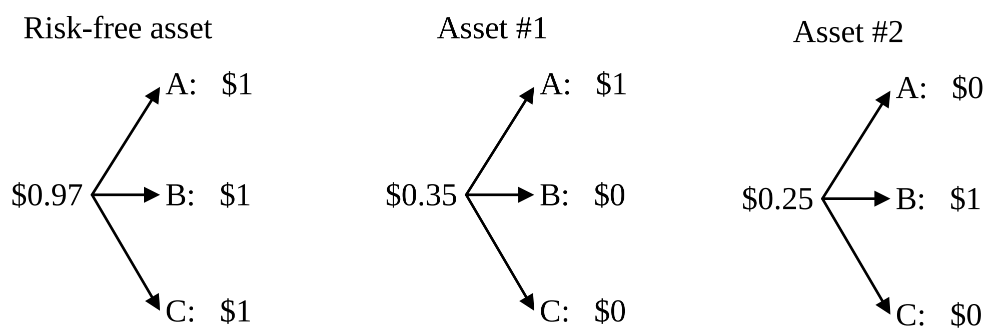
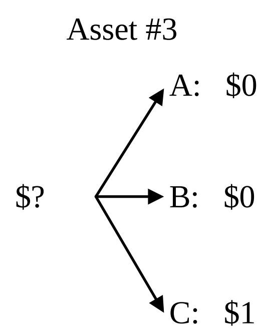
1. Assume that there are three future states of the economy. Let's label these states A, B and C. The risk-free asset pays \$1 in each of the three states and is currently selling for \$0.97. Asset #1 pays \$1 in state A and 0 in the other states. Asset # 1 is currently selling for \$0.35. Asset #2 pays \$1 in state B and 0 in the other states and is currently selling for \$0.25. In diagrams, we have:



a) What is the price of asset #3 that pays \$1 in state C and 0 in the other states?



<u>Solution:</u>

Note that asset #3 can be synthetically constructed by long 1 unit of the risk-free asset and short 1 unit of Asset #1 and 1 unit of Asset #2. As such, the price of Asset #3 must be: \$0.97 - \$0.35 - \$0.25 = \$0.37

b) What are the *risk-neutral* probabilities of each of the states A, B and C?

<u>Solution:</u>

Let's denote the risk-neutral probabilities of states A, B and C by p_A, p_B and p_C, respectively. Because these probabilities add up to 1, we only need to find p_A and p_B.

Applying the risk-neutral pricing equation to asset #1, we have:

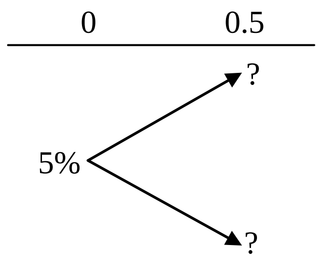
$$0.35 = \frac{p_A \times 1 + p_B \times 0 + p_C \times 0}{1+R_f} = 0.97 \times p_A \text{ which suggests } p_A = \frac{0.35}{0.97} = 0.3608$$

Applying the risk-neutral pricing equation to asset #2, we have:

$$0.25 = \frac{p_A \times 0 + p_B \times 1 + p_C \times 0}{1+R_f} = 0.97 \times p_A \text{ which suggests } p_A = \frac{0.25}{0.97} = 0.2577$$

2. Let's consider a lognormal model of interest rates with a semi-annual step size. The annualized drift from time 0 to time 0.5 is estimated to be 0.23. The standard deviation of monthly changes in log of 6-month interest rates is estimated to be 0.05. The current 6-month interest rate is 5% p.a.

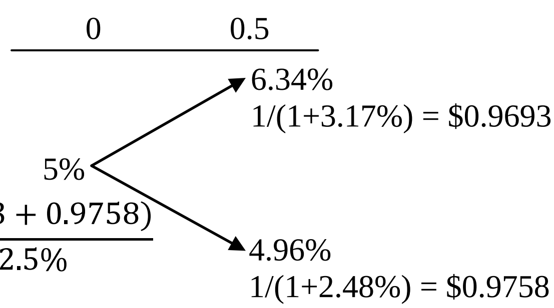
   a. Build a 1-step binomial tree of 6-month interest rates.



Solution:
We have $m_1 = 0.5 \times 0.23 = 0.115$ and $s = 0.05\sqrt{6} = 0.1225$.
The interest rate in the "up" node is: $5\%\ e^{0.115 + 0.1225} = 6.34\%$.
The interest rate in the "down" node is $5\%\ e^{0.115 - 0.1225} = 4.96\%$.

   b. Compute the per annum semi-annual compounding 1-year interest rate implied by the model

Solution:
Use the above tree to price a \$1-face 1-year zero as follows:



$$0.9488 = \frac{0.5(0.9693 + 0.9758)}{1 + 2.5\%}$$

Given the price of 0.9488, the 1-year interest rate r must be such that:

$$0.9488 = \frac{1}{\left(1 + \frac{r}{2}\right)^2}$$

Solve the above equation, we have: r = 5.32%

   c. Explain how you would estimate the drift m_2 from time 0.5 to time 1 if the 1.5-year interest rate is 7% p.a. semi-annual compounding.

Solution:
Choose the drift m_3 in a way that the mode-implied 1.5-year interest rate is equal to the observed 1.5-year interest rate of 7% p.a.

3. Please list at least 2 strengths and 2 weaknesses of the lognormal model of interest rates. In your opinion, are these weaknesses serious? Explain briefly how these weaknesses can be overcome.

Solution:

Strengths:

- Interest rates are always positive
- The model displays volatility-level effect: Interest rate volatility is high when the level of interest rates is high.

Weaknesses:

- Although the drift terms change from one period to the next, the volatility parameter remains the same.
- The lognormal model of interest rates belongs to the family of one-factor model of interest rates in which interest rates of all horizons are perfectly correlated.

The first weakness, the inflexibility in modeling volatility, can be serious if the model is used to price interest rate options whose prices are highly dependent on interest rate volatilities. This weakness is accommodated by the Black-Derman-Toy model in which the volatility parameter s, like the drift parameters, changes from one time step to another. The key is to use the historical volatility information at various horizons.

The second weakness, though certainly not desirable, is not overly serious considering how correlated interest rates at different horizons are overtime.

4. a) Explain the meaning of the phrase "price compression" in the context of callable bond pricing.

Solution:

The phrase "price compression" in the context of callable bond pricing refers to the fact that as interest rates decrease, the appreciation in price of callable bonds is must smaller than (compressed down relative to) that of their non-callable counter-parts. The reason is that as interest rates become smaller, a callable bond is more likely to be called which makes the callable feature attached to a callable bond worth more. As the value of the callable bond = value of the non-callable bond – the value of the callable feature, the increase in value of the callable feature offsets the appreciation in value of the non-callable bond. The net result is the callable bond appreciates less in value.

b) Explain intuitively why callable bonds exhibit negative convexity. Does it mean that the convexity measure of a callable bond is always negative?

Solution:

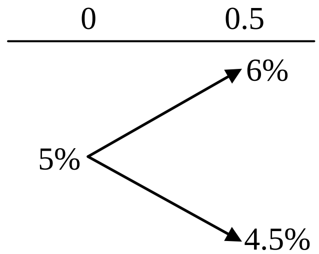
For regular bonds, dollar duration (the slope of the tangent line to the pricing function) decreases as interest rates increase. This creates positive convexity. For callable bonds, dollar duration *increases* as interest rates increase over some range of values. This is because when interest rates are very low, callable bonds are very likely to be called and as such behave like bonds with short maturities, thus having low dollar duration. When interest rates are very high, however, callable bonds behave very much like bonds with long maturities, thus having large dollar duration. Since dollar duration for bonds should change continuously as interests go from low to high, for some intermediate range, the dollar duration of a callable bond must increase with interest rates. This creates negative convexity. But it doesn't mean that the convexity measure of a callable bond is always negative. When interest rates are very low or very high, callable bonds behave like regular bonds. As such, in the extreme environments, the convexity measure of a callable bond could be positive.

c) Why is negative convexity important? What are the implications of negative convexity for banks?

Solution:

Negative convexity is important in hedging context. Mortgages often exhibit negative convexity. As such when banks buy mortgages (lend money to homeowners), they have negative convexity on their assets. In order to immunize themselves against changes in interest rates, banks would like to match the duration as well as convexity of their assets and liabilities. To match the negative convexity on the assets side, therefore, it is natural to issue callable bonds on the liabilities side in order to have a natural hedge.

5. Consider the following binomial tree of risk-free 6-month interest rates where the risk-neutral probabilities of the 'up' and 'down' branches are 50%:



The implied 1-year semi-annual compounding risk-free interest rate from the above tree is 5.1236%.

a) Consider a $100 face, 10% semi-annual coupon, 1-year callable risk-free bond which can be called at time 0.5 for a call price of $102. This bond has no default risk and liquidity risk. Based on the information given, what is the price of the bond?

<u>Solution:</u>

At the upper node, the ex-coupon price of the bond is: $\frac{105}{1+3\%} = \$101.94$. The issuer of the bond won't call the bond here since he/she would have to pay $102 for something that is worth $101.94. Together with the coupon of $5, the value of the bond here is $106.94.

At the lower node, the ex-coupon price of the bond is: $\frac{105}{1+\frac{4.5\%}{2}} = \$102.69$. The issuer will call the bond here and pay $102 for it. Together with the coupon of $5, the value of the bond here is $107.

Therefore, the price of the bond at time 0 is:

$$P = \frac{0.5 \times (106.94 + 107)}{1.025} = \$104.36$$

b) Consider **another** $100 face, 10% semi-annual coupon, 1-year callable bond which can be called at time 0.5 only for a call price of $102. If the bond's static spread is 0.34%, what price is it trading at?

<u>Solution:</u>

The price of the bond is:

$$P = \frac{5}{1 + \frac{5\% + 0.34\%}{2}} + \frac{105}{\left(1 + \frac{5.1236\% + 0.34\%}{2}\right)^2} = \$104.36$$

c) From your calculations in part a) and b) what is your best guess regarding the option-adjusted spread of the bond in part b?

<u>Solution:</u>

The price of the bond in part b) is exactly the same as the price of the risk-free bond in part a). This suggests that the bond in part b) has no default risk and liquidity risk either. As such, the option-adjusted spread of the bond should be zero.

6. Interest rates are really low and the yield curve is currently flat at 1%. What is your best guess of the dollar duration of a $100 face value, 1-year callable bond paying 10% semi-annual coupon rate, callable only at time 0.5 for a call price of $102.

Solution:

When interest rates are really low, callable bonds are very likely to be called. As such, the callable bond in the question would behave very much like a bond that pays $5+$102 at time 0.5. The dollar duration of the callable, therefore, would be very close to:

$$\frac{107}{1+\frac{1\%}{2}} \times \frac{0.5}{(1+\frac{1\%}{2})} = 52.97$$