(5) 1. Give the appropriate ϵ, δ definition of $\lim_{x \to c} f(x) = L$.

for all $\varepsilon > 0$, there is a $\delta > 0$ such that
$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

(8) 2. Define $f(x) = \dfrac{x^2 - 25}{6x - 30}$ to make the function continuous at $x = 5$.

f(x) is continuous at $x = 5$ if
$\lim_{x \to 5} f(x) = f(5)$.

$f(5) = \boxed{\dfrac{5}{3}}$

$$\lim_{x \to 5} f(x) = \lim_{x \to 5} \frac{x^2 - 25}{6x - 30} = \lim_{x \to 5} \frac{(x+5)(x-5)}{6(x-5)} = \lim_{x \to 5} \frac{x+5}{6} = \frac{10}{6}$$

So by defining $f(5)$ to be $\frac{10}{6} = \frac{5}{3}$, f is continuous.

(10) 3. Use the **limit definition of the derivative** to calculate $f'(x)$ for $f(x) = 2x^2 + 3x - 1$

$\boxed{f'(x) = 4x + 3}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{2(x+h)^2 + 3(x+h) - 1 - (2x^2 + 3x - 1)}{h}$$
$$= \lim_{h \to 0} \frac{2x^2 + 4xh + 2h^2 + 3x + 3h - 1 - 2x^2 - 3x + 1}{h}$$
$$= \lim_{h \to 0} \frac{4xh + 2h^2 + 3h}{h}$$
$$= \lim_{h \to 0} \frac{h(4x + 2h + 3)}{h}$$
$$= \lim_{h \to 0} 4x + 2h + 3$$
$$= 4x + 3$$

4. Find each of the following limits (if it exists). If the limit does not exist, explain why.

(5) (a) $\lim_{x\to 0} \frac{\sqrt{x+3}-\sqrt{3}}{x} \frac{(\sqrt{x+3}+\sqrt{3})}{(\sqrt{x+3}+\sqrt{3})}$

$= \lim_{x\to 0} \frac{x+3-3}{x(\sqrt{x+3}+\sqrt{3})}$

$= \lim_{x\to 0} \frac{1}{\sqrt{x+3}+\sqrt{3}}$

$= \frac{1}{\sqrt{3}+\sqrt{3}} = \frac{1}{2\sqrt{3}}$

$\boxed{\frac{1}{2\sqrt{3}}}$

(5) (b) $\lim_{x\to 0} \frac{|x|}{x}$

$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = \lim_{x\to 0^-} -1 = -1$

$\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} 1 = 1$

Since the one-sided limits are not equal, the limit does not exist.

$\boxed{\text{D.N.E.}}$

(5) (c) $\lim_{x\to 0} \frac{\sin 2x}{5x}$

$= \lim_{x\to 0} 2\frac{\sin 2x}{5\cdot 2x}$

$= \frac{2}{5} \cdot \lim_{x\to 0} \frac{\sin 2x}{2x}$

$= \frac{2}{5} \cdot 1 = \frac{2}{5}$

$\boxed{\frac{2}{5}}$

(10) 5. Find the equation of the tangent line to the graph of $\frac{x^4}{x^2+1}$ at the point where $x = 1$.

$y = \frac{x^4}{x^2+1}$.

$D_x y = \frac{4x^3(x^2+1) - x^4(2x)}{(x^2+1)^2}$

when $x=1$, $D_x y$ is $\frac{4\cdot 2 - 1\cdot 2}{2^2} = \frac{8-2}{4} = \frac{6}{4}$

$\boxed{y - \frac{1}{2} = \frac{6}{4}(x-1)}$

when $x=1$, $y = \frac{1}{1+1} = \frac{1}{2}$

(10) 6. Find the **coordinates** on the curve $y = (x-2)^2$ at which the tangent line is perpendicular to the line $y = 2x + 3$.

$D_x y = 2(x-2)$

want slope to be $-\frac{1}{2}$: $2(x-2) = -\frac{1}{2}$

$x - 2 = -\frac{1}{4}$

$x = 1\frac{3}{4} = \frac{7}{4}$

at $x = \frac{7}{4}$, $y = \left(\frac{7}{4} - 2\right)^2 = \left(-\frac{1}{4}\right)^2 = \frac{1}{16}$

$\boxed{\left(\frac{7}{4}, \frac{1}{16}\right)}$

7. Calculate $f'(x) = \dfrac{dy}{dx}$ for each of the following functions. **Do Not Simplify!**

(5) (a) $f(x) = 2x^3 - \dfrac{3}{x^2} + \pi$

$f'(x) = 2 \cdot 3x^2 - 3 \cdot -2x^{-3}$

$= 6x^2 + \dfrac{6}{x^3}$

$\boxed{6x^2 + \dfrac{6}{x^3}}$

(5) (b) $f(x) = (3x^4 + 14x^3 - x^2 - 1)(4x^3 - 5)$

$f'(x) = (3 \cdot 4x^3 + 14 \cdot 3x^2 - 2x)(4x^3 - 5) +$

$(3x^4 + 14x^3 - x^2 - 1) \cdot (4 \cdot 3x^2)$

$\boxed{= (12x^3 + 42x^2 - 2x)(4x^3 - 5) + (3x^4 + 14x^3 - x^2 - 1)(12x^2)}$

(5) (c) $f(x) = \dfrac{4x^3 + x}{x^2 - 3}$

$f'(x) = \dfrac{(x^2-3)(4 \cdot 3x^2 + 1) - (4x^3 + x) \cdot (2x)}{(x^2-3)^2}$

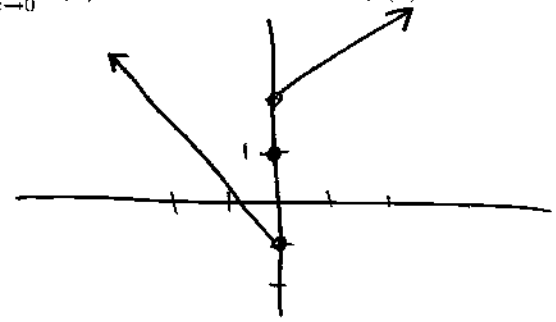
$= \boxed{\dfrac{(x^2-3)(12x^2+1) - 2x(4x^3 + x)}{(x^2-3)^2}}$

Please go on to the next page...

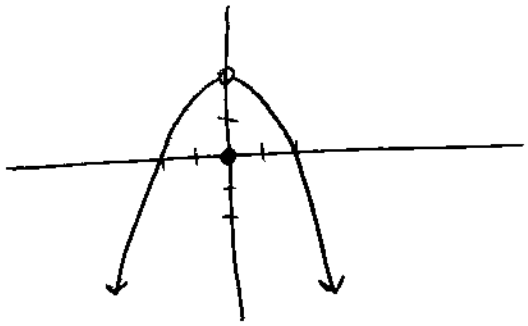
These are merely examples. Solutions will vary.

8. For each of the following, draw a graph of a function $f(x)$ which is described.

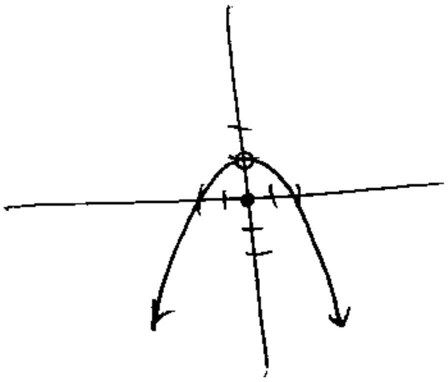
(3) (a) $\lim_{x \to 0} f(x)$ does not exist, and $f(0) = 1$



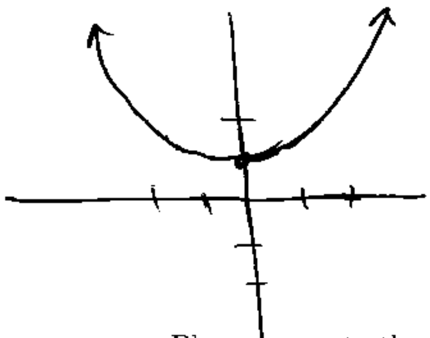
(3) (b) $\lim_{x \to 0} f(x)$ exists, but $f(x)$ is not continuous at 0.



(3) (c) $\lim_{x \to 0} f(x) = 1$, but $f(x)$ is not differentiable at 0



(3) (d) $f'(0) = 0$



9. Find the following limits (if it exists). If it does not exist, state why.

(5) (a) $\lim_{x\to\infty} \frac{4x^3+3x-1}{-3x^4+x^2+2}$

Divide all by x^4:

$$= \lim_{x\to\infty} \frac{\frac{4}{x}+\frac{3}{x^3}-\frac{1}{x^4}}{-3+\frac{1}{x^2}+\frac{2}{x^4}}$$

$$= \frac{0}{-3}$$

$$= 0$$

$\boxed{0}$

(5) (b) $\lim_{t\to 4} \frac{t^2-5t-6}{t-4}$

$\lim_{t\to 4} t^2-5t-6 = 16-20-6=-10$

$\boxed{\text{D.N.E.}}$

$\lim_{t\to 4} t-4 = 0.$

As $t\to 4^-$, $t-4\to 0^-$. So $\lim_{t\to 4^-} \frac{t^2-5t-6}{t-4} = \infty$

As $t\to 4^+$, $t-4\to 0^+$. So $\lim_{t\to 4^+} \frac{t^2-5t-6}{t-4} = -\infty$

Not equal, so the limit does not exist.

(5) (c) $\lim_{x\to 3^-} \frac{x+5}{\sin(|x-3|)}$

$\lim_{x\to 3^-} x+5 = 8$

$\boxed{\infty}$

$\lim_{x\to 3^-} |x-3| = 0^+$

$\lim_{x\to 3^-} \sin|x-3| = 0^+$

So $\lim_{x\to 3^-} \frac{x+5}{\sin|x-3|} = \infty$