Estimating Multilevel Linear Models as Structural Equation Models

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Presentation at the
2002 Meeting of the Psychometric Society

June 21-June 23, 2002
Outline

- Describe Motivation
- Introduce Multilevel Linear Model (MLM)
- Show that MLM can be estimated as SEM
- Show that we can extend MLM within SEM
Motivation

Strengths/Limitations of MLMs

• Optimal For
  o Obtaining correct SEs for nested data
  o Estimating & predicting random effects

• Difficult For
  o Estimating measurement models
  o Obtaining explicit tests of mediation

Strengths/Limitations of SEM

• Opposite of above

Goals are to combine the strengths of the two models and bridge modeling traditions
A 2-Level MLM w/ L1 Covariates

**Level 1 Model**

\[ y_{ij} = \pi_{0j} + \sum_{p=1}^{P} \pi_{pj} x_{pj} + r_{ij} \]

where

\[ r_{ij} \sim MVN(0, \Sigma_{r_j}) \]

**Level 2 Model**

\[ \pi_{0j} = \beta_{00} + u_{0j} \]
\[ \pi_{1j} = \beta_{10} + u_{1j} \]
\[ \vdots \]
\[ \pi_{pj} = \beta_{p0} + u_{pj} \]

where

\[ \pi_{0j}, \pi_{1j}, \ldots, \pi_{0p} \sim MVN(\beta, T) \]
A 2-Level MLM w/ L1 Covariates

**Reduced-Form Equation**

\[ y_{ij} = \beta_{00} + \sum_{p=1}^{P} \beta_{p0} x_{p_{ij}} + u_{00} + \sum_{p=1}^{P} u_{pj} x_{p_{ij}} + r_{ij} \]

Fixed Coefficients

Random Coefficients

This is a special case of the linear mixed-effects model of Laird & Ware (1982)

\[ y_j = X_j \beta + Z_j u_j + r_j \]

where

- \( X_j \) is the design matrix for the fixed effects \( \beta \)
- \( Z_j \) is the design matrix for the random effects \( U_j \)

implying that

\[ y_j \sim MVN(X_j \beta, Z_j T Z_j' + \Sigma_{r_j}) \]
From MLM to SEM

In our case, $X_j = Z_j$, so

$$y_j \sim MVN(X_j \beta, X_j TX_j' + \Sigma_{r,j})$$

Further, if the design is balanced then $X_j = X$

and

$$y_j \sim MVN(X \beta, XTX' + \Sigma_r)$$

This is the same structure as a CFA model where

$$X = \Lambda$$

$$\beta = \kappa$$

$$T = \Phi$$

$$\Sigma_r = \Theta \delta$$
A Classic Case: The Growth Model

Multilevel Linear Growth Model

Level 1: \( y_{ti} = \pi_{0i} + \pi_{1i} x_{ti} + r_{ti} \)
\[ \Sigma_r = \text{DIAG}(\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4) \]

Level 2: \( \pi_{0i} = \beta_{00} + u_{0i} \)
\( \pi_{1i} = \beta_{10} + u_{1i} \)
\( T = \begin{pmatrix} \tau_{00} \\ \tau_{10} \quad \tau_{11} \end{pmatrix} \)

Linear Latent Curve Model in SEM

\( X_i = X = \Lambda \) because assuming balanced design. Random coefficients are represented as latent variables.
Balanced Data

Example

3 male & 3 female students per school to evaluate effect of sex on language ability

Multilevel Linear Model

Level 1: \( \text{lang}_{ij} = \pi_{0j} + \pi_{1j}\text{sex}_{ij} + r_{ij} \)

Level 2: \( \pi_{0j} = \beta_{00} + u_{0j} \)
\( \pi_{1i} = \beta_{10} + u_{1j} \)

Equivalent SEM

\[
\begin{pmatrix}
\sigma
\end{pmatrix}
\begin{pmatrix}
r_{m1}
\end{pmatrix}
\begin{pmatrix}
r_{m2}
\end{pmatrix}
\begin{pmatrix}
r_{m3}
\end{pmatrix}
\begin{pmatrix}
r_{f1}
\end{pmatrix}
\begin{pmatrix}
r_{f2}
\end{pmatrix}
\begin{pmatrix}
r_{f3}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\text{Lang}_{m1} \\
\text{Lang}_{m2} \\
\text{Lang}_{m3} \\
\text{Lang}_{f1} \\
\text{Lang}_{f2} \\
\text{Lang}_{f3}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\pi_0 \\
\pi_1
\end{pmatrix}
\begin{pmatrix}
(\beta_{00}) \\
(\beta_{10})
\end{pmatrix}
\]

\[
\begin{pmatrix}
\tau_{00} \\
\tau_{10}
\end{pmatrix}
\]

Order of the 3 males, 3 females w/in units \( j \) is arbitrary
Strategies for Imbalanced Data

*Treat as missing*

- Construct complete-data $\hat{\Sigma}(\theta), \hat{\mu}(\theta)$
- Compare each $y_j$ to submatrices $\hat{\Sigma}(\theta)_j, \hat{\mu}(\theta)_j$

*Example:* $M = \max$ # male students & $F = \max$ # female students in any given school.

![Diagram of a statistical model representing the strategies for imbalanced data.](image-url)
Strategies for Imbalanced Data

Compute $\hat{\Sigma}(\theta)_j, \hat{\mu}(\theta)_j$ directly from $\Lambda_j$

- Truer to multilevel approach: $\Lambda_j = X_j$
- $X_j$ referred to as definition variables for $\Lambda_j$
- Due to Neale

Example: $S = \text{max } \# \text{ students in any given school}$
What to do if you’re imbalanced?

Both approaches provide computationally equivalent results but

- **Strategy 1** is better for few discrete covariates & complex residual structures.

- **Strategy 2** is better for continuous covariates (highly imbalanced data) & homogeneity of error variance.
Adding Higher-Level Predictors

Adding Level 2 Covariates

Problem is \( X_j \neq Z_j \) but one \( \Lambda_j \)

*Rovine & Molenaar Solution:*

- Fixed effects factors have means, no variance
- Random effects factors have variance, no means
- Define \( \Lambda_j = BLOCK(X_j, Z_j) \)
- True to mixed-effects model, non-intuitive.

*Alternative Solution:*

- Extends approach used w/ latent curve models
- L2 predictors are ‘fixed X’ covariates
  - Effects contained in \( \Gamma \)
- Computationally equivalent to R & M Solution

*Both solutions can be extended to 3+ Level models*
Expanding the Model: A New Approach to Multilevel CFA

Adding a measurement model for item level outcomes

**Example:**

Data from High-School & Beyond: Teacher Survey

- 456 schools; 10,365 teachers
  - Imbalanced: # teachers/school ranges from 1 to 30
  - Let max # teachers = T = 30

- 9 item measure of teacher perceptions of control
  - 4 items on control of school policy
  - 5 items on control of classroom teaching/planning
  - 6 point Likert scales; Centered at mean

Estimating 2-Factor Model
High-School & Beyond 2-Factor Model
# Empirical Validation

Comparing SEM and MLM estimates

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