for any vector $z_t \in \Omega_t$. For example, Eichenbaum estimates the parameters using the lagged values of inventories and sales, $\{S_{t-i}, I_{t-i}; i = 1, 2, \ldots k\}$, in z_t.

Maximum Likelihood would involve estimation of the bivariate vector autoregressive system for (S_t, I_t) subject to the nonlinear cross equation restrictions on the parameters implied by the model. This is likely to be more computationally burdensome with the exact degree depending on the choice of distribution. Unfortunately, economic theory provides no guidance on this choice. Once again, unless the chosen distribution is correct then the resulting MLE's are unlikely to have the anticipated optimal properties.

1.3.5 Stochastic Volatility Models of Exchange Rates

The preceding models have all been developed from economic theory. In some circumstances, it may be desired to capture the time series properties of an economic variable using a purely statistical model. An example of such a model would be the autoregressive integrated moving average (ARIMA) class developed by Box and Jenkins (1976). However, ARIMA models are not particularly appropriate for many financial assets because they do not allow the conditional variance to change over time. This has led to considerable interest in statistical models which can capture this type of behaviour. The most prominent of these models are the autoregressive conditional heteroscedasticity (ARCH) models introduced by Engle (1982), which have been applied very widely in finance, see the survey by Bollerslev, Chou, and Kroner (1992). More recently, a second class is receiving considerable attention and these are known as stochastic volatility models; see the survey by Ghysels, Harvey, and Renault (1996).

In this section we describe the stochastic volatility model used by Melino and Turnbull (1990) to analyze daily exchange rates. The model has its origins in a stochastic differential equation for the evolution of the exchange rate over time. However, we focus directly on the discrete time stochastic process which is used to approximate this underlying continuous time process. Let $y(\tau)$ denote the exchange rate at time τ and assume that the exchange rate is observed at times $\{\tau_1, \tau_2, \ldots \tau_T\}$. These observations are not at evenly spaced intervals because there are days on which no trading occurs, such as weekends and holidays. To accomodate these effects, it is useful to denote the distance between observations by $d_t = \tau_t - \tau_{t-1}$, and the minimum distance by $d = min_t(d_t)$. The discrete time approximation takes the form

$$\begin{aligned} y(\tau_t) \;=\; \alpha_0 d_t \;&+\; (1 + \beta_0 d_t) y(\tau_{t-1}) \\ &+ x(\tau_{t-1}) y(\tau_{t-1})^{\gamma_0/2} d_t^{1/2} e(\tau_t) \end{aligned} \tag{1.45}$$

where the latent process $x(\tau_t)$ is generated by

$$ln[x(\tau_t)] \;=\; \delta_0 d \;+\; (1 + \eta_0 d) ln[x(\tau_t - d)] \;+\; \zeta_0 d^{1/2} u(\tau_t) \tag{1.46}$$

and

$$\left[\begin{array}{c} e(\tau_t) \\ u(\tau_t) \end{array} \right] \sim IN\left(\left[\begin{array}{c} 0 \\ 0 \end{array} \right] \left[\begin{array}{cc} 1 & \rho_0 \\ \rho_0 & 1 \end{array} \right] \right) \tag{1.47}$$

Given that the model includes a distributional assumption, it is natural to use Maximum Likelihood. However, the evaluation of the conditional likelihood at time t involves a T-dimensional numerical integration which is computationally extremely burdensome – if not infeasible – on many currently available computer systems. However the normality assumption implies various population moment conditions which can form the basis of GMM estimation of the parameter vector $\theta_0 = (\alpha_0, \beta_0, \delta_0, \eta_0, \zeta_0, \rho_0)$.[30] For example, Melino and Turnbull (1990) show that the following population moment conditions hold:[31]

$$\begin{aligned} E[w_t(\theta_0)] &= 0 \\ E[w_t^2(\theta_0)] \;-\; exp[2\mu_x \;+\; 2\sigma_x^2] &= 0 \\ E[w_t^3(\theta_0)] &= 0 \\ E[w_t^4(\theta_0)] \;-\; 3exp[4\mu_x \;+\; 8\sigma_x^2] &= 0 \\ E[|w_t(\theta_0)|] \;-\; (2/\pi)^{1/2} exp[\mu_x + 0.5\sigma_x^2] &= 0 \\ E[|w_t(\theta_0)|^3] \;-\; 2(2/\pi)^{1/2} exp[3\mu_x + 4.5\sigma_x^2] &= 0 \\ E[|w_t(\theta_0)| w_t(\theta_0)] &= 0 \\ E[w_t(\theta_0) w_{t-j}(\theta_0)] &= 0 \\ E[|w_t(\theta_0) w_{t-j}(\theta_0)|] - \ell_{1,j}(\theta_0) \;+\; \ell_{2,j}(\theta_0) &= 0 \\ E[|w_t(\theta_0)| w_{t-j}(\theta_0)] \;-\; m_j(\theta_0) &= 0 \\ E[w_t^2(\theta_0) w_{t-j}^2(\theta_0)] \;-\; n_j(\theta_0) &= 0 \end{aligned} \tag{1.48}$$

for $j = 1, 2, \ldots$ where

$$w_t(\theta_0) \;=\; \frac{y(\tau_t) - \alpha_0 d_t - (1 + \beta_0 d_t) y(\tau_{t-1})}{[d_t \{y(\tau_{t-1})\}^{\gamma_0}]^{1/2}} \tag{1.49}$$

and

$$\begin{aligned} \ell_{1,j}(\theta_0) &= (2/\pi)^{1/2} exp[2\mu_x + \sigma_x^2(1 + (1 + \eta_0 d)^j) - 0.5\rho_0^2\zeta_0^2 d(1 + \eta_0 d)^{2(j-1)}] \\ \ell_{2,j}(\theta_0) &= (2/\pi)^{1/2} \rho_0 \zeta_0 d^{1/2} (1 + \eta_0 d)^{j-1} (1 - 2\Phi(\rho_0 \zeta_0 d^{1/2} (1 + \eta_0 d)^{j-1}) \\ &\qquad \times exp[2\mu_x + \sigma_x^2(1 + (1 + \eta_0 d)^j)] \\ m_j(\theta_0) &= (2/\pi)^{1/2} \rho_0 \zeta_0 d^{1/2} (1 + \eta_0 d)^{j-1} exp[2\mu_x + \sigma_x^2(1 + (1 + \eta_0 d)^j)] \\ n_j(\theta_0) &= \{4\rho_0^2 \zeta_0^2 d(1 + \eta_0 d)^{2(j-1)} + 1\} exp[4\mu_x + 4\sigma_x^2(1 + (1 + \eta_0 d)^j)] \\ \mu_x &= -\delta_0/\eta_0 \\ \sigma_x^2 &= \zeta_0^2 d/[1 - (1 + \eta_0 d)^2] \end{aligned}$$

and $\Phi(.)$ denotes the cumulative distribution function of a standard normal random variable.

[30] In their estimations, Melino and Turnbull (1990) fix the value of γ_0 and so we omit this term from θ_0. See Section 9.4 for further discussion of this issue.

[31] These expressions are not actually presented in the published version of Melino and Turnbull's paper but are contained in an unpublished appendix by Ken Vetzal which was kindly sent to the author by Angelo Melino.