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The Role of Conditioning Information in Deducing Testable Restrictions Implied by Dynamic Asset Pricing Models

Econometrica, 55, 1987

Also based on Cochrane Chapter 5

BIG PICTURE RESULTS EXISTENCE OF SDF

Law of one price

If two portfolios have same payoffs in all states of nature then they must have the same price

$\exists$ m : p = E m x iff law holds

Absence of arbitrage

No arbitrage opportunities

Iff : $\exists$ m > 0 p = E m x

Back to the basic questions

Instead of starting from general equilibrium consumption - based asset pricing model.

Let us view discount factor as stochastic process satisfying

$$p = E\, m\, x$$

- Does there always exist such an m ?
- What markets structure supports this fundamental equation?

- Many aspects of payoff space can be conveniently captured by restrictions on the discount factor.
- Check and impose restrictions on discount factor is easier than on all possible portfolios priced by discount factor.

CONTINGENT CLAIMS

- Suppose S states of nature tomorrow
- Contingent claim pays $1 (1 unit of consumption)
- pc(s) : price today of contingent claim state “s”

COMPLETE MARKET

- Any contingent claim can be bought
- Note that each contract does not have to be traded explicitly
- It can be spanned by other securities (so called synthesized securities e.g. European options with every possible strike)

If a complete set of contingent claims exists, then a discount factor exists and it is equal to the contingent claim prices divided by state probabilities

Let x (s) denote Payoff $\Rightarrow$ p(x) $= \sum_s$ pc(s) x(s)

$$p(x) \;= \sum_s \pi(s) \left[\frac{\mathrm{pc}(s)}{\pi(s)} \right] x(s)$$

Where π (s) is probability of states

$$m(s) = \frac{pc(s)}{\pi(s)}$$

Then p $= \sum_s \pi$ (s)m(s) x (s) = Em x

Therefore in complete markets the stochastic discount factor m exists with p=Emx

m(s) is state price pc(s) rescaled by probability of state, therefore m(s) : State Price Density

Risk neutral probabilities

$$\pi^{*}(s) \equiv R^{f} m(s)\pi(s) = R^{f} pc(s) = \frac{pc(s)}{E(m)}$$

$$and \quad p(x) = \frac{E^{*}(x)}{R^{f}}$$

- Asset pricing becomes as if there is risk neutrality with re-scaled probabilities π^*
- Deep issue : risk aversion is equivalent to paying more attention to certain (unpleasant) states

$$\pi^*(s) > \pi(s)$$

In “bad” states high subjective probabilities

$$\pi^*(s) = \frac{m(s)}{E(m)}\pi(s)$$

- In incomplete markets there are many m, but one is positive
- m>0 but is not unique
- When m is not unique there might be some negative discount factors.

Hilbert Spaces

A non-empty set H is called a Hilbert space if H is a real linear vector space, together with a real-valued function $<.,.>$ from H x H into C having the following properties:

- $<x,x>$ is nonnegative and $<x,x> = 0$ iff $x = 0$
- $<x+y,z> = <x,z> + <y,z>$ for all x,y,z
- $<ax,y> = a<x,y>$ for all x,y in H and *a* real-valued
- $<x,y> = <y,x>$ for all x,y in H
- If $\{x_n\}$ is a sequence in H with $\lim < x_n - x_m , x_n - x_m > \to 0$ as $n,m \to \infty$ then there is an element x in H such that $\lim < x_n - x , x_n - x > = 0$.

Inner product and norm

- The function <.,.> is called the inner product
- The real-valued function $\|x\| = < x, x >^{1/2}$ is called the norm.
- Schwartz inequality $|< x, y >| \leq \|x\| \|y\|$
- Two points x and y in a Hilbert space are orthogonal if <x,y> = 0
- Riesz representation theorem: for every bounded linear functional x^* on H there is a unique z in H such that $x^*(x) = <x,z>$ for all x in H and $\|x^*\| = \|z\|$

Notations:

G_t - all the info at time t

I_t - all the random variables with outcomes in G_t (i.e. measurable functions)

P_{t+1} - all payoffs p_{t+1} on assets at time t + 1

$\pi_t(p_{t+1})$: $P_{t+1} \rightarrow I_t$ price of p at time t

w_t in I_t: portfolio weights

$$p_1, p_2 \text{ in } P \rightarrow w_1 p_1 + w_2 p_2 \text{ in } P$$

Objectives

- Construct a general asset pricing function
- This construction will allow for a very nice decomposition of any return into a sum of three orthogonal components
- The decomposition yields an easy solution to the mean-variance problem
- Discuss conditional vs.unconditional approach
- Bridge to empirical work

One-period setting

- One period setting suffices because of stationarity
- For any p in I_1 consider $P^+ = \{p \in I_1 : E[p^2 \mid G] < \infty\}$
- Conditional counterpart of inner product on P^+ for any p_a and p_b in P^+ $< p_a, p_b >_G = E[p_a p_b \mid G]$
- We introduce a subset P of P^+ that is restricted to satisfy conditional counterparts of linearity and completeness (i.e. linear combinations of elements of P belong to P and conditional Cauchy sequences converge in P)

Technical (and Economic) Assumptions

Assumption A.1 (Law of one price)

For any p_1, p_2 in P and w_1, w_2 in I

$$\pi(w_1 p_1 + w_2 p_2) = w_1 \pi(p_1) + w_2 \pi(p_2)$$

Assumption A.2 For any $p \geq 0$, $\pi(p) \geq 0$

Assumption A.3 $\exists\, p_o$: $\Pr\{\pi(p_o) = 0\} = 0$

Technical Assumption Implications

The conditional version of the Riesz representation theorem allows us to write $\pi(p) = E(pp^*|G)$ for some $p^* > 0$ with $Pr = 1$.

Moreover, we have that

A.1: $\pi(w_1p_1 + w_2p_2) = E((w_1p_1 + w_2p_2)p^*|G)$
$= E(w_1p_1p^* \mid G) + E(w_2p_2p^*|G)$
$= w_1\pi(p_1) + w_2\pi(p_2)$

A.2: $p \geq 0 \quad \pi(p) = E(pp^*|G) \geq 0$

A.3: $p_o = 1 \quad \pi(p_o) = E(p^*|G) > 0$

Theorem

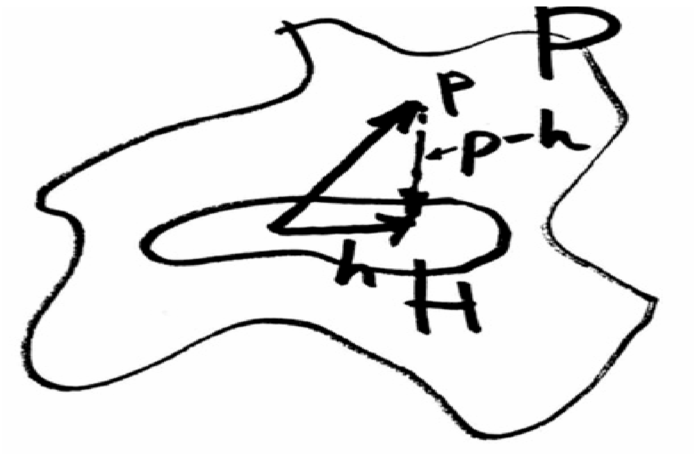
A.1-3 + conditional Riesz theorem are satisfied $\Rightarrow$ exists unique p^* in P:

$$\pi(p) = E(pp^*|G) = \langle p,p^* \rangle_G \text{ for all p in P}$$

$$\Pr\{\pi(p^*) = 0\} = 0$$

i.e. ALL pricing functions have such a representation

Lemma (Conditional Projection Theorem)



Find h_o that minimizes

$||p - h||_G$

h_o is unique iff

$p - h_o$ is orthogonal to H

i.e. $E((p-h_o)\ h|G) = 0$

call h_o is the projection of p onto H

Sketch of the proof:

Take p_o that satisfies A.3

z_o is the projection of p_o onto Z the conditionally complete linear subspace of P satisfying {p in P: $\pi(p) = 0$} and hence $\pi(z_o) = 0$.

$$r^b := \frac{p_0 - z_0}{\pi(p_0 - z_0)} = \frac{p_0 - z_0}{\pi(p_0)} \Rightarrow r^b \text{ orthogonal to Z}$$

Note that $\pi(r^b) = 1$ and r^b is called the benchmark return and r^b is conditionally orthogonal to Z. Note also that:

$p - \pi(p)\, r^b \in Z$: $\pi(p - \pi(p) r^b) = \pi(p) - \pi(p)\, \pi(r^b) = 0$

Since $p - \pi(p)\, r^b \in Z$

$$\Rightarrow E(([p - \pi(p)r^b]\, r^b \,|G) = \langle p - \pi(p)r^b, r^b\rangle_G = 0$$

and therefore

$$E(pr^b|G) = \langle p, r^b\rangle_G = \pi(p)\|r^b\|_G^2$$

moreover, the technical assumptions imply that $\Pr\{\|r^b\|_G = 0\} = 0$

$$\pi(p) = E\left(p\, \frac{r^b}{\|r^b\|_G^2} \mid G\right)$$

$$p^* \uparrow$$

Definition:

π has no arbitrage opportunities if for any $p \in P$

$$\Pr \{p \geq 0\} = 1 \qquad\qquad \Pr(\{\pi(p) \leq 0\} \cap \{p > 0\}) = 0$$

Lemma: π has no arbitrage iff

$$\Pr \{p^* > 0\} = 1$$

$$r^* := \frac{p^*}{\pi(p^*)}$$ - return on p^*

There are three sets that are of special interest:

R = $\{p \text{ in } P: \ \pi(p) = 1\}$ space of returns with price 1

Z = $\{p \text{ in } P: \ \pi(p) = 0\}$ space of returns with price 0

N = $\{z \text{ in } Z: \ E[z|G] = 0\}$ space of idiosyncratic risk

We will show that

$R = \{r : r = r^* + wz^* + n\}$

Where $r^* = p^*/ \pi(p^*)$ is return on benchmark asset, projection of stochastic discount factor onto space of assets.

$(wz^* + n)$ is in Z and n is in N

Lemma: (P,π) satisfies A.1-3 + technical assumptions

(i) $E(r^* z|G) = 0$

(ii) $\|r^*\|_G \leq \|r\|_G$, $r \in R$

$R = \{r : r = r^* + z\}$

Some implications of results obtained so far:

1. $\|r^*\|_G \leq \|r\|_G$, $r \in R$ implies that r^* conditional minimum mean-variance portfolio (more on this later)

2. When Z is non-empty asset pricing does not coincide with risk neutral pricing

Problem C $\min \|r\|_G^2$

$$\text{s.t. } E(r|G) = w \in I$$

$$w^* = \frac{w - E(r^*|G)}{E(z^*|G)}$$

$$E(r|G) = w \Leftrightarrow r = r^* + w^* z^* + n$$

$\Rightarrow n = 0$ solves the problem

What is the effect of omitting the info?

Consider R^*, Z^*, N^*

$$R^* = \{r : r = r^* + cz^* + n^*\}$$

Problem U $\min \|r\|^2$

s.t. $E(r) = c$

$$c^* = [c - E(r^*)] / E(z^*)$$

$E(r) = c \Leftrightarrow r = r^* + c^* z^* + n^*$

$n^* = 0$ solves the problem

Back to Big Picture

Want to test H_0 : r_i is on the frontier

Method: Test CAPM via regression

However, when we do not take into account conditioning information Hansen and Richard show that rejection of CAPM does not necessarily mean that a portfolio is not mean-variance efficient.

Conditional vs unconditional factor models in discount factor language

Consider CAPM

$$r^* = m^* = a - bR^W$$

Where R^W is the return on the market or wealth portfolio

$$1 = E_t m^*_{t+1} R^W_{t+1}$$

$$1 = E_t m^*_{t+1} R^f_{t+1}$$

We can solve for a and b

$$a = \frac{1}{R_t^f} + bE_t(R^W_{t+1})$$

$$b = (E_t(R^W_{t+1}) - R_t^f) / R_t^f \sigma_t^2(R^W_{t+1})$$

But this means a and b vary over time

If it is to price assets conditionally, the CAPM model must be linear factor model with time-varying weights:

$$m^*_{t+1} = a_t + b_t R^W_{t+1}$$

Therefore

$$1 = E_t[(a_t + b_t R^W_{t+1}) R_{t+1}]$$

While we can say:

$$1 = E[(a_t + b_t R^W_{t+1}) R_{t+1}]$$

We cannot say

$$1 = E[(a + bR_{t+1}^W)R_{t+1}]$$

Instead:

$$1 = E(a_t)E(R_{t+1}) + E(b_t)ER_{t+1}^W R_t$$

$$+ Cov(a_t, R_{t+1})$$

$$+ Cov(b_t, R_{t+1}^W R_{t+1})$$

Only when the covariances are zero then,

$$1 = E[E(a_t) + E(b_t)R_{t+1}^W)R_{t+1}]$$

In general

$$Cov\left(\frac{1}{R_t^f} + b_t E_t(R_{t+1}^W), R_{t+1}\right) \neq 0$$

$$Cov\left(\frac{E_t(R_{t+1}^W) - R_t^f}{R_t^f \sigma_t^2(R_{t+1}^W)}, R_{t+1} R_{t+1}^W\right) \neq 0$$

Of course if:

$$a_t = a, b_t = b$$

Then,

$$1 = E(a + bR_{t+1}^W)R_{t+1}$$

CONDITIONAL VS CONDITIONAL IN AN EXPECTED RUTURN/BETA MODEL

$$E_t(R^i) = R_t^f + \beta_t \lambda_t$$

Does not imply

$$E(R^i) = \alpha + \beta\lambda$$

If returns & factors are i.i.d. then Cov(.)= Cov_t(.), Var(.) = Var_t(.)

So that,

$$\beta_t = \beta$$