Math 31 Sec 7 EXAM I 26 September 2003

Name: KEY

I pledge that I have neither given nor received any unauthorized assistance on this exam.

____
(signature)

DIRECTIONS

1) Print your name and sign the honor pledge above. If the pledge is not signed, your exam will ***not*** be graded.

2) Check now that your test contains all ***6 pages*** and ***9 problems***.

3) You may use a calculator (except symbolic manipulators such as a TI-89, TI-92, or similar), but your answers must be given in their ***exact*** form. (i.e. $\sqrt{3}$ and not 1.73, π and not 3.14)

4) All work must be shown on this exam. ***No credit will be given for a correct answer without supporting work that leads to the answer***. When it is indicated that calculators are not to be used, clear non-calculator work must be shown.

5) Place a box around ***all*** of your final answers. Include units when necessary.

6) Notation and clarity count. Your job is to communicate mathematically; make what you are thinking clear.

7) Work quickly but thoroughly through the test. If you get stuck on a problem, move on to the next and return to it later after you've completed the problems you know how to do. ***Good Luck.***

(5) 1. Let $g(x) = \begin{cases} \frac{1-\cos x}{x} & x \neq 0 \\ c & x = 0 \end{cases}$ For what value of c is $g(x)$ continuous at 0? Explain.

$g(x)$ is continuous at 0 if $\lim\limits_{x\to 0} g(x) = g(0)$.

$\lim\limits_{x\to 0} g(x) = \lim\limits_{x\to 0} \frac{1-\cos x}{x} = 0$.

$\boxed{\text{So } g(x) \text{ is continuous at } 0 \text{ if } c = 0.}$

(10) 2. Write the equations for the horizontal asymptotes of the graph of $f(x) = \dfrac{x-3}{\sqrt{3x^2+1}}$

Horizontal asymptotes occur at $y = L$ if $\lim\limits_{x\to\infty} f(x) = L$ or $\lim\limits_{x\to-\infty} f(x) = L$.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x-3}{\sqrt{3x^2+1}} = \lim_{x\to\infty} \frac{(x-3)/x}{\sqrt{3x^2+1}/x} = \lim_{x\to\infty} \frac{1-\frac{3}{x}}{\frac{\sqrt{3x^2+1}}{\sqrt{x^2}}},$$

Since as $x\to\infty$, $x>0$ and $\sqrt{x^2} = |x| = x$.

So: $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{1-\frac{3}{x}}{\sqrt{3+\frac{1}{x^2}}} = \dfrac{1}{\sqrt{3}}$

as $x\to-\infty$, $x<0$, and $\sqrt{x^2} = |x| = -x$, so we have

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{(x-3)/x}{\sqrt{3x^2+1}/x} = \lim_{x\to-\infty} \frac{1-\frac{3}{x}}{-\frac{\sqrt{3x^2+1}}{\sqrt{x^2}}} = \lim_{x\to-\infty} \frac{1-\frac{3}{x}}{-\sqrt{3+\frac{1}{x^2}}}$$

$$= \frac{1}{-\sqrt{3}}.$$

$\boxed{\text{Horizontal asymptotes occur at } y = \frac{1}{\sqrt{3}} \text{ and } y = \frac{-1}{\sqrt{3}}}$

(6 each) 3. Calculate the following limits (if they exist) or state why it does not exist.

(a) $\lim_{t \to -2} \frac{t^2 - 3t - 10}{t^2 - 4}$ $= \lim_{t \to -2} \frac{(t+2)(t-5)}{(t+2)(t-2)} = \lim_{t \to -2} \frac{t-5}{t-2} = \frac{-7}{-4} = \boxed{\frac{7}{4}}$

(b) $\lim_{x \to 1} \frac{|x-1|}{x-1}$ Consider the one-sided limits:

$\lim_{x \to 1^-} \frac{|x-1|}{x-1} = \lim_{x \to 1^-} \frac{-(x-1)}{x-1} = \lim_{x \to 1^-} -1 = -1$

$\lim_{x \to 1^+} \frac{|x-1|}{x-1} = \lim_{x \to 1^+} \frac{x-1}{x-1} = \lim_{x \to 1^+} 1 = 1$

Since the one-sided limits are not equal, the limit $\boxed{\text{does not exist.}}$

(c) $\lim_{\theta \to 0} \frac{\pi\theta \cos(3\theta)}{sin(4\theta)}$ $= \lim_{\theta \to 0} \frac{\pi \cdot \cos 3\theta}{4} \cdot \frac{4\theta}{\sin(4\theta)} = \lim_{\theta \to 0} \frac{\pi \cdot \cos 3\theta}{4} \cdot \lim_{\theta \to 0} \frac{4\theta}{\sin 4\theta}$

$= \frac{\pi \cdot 1}{4} \cdot 1 = \boxed{\frac{\pi}{4}}$

(d) $\lim_{x \to 3^+} \frac{x^2 - 7x + 10}{x^2 - 9}$ $= \lim_{x \to 3^+} \frac{\overbrace{(x-2)}^{1}\overbrace{(x-5)}^{-2}}{\underbrace{(x-3)}_{0^+}\underbrace{(x+3)}_{6}} = \boxed{-\infty}$

(10) 4. Sketch the graph of a function $f(x)$ satisfying all of the following:

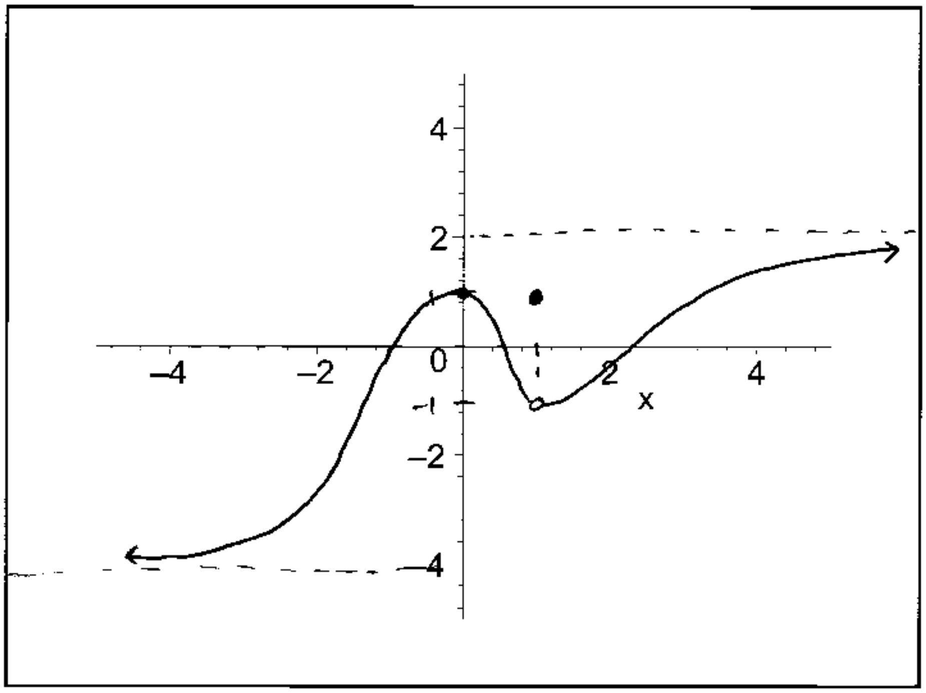
A. $\lim_{x \to \infty} f(x) = 2$

B. $\lim_{x \to -\infty} f(x) = -4$

C. $f(0) = 1$ and $f'(0) = 0$

D. $\lim_{x \to 1} f(x) = -1$, but f is not continuous at 1

Solutions will vary, this is one example.



(12) 5. Write the equation of the tangent line to the graph of $f(x) = \dfrac{x^2}{x+4}$ at the point $(2, \frac{2}{3})$.

$$m_{tan} = f'(x) = \frac{(x+4)\cdot 2x - x^2(1)}{(x+4)^2}$$

$$f'(2) = \frac{6\cdot 4 - 4}{6^2} = \frac{20}{36} = \frac{5}{9}.$$

The line with slope $\frac{5}{9}$ through the point $(2, \frac{2}{3})$ has the equation:

$$\boxed{y - \frac{2}{3} = \frac{5}{9}(x-2)}$$

(5 each) 6. (a) State the limit definition of $f'(x)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

(b) Use the limit definition to compute $f'(x)$ for $f(x) = \sqrt{8x+3}$.
You **must** use the limit definition for this; using any other method will result in zero credit.

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{8(x+h)+3} - \sqrt{8x+3}}{h} \cdot \frac{\sqrt{8(x+h)+3} + \sqrt{8x+3}}{\sqrt{8(x+h)+3} + \sqrt{8x+3}}$$

$$= \lim_{h\to 0} \frac{8(x+h)+3-(8x+3)}{h\left(\sqrt{8(x+h)+3}+\sqrt{8x+3}\right)}$$

$$= \lim_{h\to 0} \frac{8x+8h+3-8x-3}{h\left(\sqrt{8(x+h)+3}+\sqrt{8x+3}\right)}$$

$$= \lim_{h\to 0} \frac{8h}{h\left(\sqrt{8(x+h)+3}+\sqrt{8x+3}\right)}$$

$$= \lim_{h\to 0} \frac{8}{\sqrt{8(x+h)+3}+\sqrt{8x+3}}$$

$$= \frac{8}{\sqrt{8(x+0)+3}+\sqrt{8x+3}} = \frac{8}{2\sqrt{8x+3}} = \boxed{\frac{4}{\sqrt{8x+3}}}$$

(5) 7. Use the Intermediate Value Theorem to show that $f(x) = -3\sin^2 x - 5x + 6$ has a root in the interval $[0, \pi]$. Do not actually find this root!

$f(0) = -3\cdot 0 - 5\cdot 0 + 6 = 6$

$f(\pi) = -3\cdot 0 - 5\pi + 6 = 6 - 5\pi \approx -9.708$

f is continuous, $f(0) > 0$, $f(\pi) < 0$; So by the Intermediate Value Theorem, there is an x, $0 \le x \le \pi$, such that $f(x) = 0$.

(6 each) 8. Find $D_x y$ for the following y.

(a) $y = (3x^2 + 2x^{-3}) \cdot (\frac{4}{x} - 3x^4)$

$D_x y = (3 \cdot 2x + 2 \cdot -3x^{-4})(\frac{4}{x} - 3x^4) + (3x^2 + 2x^{-3})(-\frac{4}{x^2} - 3 \cdot 4x^3)$

$= \boxed{(6x - 6x^{-4})(\frac{4}{x} - 3x^4) + (3x^2 + 2x^{-3})(-\frac{4}{x^2} - 12x^3)}$

(b) $y = 3\pi^2 x - 4x^3$

$D_x y = 3\pi^2 - 4 \cdot 3x^2$

$= \boxed{3\pi^2 - 12x^2}$

(6 each) 9. Suppose an arrow is shot upward with velocity of 58 m/s, and its height in meters after t seconds given by $h(t) = 58t - .83t^2$.

(a) Find the average velocity of the arrow between 1 and 3 seconds.

average velocity is $\frac{h(3) - h(1)}{3-1} \frac{m}{sec} = \frac{58(3) - .83 \cdot 9 - 58 + .83}{2}$ m/s

$= \frac{109.36}{2}$ m/s $= \boxed{54.86 \text{ m/s}}$

(b) Find the instantaneous velocity when the arrow lands ($h(t) = 0$).

instantaneous velocity is $h'(t) = 58 - 2 \cdot .83t$ m/s
$= 58 - 1.66t$ m/s

$h(t) = 0$ for $58t - .83t^2 = 0$

$t(58 - .83t) = 0$

$t = 0$ or $t = 58/.83$.

$h'(58/.83) = 58 - 1.66 \cdot 58/.83$ m/s

$= \boxed{-58 \text{ m/s}}$