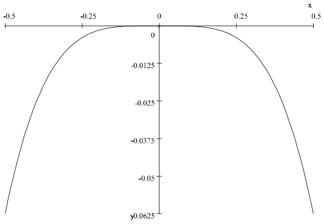
$$f'(z)\,(y - x) = f(y) - f(x).$$

But $f'' \leq 0$ and $x < z$ implies $f'(x) \geq f'(z)$. Therefore

$$f'(x)\,(y - x) \geq f(y) - f(x).$$

To prove part (b) observe that if $f'' < 0$, then the inequality is strict and the theorem follows from Theorem 1(b). ■

Note that f strictly concave does not imply that $f''(x) < 0$ for **all** x. For instance $f(x) = -x^4$ is a strictly concave function. However $f''(x) = -12x^2$ which implies that $f''(0) = 0$.



Some important concave functions.

Theorem 2 implies that the following functions $f : \mathbb{R}_{++} \to \mathbb{R}$ are concave:

i. $f(x) = \frac{x^\alpha}{\alpha}$ for $\alpha \neq 0$, $\alpha < 1$.

ii. $f(x) = \log x$.

Characterizations of Multivariate Concave Functions

We may extend Theorems 1 and 2 to multivariate functions by adopting the same approach we used for deriving the multi-variate first and second order conditions for a maximum.

For the remainder of this handout, we suppose that $X \subset \mathbb{R}^n$ is a convex set and f is a function $f : X \to \mathbb{R}$. For any $x, y \in X$, we define $h_{xy} : [0, 1] \to \mathbb{R}$ by

$$h_{xy}(\lambda) \equiv f(\lambda(y - x) + x) = f(\lambda y + (1 - \lambda)\,x).$$

The proofs that follow exploit the following relation between h_{xy} and f, which is proved in the Appendix.

Lemma 1: $f : X \to \mathbb{R}$ is (strictly) concave if and only if each h_{xy} is (strictly) concave.

In an earlier lecture, we also established the following relations between the first and second derivatives of f and each h_{xy}.

Lemma 01: (a) If f is continuously differentiable, then for any $\lambda \in [0, 1]$, and $z = \lambda y + (1 - \lambda)\,x$

$$h'_{xy}(\lambda) = Df(z) \cdot (y - x)$$

(b) If f is twice continuously differentiable, then

$$h''_{xy}(\lambda) = (y-x)^T \, D^2 f(z) \cdot (y-x).$$

The extension of Theorem 1 to multivariate functions may be stated as:

Theorem 3: Suppose f is continuously differentiable. (a) f is concave if and only if for all $x, y \in X$ with $x \neq y$, we have

$$f(y) - f(x) \leq Df(x) \cdot (y-x).$$

(b) f is strictly concave if and only if the inequality is always strict.

Proof. (only if) Suppose f is concave. Then for any $x \neq y$, Lemma 01 implies that h_{xy} is concave. Therefore

$$\begin{aligned} f(y) - f(x) &= h_{xy}(1) - h_{xy}(0) && \text{(by definition of } h_{xy}) \\ &\leq h'_{xy}(0) \cdot 1 && \text{(by Theorem 1)} \\ &= Df(x) \cdot (y-x) && \text{(by Lemma 01(a))} \end{aligned}$$

with the inequality strict if h_{xy} is strictly concave.

(if) Suppose $f(w) - f(z) \leq Df(z) \cdot (w-z)$ for all $z, w \in X$. Choose any $x, y \in X$ and consider any $\lambda, \mu \in (0,1)$. Then for $w = \lambda(y-x) + x$ and $z = \mu(y-x) + x$, we have

$$\begin{aligned} h_{xy}(\lambda) - h_{xy}(\mu) &= f(w) - f(z) \leq Df(z) \cdot (w-z) \\ &= Df(z) \cdot (y-x)(\lambda - \mu) && \text{(since } w - z = (\lambda-\mu)(y-x)) \\ &= h'_{xy}(\mu)(\lambda - \mu) && \text{(by Lemma 01(a))} \end{aligned}$$

implies that h_{xy} is concave. But if h_{xy} is concave for all $x, y \in X$, then Lemma 1 implies that f is concave. If the inequality is strict, then Theorem 1(b) implies that each h_{xy} is strictly concave, in which case Lemma 1 implies that f is strictly concave. ■

[Aside: The straight line in the figure illustrating Theorem 1 represents the graph of $h : \mathbb{R} \to \mathbb{R}$ defined by $h(z) \equiv f(x) + f'(x)(z-x)$. For a function f of two variables, the corresponding graph would be a plane embedded in $\mathbb{R}^3$. For a function of n variables, it would correspond to an n-dimensional plane *hyperplane* in $\mathbb{R}^{n+1}$ defined by the graph of $h(z) \equiv f(x) + Df(x) \cdot (z-x)$. In this case, it is called a *supporting hyperplane* because f lies on or below the hyperplane at all points and lies on the hyperplane at at least one point.]

The extension of Theorem 2 to multivariate functions may be stated as:

Theorem 4: Suppose f is twice continuously differentiable. Then (a) f is concave if and only if for all $x, y \in X$, we have

$$(y-x)^T \cdot D^2 f(x) \cdot (y-x) \leq 0. \tag{2}$$

(b) If the inequality is strict for any $x \neq y$, then f is strictly concave.

Proof. (only if) Suppose f is concave. Then for any $x, y \in X$, Lemma 1 implies that h_{xy} is concave. It then follows from Theorem 02 and Lemma 01(b) that

$$(y-x)^T \cdot D^2 f(x) \cdot (y-x) = h''_{xy}(x) \leq 0.$$

(if) Suppose for all $z, y \in X$, we have

$$(y-z)^T \cdot D^2 f(z) \cdot (y-z) \leq 0.$$

Fix $x, y \in X$ and consider any $\lambda \in [0,1]$. If $\lambda = 1$, then Lemma 01(b) implies

$$h''_{xy}(1) = (y-x)^T \cdot D^2 f(y) \cdot (y-x) = (x-y)^T \cdot D^2 f(y) \cdot (x-y) \leq 0.$$

If $\lambda < 1$, let $z \equiv \lambda y + (1-\lambda)\, x$. In this case, we have $y - z = (1-\lambda)\,(y-x)$ which implies

$$h''_{xy}(\lambda) = (y-x)^T \cdot D^2 f(z) \cdot (y-x) = (1-\lambda)^{-2} \left((y-z)^T \cdot D^2 f(z) \cdot (y-z) \right) \leq 0. \tag{3}$$

It then follows from Theorem 02 that each h_{xy} is concave and therefore from Lemma 1 that f is concave.

(b) If inequality (2) is strict, then inequality (3) is strict, in which case Lemma 1 and Theorem 2 imply that f is strictly concave. ■

Sufficient Conditions for a Maximum

We may use Theorem 3 to provide sufficient FOC for a global maximum.

Theorem 5: Suppose $f : \mathbb{R}^n_+ \to \mathbb{R}$ is concave. Then x maximizes f if and only if

$$Df(x) \leq 0,\ x \geq 0,\ \text{and } Df(x) \cdot x = 0$$

Proof. We have shown the necessity of these conditions in earlier notes. To show that these conditions are sufficient for concave functions, observe that f concave implies that for any $z \in \mathbb{R}^n_+$, we have

$$\begin{aligned} f(z) &\leq Df(x)\cdot(z-x) + f(x) \quad \text{(by Theorem 3)} \\ &= Df(x)\cdot z + f(x) \quad \text{(since } Df(x)\cdot x = 0\text{)} \\ &\leq f(x) \quad \text{(since } Df(x) \leq 0 \text{ and } z \geq 0 \text{ implies } Df(x)\cdot z \leq 0\text{)} \end{aligned}$$

■

- In earlier notes we also proved that if f is strictly concave it has a unique maximizer. Therefore if f is strictly concave, once we have found a solution to the FOC of Theorem 3, we have found all of the maximizers of f.

Convex Functions

Recall that $f : X \to \mathbb{R}$ is a *convex* function if for any $x, y \in X$, we have, for all $\lambda \in (0, 1)$,

$$f(\lambda x + (1 - \lambda) y) \leq \lambda f (x) + (1 - \lambda) f(y)$$

$f : X \to \mathbb{R}$ is a *strictly convex* function if for any $x, y \in X$, we have, for all $\lambda \in (0, 1)$,

$$f(\lambda x + (1 - \lambda) y) < \lambda f (x) + (1 - \lambda) f(y)$$

As we observed in the notes on concave functions, f is a (strictly) convex function if and only if $-f$ is a (strictly) concave function. Consequently,

- If $f : X \to \mathbb{R}$ is convex and differentiable, then $f(z) - f(x) \geq Df(x) (z - x)$ for all $x, z \in X$.
- If f is twice differentiable, then f is convex if and only if $D^2 f(x)$ is positive semi-definite for all $x \in X$.
- If $D^2 f(x)$ is positive definite for all $x \in X$, then f is strictly convex.
- If f is (strictly) convex then $\alpha f + \beta$ with $\alpha > 0$ is (strictly) convex.

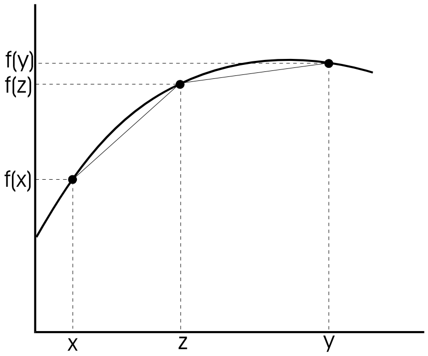
Appendix

For the proof of the results below, we will use the following characterization of a univariate concave function. As in the text, we suppose that I is an interval of $\mathbb{R}$.

Lemma A1: Suppose $f : I \to \mathbb{R}$. Then f is concave if and only if for any $x < z < y$, we have

$$\frac{f(y)-f(z)}{y-z} \leq \frac{f(y)-f(x)}{y-x} \leq \frac{f(z)-f(x)}{z-x}.$$

f is strictly concave if and only if the inequalities are strict.



Proof. Choose any $x < z < y$ and define $\lambda = \frac{z-x}{y-x}$. Then $(1-\lambda) = \frac{y-z}{y-x}$ and $z = \lambda y + (1-\lambda)\,x$. Then

$$f(z) \geq \lambda f(y) + (1-\lambda)\,f(x)$$

is equivalent to

$$(1-\lambda)\,(f(z) - f(x)) \leq \lambda\,(f(y) - f(z))$$

Substituting for λ yields and multiplying by $(y-x)$ yields the equivalent statement.

$$(y-z)\,(f(z) - f(x)) \geq (z-x)\,(f(y) - f(z))$$

Adding $(z-x)\,(f(z) - f(x))$ to each side and simplying terms we obtain the equivalent statement

$$(y-x)\,(f(z) - f(x)) \geq (z-x)\,(f(y) - f(x))$$

which is equivalent to

$$\frac{f(z)-f(x)}{z-x} \geq \frac{f(y)-f(x)}{y-x}$$

Similarly, adding $(y-z)\,(f(y) - f(z))$ to each side yields, we obtain the equivalent statement

$$(y-z)\,(f(y) - f(x)) \geq (y-x)\,(f(y) - f(x))$$

and therefore the equivalent statement.

$$\frac{f(y)-f(x)}{y-x} \geq \frac{f(y)-f(z)}{y-z}.$$

The equivalence of each statement also holds if each inequality is strict.

Since f is concave if and only if the first statement is true for all $x < z < y$, the theorem is proved. ■

Lemma A2: Suppose $f : I \to \mathbb{R}$ is concave. Then $x \in int(I)$ implies $f^{-}(x)$ and $f^{+}(x)$ exist and $f^{-}(x) \geq f^{+}(x)$.

Proof. Note first that Lemma A1 implies that $\frac{f(y)-f(x)}{y-x}$ is nonincreasing in x and y for $x \neq y$. Therefore for all $x < z < y$, we have

$$f^{+}(z) \equiv \lim_{y \downarrow z} \frac{f(y)-f(z)}{y-z} \leq \frac{f(z)-f(x)}{z-x}.$$

and

$$f^{+}(z) \leq \lim_{x \uparrow z} \frac{f(z)-f(x)}{y-x} \equiv f^{-}(z)$$

■

Theorem A1: (a) $f : I \to \mathbb{R}$ is concave if and only if for all $z \in int(I)$ and all $x, y \in I$ with $x < z < y$, we have

$$\begin{aligned} f(z)-f(x) &\geq f^{-}(z)\,(z-x) \\ f(y)-f(z) &\leq f^{+}(z)\,(y-z)\,. \end{aligned}$$

(b) f is strictly concave if and only if the inequalities are strict for $x, y \neq z$.

Proof. (only if) Suppose f is concave. The statement is trivial if $y = x$, so suppose that $y \neq x$. Then Lemma A1 implies

$$f^{-}(y) \equiv \lim_{z \uparrow y} \frac{f(y)-f(z)}{y-z} \leq \frac{f(y)-f(x)}{y-x} \leq \lim_{z \downarrow x} \frac{f(z)-f(x)}{z-x} \equiv f^{+}(x)$$

Multiplying through by $(y-x)$ then yields the result.

If f is strictly concave, then Lemma A1 implies that the inequalities are strict.

(if) Choose any $x, y \in I$ with $x > y$ and $\lambda \in (0,1)$. Define $z = \lambda x + (1-\lambda)\,y$. By assumption

$$\begin{aligned} f^{+}(z)\,(y-z) &\geq f(y)-f(z) \\ f^{-}(z)\,(z-x) &\leq f(z)-f(x) \end{aligned}$$

But $y - z = \lambda(y-x)$ and $z - x = (1-\lambda)\,(y-x)$. Therefore

$$\begin{aligned} \lambda f^{+}(z)\,(y-x) &\geq f(y)-f(z) \\ (1-\lambda)\,f^{-}(z)\,(y-x) &\leq f(z)-f(x) \end{aligned}$$

Now if we multiply the first relation by $(1-\lambda)$ and the second relation by λ, and subtract the second from the first, we have

$$0 \geq (1-\lambda)\,\lambda\left(f^{+}(z)-f^{-}(z)\right)(y-x) \geq (1-\lambda)\,f(y)+\lambda f(x)-f(z)$$

which, from the definition of z, implies

$$f(\lambda x+(1-\lambda)\,y) \geq \lambda f(x)+(1-\lambda)\,f\,(y)\,.$$

If the initial inequalities are strict, then the concavity is strict. ■

Corollary A1: If $f : I \to \mathbb{R}$ is concave, then for any $x \in int(I)$, there is an $a \in \mathbb{R}$ such that $f(z) - f(x) \leq a(z - x)$ for all $z \in I$.

Proof. Choose a such that $f^{+}(x) \geq f^{-}(x)$. The corollary then follows from Theorem A1.

For the next lemma, we assume that $X \subset \mathbb{R}^n$ is convex and h is defined as in the text. ■

Lemma 1: $f : X \to \mathbb{R}$ is (strictly) concave if and only if each h_{xy} is (strictly) concave.

Proof. (if) Suppose each h_{xy} is concave. Then, for any $\lambda \in (0, 1)$, we have

$$f(\lambda y + (1 - \lambda)\, x) = h_{xy}(\lambda) \geq \lambda h_{xy}(1) + (1 - \lambda)\, h_{xy}(0) = \lambda f(y) + (1 - \lambda)\, f(x)$$

which implies that f is concave. If each h_{xy} is strictly concave then each inequality is strict for all $\lambda \in (0, 1)$ and therefore f is strictly concave.

(only if) Suppose f is concave. To show that any h_{xy} is concave, choose any $\mu_1, \mu_2 \in [0, 1]$ and let $z^i \equiv \mu_i y + (1 - \mu_i)\, x$ (so that $h_{xy}(\mu_i) = f(z^i)$).for $i = 1, 2$. Now consider any $\lambda \in (0, 1)$ and let $\mu \equiv \lambda\mu_2 + (1 - \lambda)\, \mu_1$. Then

$$\begin{aligned} \lambda x^2 + (1 - \lambda)\, x^1 &= \lambda\, (\mu_2 y + (1 - \mu_2)\, x) + (1 - \lambda)\, (\mu_1 y + (1 - \mu_1)\, x) \\ &= \mu y + (1 - \mu)\, x \end{aligned}$$

and therefore

$$\begin{aligned} h_{xy}(\lambda\mu_1 + (1 - \lambda)\, \mu_2) &= h_{xy}(\mu) \equiv f(\mu y + (1 - \mu)\, x) \quad \text{(by definition of } h_{xy}\text{)} \\ &= f(\lambda z^2 + (1 - \lambda)\, z^1) \quad \text{(from the relation above)} \\ &\leq \lambda f(z^2) + (1 - \lambda)\, f(z^1) \quad \text{(by the concavity of } f\text{)} \\ &= \lambda h_{xy}(\mu_2) + (1 - \lambda)\, h_{xy}(\mu_1) \quad \text{(by definition of } z^1, z^2\text{).} \end{aligned}$$

which implies that h_{xy} is concave. If f is strictly concave, then the inequality is strict and h_{xy} is strictly concave. ■