I. Asset Pricing Theory -- A Brief Review

Asset Pricing Theory
A Brief Review

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1. Introduction

Fundamental Issue of Asset Pricing Theory

How Do We Price Uncertain Future Flows of Income?

- Absolute Pricing
  - Price Each Asset by Reference to Its Exposure to Fundamental Sources of (Macroeconomic) Risk
  - Best Examples: G.E. Models, Factor Models
- Relative Pricing
  - Best Example: B.S. Option Pricing

- Asset pricing theory tries to understand the prices or values of claims to uncertain payments
- A low price implies a high rate of return, so we can also think of the theory as explaining why some assets pay a higher average returns than others
- OBJECTIVE: To explain the variation in expected returns across stocks and the variation in the equity premium (expected market return minus risk-free rate) through time as a tradeoff between risk and return

- As already noted two approaches:
- *Absolute pricing*: we price each asset by reference to its exposure to fundamental sources of macroeconomic risk → we want to understand and measure the sources of aggregate or macroeconomic risk that drive asset prices
- *Relative pricing*: how can we value an asset given the prices of some other assets (derivatives)

Basic Equation

$$\begin{cases} P_t = E_t\left[m_{t+1} x_{t+1}\right] \\ m_{t+1} = f(\text{data, parameters}) \end{cases}$$

P_t : Asset Price

x_{t+1} : Payoff

m_{t+1} : Disc. Factor

2. Consumption-Based Model and Overview

2.1 Basic Pricing Equation

$$P_t = E_t\left[\beta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1}\right]$$

Basic Task: Find Value at Time t of a Payoff x_{t+1}

We Start With:

$$u(c_t, c_{t+1}) = u(c_t) + \beta E_t\left[u(c_{t+1})\right]$$

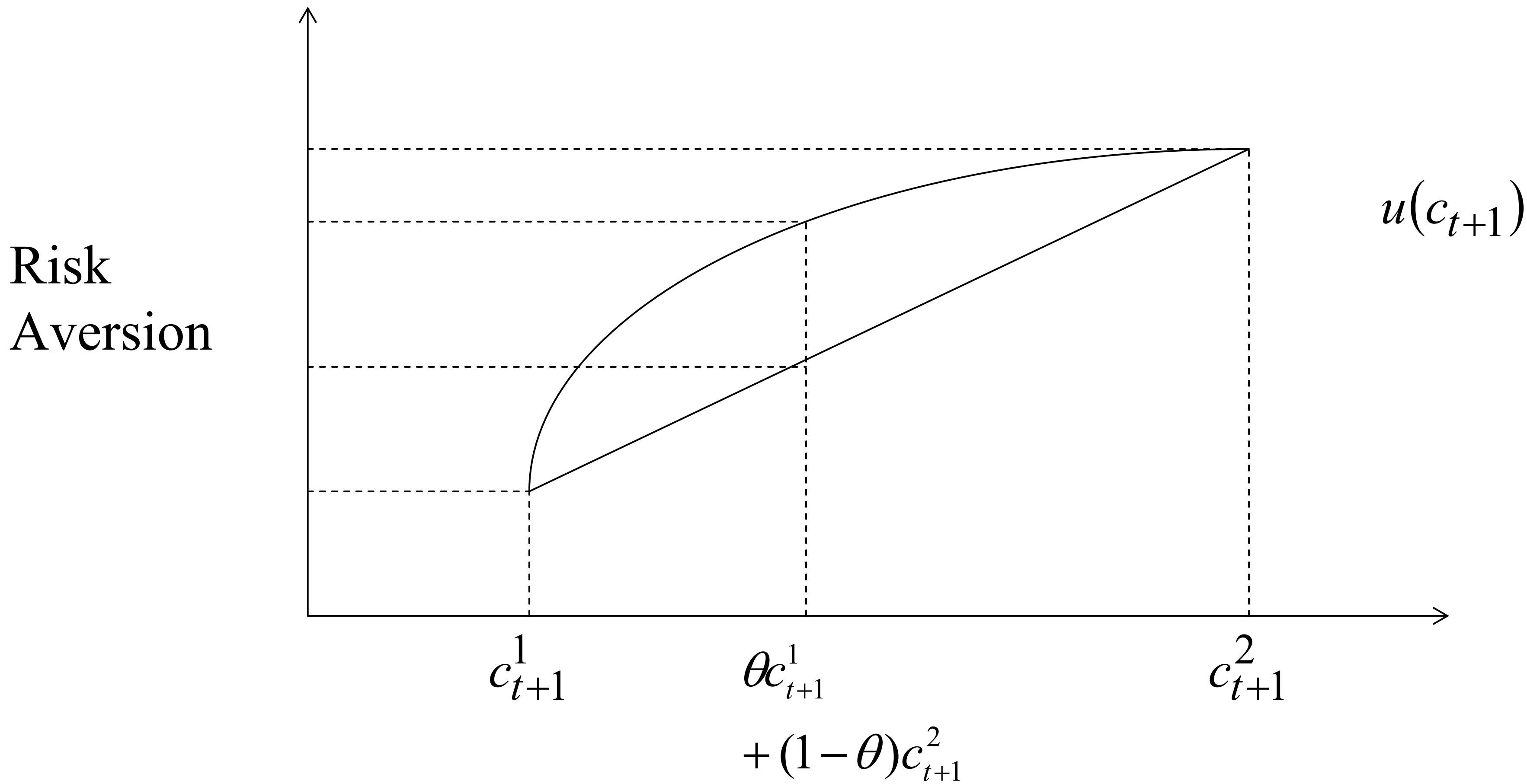
Specific Utility Functions Often Used:

$$u(c_t) = \frac{1}{1-\gamma} c_t^{1-\gamma} \qquad \gamma \neq 1 \ \ (CRRA\ )$$

$$u(c_t) = \ln\ c_t \qquad \gamma = 1$$

Period Utility $u(.)$ increasing & concave

The parameter β is Subjective Discount Factor



Curvature: Risk Aversion & Intertemporal Substitution

Investor's Maximization Problem

$$\underset{\{\xi_t\}}{MAX} \left[u\left(c_t\right) + E_t \beta u\left(c_{t+1}\right)\right]$$

s.t. $c_t = e_t - p_t \xi_t$

ξ_t : Amount of Asset to Buy (at Price p_t in Period t)

e_t : Endowment shocks at time t

Euler Equation — First Order Condition

$$P_t \, u'(c_t) = E_t\left[\beta \, u'\left(c_{t+1}\right) x_{t+1}\right] \qquad (1)$$

and therefore

$$P_t = E_t\left[\beta \frac{u'\left(c_{t+1}\right)}{u'\left(c_t\right)} x_{t+1}\right] \qquad (2)$$

(1) Intertemporal Substitute Condition

(2) Stochastic Discount Factor Fundamental Equation

Note That We Don't Have Closed Form Solutions

$$c_{t+1} = c_{t+1}\left(e_t, e_{t+1}, x_{t+1}\right)$$

$$p_t = p_t\left(e_t, e_{t+1}, x_{t+1}\right)$$

Therefore, with the Fundamental Equation:

$$p_t = E_t m_{t+1} x_{t+1}$$

Endogenous

2.2 Marginal Rate of Substitutions/Stochastic Discount Factor

$$p_t = E_t m_{t+1} x_{t+1}$$

$$m_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)}$$

If there is no uncertainty: $p_t = \frac{1}{R^{\mathrm{f}}} x_{t+1}$

Where R^f: Risk free rate

Some Simplified Notation

$$P = Emx$$

Namely, Drop Time Subscribs

Deep Issue: There is One Stochastic Discount Factor for All Risky Assets

Therefore, we avoid:

$$p_t^i = \frac{1}{R^i} E_t x_{t+1}^i$$

i: Different Asset Category

2.3 Prices, Payoffs and Notation

	Price p_t	Payoff x_{t+1}
Stock	p_t	$p_{t+1} + d_{t+1}$
Return	1	R_{t+1}
Price-dividend ratio	$\frac{p_t}{d_t}$	$\left(\frac{p_{t+1}}{d_{t+1}} + 1\right)\frac{d_{t+1}}{d_t}$
Excess returns	0	$R^e_{t+1} = R^a_{t+1} - R^b_{t+1}$
Managed portfolio	z_t	$z_t R_{t+1}$
Moment condition	$E(p_t z_t)$	$x_{t+1} z_t$
One-period bond	p_t	1
Risk free rate	1	R^f
Option	C_t	$\max(p_{t+1} - K, 0)$

- For Stocks $x_{t+1} = p_{t+1} + d_{t+1}$

- Gross Returns $\mathrm{R}_{t+1} = \frac{p_{t+1} + d_{t+1}}{p_t} \Rightarrow 1 = E_t m_{t+1} R_{t+1}$

Notation

Small → r (Net Returns)

Capital → R (Gross Returns)

$$r = R - 1$$

Net Returns

Gross Returns

$R \approx 1.05$

$r \approx 0.05$

2.4 Intuition — Implications and Classic Issues in Finance

Arrangements Yield

- Determinants of Interest Rate
- Risk Correction
- Idiosyncratic vs. Systematic Risk
- Beta Pricing Models
- Mean-Variance Frontiers

2.4.1 Risk Free Rate

$R_f = 1 / E_t m_{t+1}$

Specific Assumptions About Economy

- $u'(c) = c^{-\gamma}$
- $c_{t+1} / c_t \sim$ Log Normal

$$r_f = \ln R_f = \delta + \gamma E_t \Delta \ln c_{t+1} - \frac{\gamma^2}{2} \sigma_t^2 (\Delta \ln c_{t+1})$$

Where

$$\beta = e^{-\delta}$$

$$\Delta \ln c_t = \ln c_t = \ln c_{t-1}$$

$$E(e^z) = e^{E(z) + \frac{1}{2}\sigma^2(z)}$$ because of log normality

$r_f \nearrow$ When $\beta \nearrow$ (impatience)

$r_f \nearrow$ When $E_t\, \Delta \ln c_{t+1} \nearrow$

with slope $\gamma \longrightarrow$ Inverse of Elast. of Intertemporal Substitution

$r_f \searrow$ When $\sigma^2 \nearrow$

Precautionary Savings Drives Down r_f

2.4.2 Risk Corrections

$$\text{Cov}(m,x) = Emx - E(m)E(x)$$

$$\Rightarrow p = E(m)E(x) + Cov(m,x)$$

$$p = \frac{E(x)}{R_f} + Cov(m,x)$$

Standard Risk Neutral — Risk Adjustment

Interpretation of Risk Adjustment

Recall $$m_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)}$$

Therefore $$p_t = \frac{E_t x_{t+1}}{R_f} + Cov_t\left(\beta \frac{u'(c_{t+1})}{u'(c_t)}, x_{t+1}\right)$$

p_t ↘ With Pos. Cov.

p_t ↗ With Neg. Cov.

- Operates Like an "Insurance Premium"
- Assets With High Payoff in Low Consumption States Sell at Premium

2.4.3 Risk Corrections to Expected Returns

Start from Return Equation

$$1 = E_t m_{t+1} R_{t+1}$$

$$1 = \left(E_t m_{t+1}\right)\left(E_t R_{t+1}\right) + Cov_t\left(m_{t+1}, R_{t+1}\right)$$

$$E_t R_{t+1} = \frac{1}{E_t m_{t+1}} - \frac{Cov_t\left(m_{t+1}, R_{t+1}\right)}{E_t m_{t+1}}$$

$$E_t R_{t+1} = R_f - \frac{Cov_t\left(\beta u'\left(c_{t+1}\right), R_{t+1}\right)}{E_t \beta u'\left(c_{t+1}\right)}$$

Sometimes referred to as equity premium

"Riskier" Asset

- Sell at Lower Price
- Have Higher Expected Returns

2.4.4 Idiosyncratic Risk Does Not Affect Price

Suppose with Payoff Uncorrelated to m

$$Cov(m,x)=0 \Rightarrow p=\frac{E(x)}{R_f}$$

More General

$$Cov\,(m,x)=Cov\left(m, proj\,(x|m)+\varepsilon\right)=Cov\left(m, proj\,(x|m)\right)$$

2.4.5 Expected Return-Beta Representation

Recall

$$E(R) = \frac{1}{E(m)} - \frac{Cov(m,R)}{E(m)}$$

So for risky asset I we have:

$$ER^{i} = R_{f} + \frac{Cov(R^{i}, m)}{Var(m)}\left(-\frac{Var(m)}{E(m)}\right)$$

$$ER^{i} = R_{f} + \beta_{i,m}\lambda_{m}$$

Where:
- $\beta_{i,m}$ is "Beta"
- λ_{m} is Price of β Risk

Note that m is ‘market portfolio and expectations are conditional

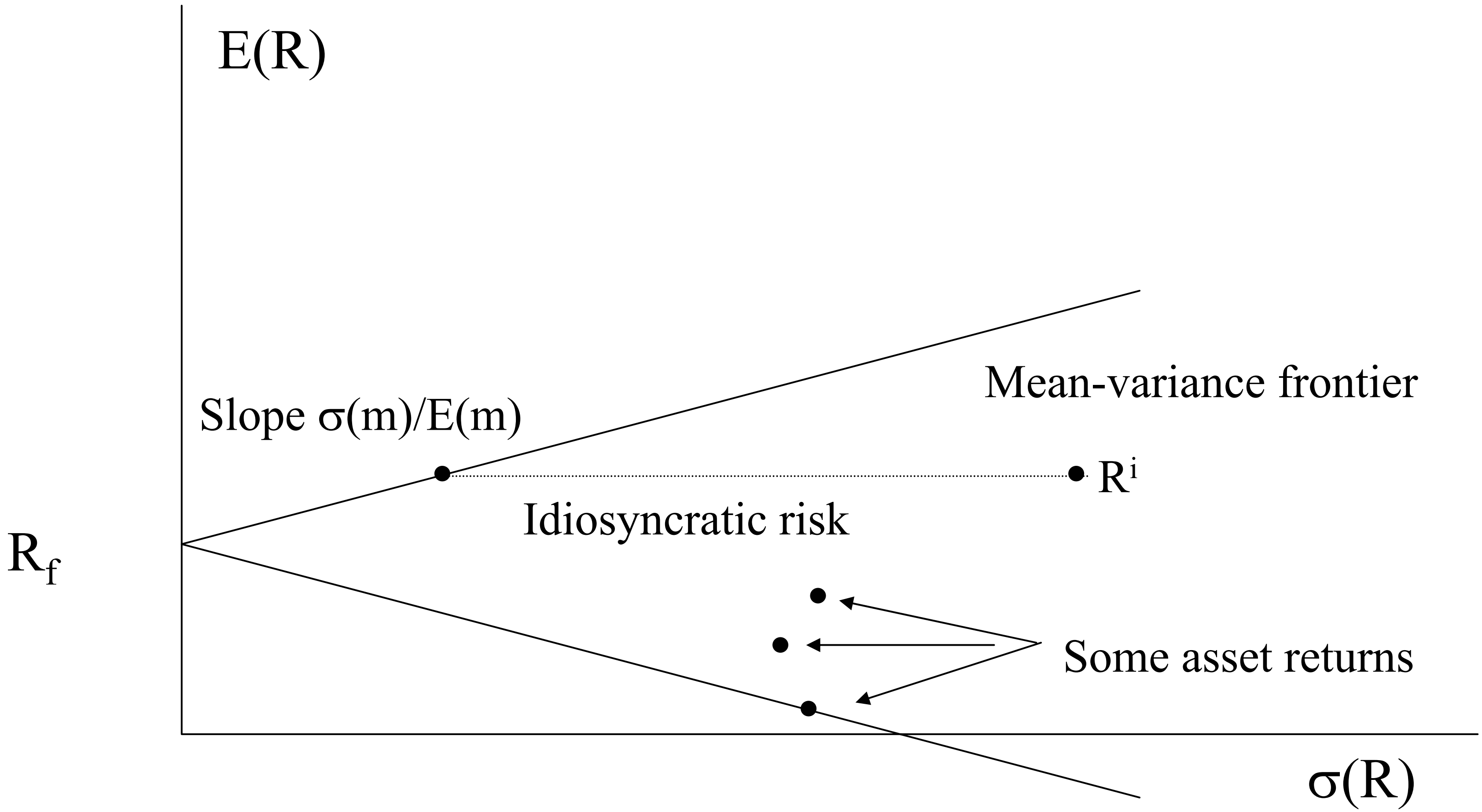
2.4.6 Mean-Variance Frontier

$$\left| E(R^i) - R_f \right| \leq \frac{\sigma(m)}{E(m)} \sigma(R^i)$$

Because

$$1 = E(mR^i) = E(m)E(R^i) + \rho_{m,R^i} \sigma(R^i)\sigma(m)$$

$$E\left(R^i\right) = R_f - \rho_{m,R^i} \frac{\sigma(m)}{E(m)} \sigma(R^i)$$


E(R)
Mean-variance frontier
Slope σ(m)/E(m)
R^i
Idiosyncratic risk
R_f
Some asset returns
σ(R)

- All Returns on Frontier are Perfectly Correlated with Discount Factor, i.e. $\left| \rho_{m,R^i} \right| = 1$

All Frontier Returns are Perfectly Correlated With Each Other

→ Two Returns Span Frontier R_f and R^m

$R^{mv} = R_f + a(R^m - R_f)$

Equivalently $\exists$ a, b, d, e

$m = a + bR^m$

$R^{mv} = d + em$

- Any Means Variance Efficient Frontier Return Other Than R_f Carries All Pricing Information

$$\Rightarrow E(R^i) = R_f + \beta_{i,mv} x \left[R(R^{mv}) - R_f \right]$$

$\uparrow$

Any Mean-Variance Frontier Return

2.4.7 Slope of the Mean- Standard Dev. Frontier

$$\left|\frac{E(R^p)-R_f}{\sigma(R^p)}\right|=\frac{\sigma(m)}{E(m)}=\sigma(m)R_f$$

For Any Portfolio "P" on the Frontier

Slope is "Price of Risk" or Sharpe Ratio of Stochastic Discount Factor $\frac{\sigma(m)}{E(m)}$

If We Take $u'(c) = c^{-\gamma}$

Then

$$\left|\frac{E(R)-R_f}{\sigma(R)}\right| = SQRT\left(e^{\gamma^2\sigma^2(\Delta \ln c_{t+1})}-1\right) \approx \gamma\sigma(\Delta \ln c)$$

Slope ↗

- If $\sigma(\Delta \ln c)$ ↗
- Risk Aversion γ ↗

2.4.8 Time-Varying Expected Returns and Random Walks

Time Series as Well as Cross-Sectional Interpretations of Fundamental Equation

For Instance: $1 = E_t(m_{t+1} R_{t+1})$ implies

$$E_t R_{t+1} = R_t^f - \frac{Cov_t(m_{t+1}, R_{t+1})}{E_t(m_{t+1})}$$

→ Expected Returns are Predictable

→ Expected Returns Vary Over Time

However, Predictability Has To Be <u>Explained</u> By:

- Changing Consumption
- Changing Conditional Variance

Where is the Martingale Hypothesis?

$$p_t u'(c_t) = E_t\left[\beta u'(c_{t+1})(p_{t+1} + d_{t+1})\right]$$

Rather Than $P_t = E_t(P_{t+1})$