Supplement to "Information acquisition and mutual funds"

Diego García and Joel M. Vanden

Proof of Proposition 1.

Proposition 1 characterizes our equilibrium at date 2, which is the asset trading stage of the model. At this stage we take the sets $\mathcal{A} = \left\{\alpha_{ik} : i = 1, \ldots, m_k; k \in \left[0, \bar{k}\right]\right\}$ and $\mathcal{M} = \left\{m_k : k \in \left[0, \bar{k}\right]\right\}$ as given. Proposition 1 generalizes the standard noisy rational expectations equilibrium to the case in which m_k informed fund managers establish mutual funds and market their private information to the uninformed households in group k. The ith informed fund manager in group k is compensated by the proportional fee α_{ik}.

The optimization problem of the ith fund manager in group k is

$$\max_{\gamma_{ik}} \quad -e^{\tau c - \tau \alpha_{ik} P_{ik} + \tau \alpha_{ik} \gamma_{ik} P_x} \mathbb{E}\left[e^{-\tau \alpha_{ik} \gamma_{ik} X} \middle| P_x, Y_{ik} \right]. \tag{A-1}$$

Using standard properties of Gaussian random variables, it is straightforward to evaluate the conditional expectation in (A-1) to get

$$\mathbb{E}\left[e^{-\tau \alpha_{ik} \gamma_{ik} X} \middle| P_x, Y_{ik} \right] = e^{-\tau \alpha_{ik} \gamma_{ik} \mathbb{E}[X|P_x, Y_{ik}] + 0.5 \tau^2 \alpha_{ik}^2 \gamma_{ik}^2 \mathrm{var}[X|P_x, Y_{ik}]},$$

where

$$\mathbb{E}\left[X|P_x, Y_{ik}\right] = \mu_x + \left[\frac{(Y_{ik} - \mu_x)}{\sigma_\epsilon^2} + \frac{b\left(P_x - \bar{P}_x\right)}{d^2 \sigma_u^2} \right] \mathrm{var}(X|P_x, Y_{ik}), \tag{A-2}$$

$$\mathrm{var}(X|P_x, Y_{ik}) = \left[\frac{1}{\sigma_x^2} + \frac{1}{\sigma_\epsilon^2} + \frac{b^2}{d^2 \sigma_u^2} \right]^{-1}, \tag{A-3}$$

and $\bar{P}_x = \mathbb{E}\left[P_x\right]$. Solving (A-1) then produces the familiar mean-variance expression

$$\hat{\gamma}_{ik} = \frac{\mathbb{E}\left[X|P_x, Y_{ik}\right] - P_x}{\alpha_{ik} \tau \, \mathrm{var}(X|P_x, Y_{ik})}, \tag{A-4}$$

which corresponds to (15) in the paper. It is easy to verify that the second-order condition for a maximum is satisfied. Using (A-2)-(A-4), note that the demand of the ith manager in group k can be rewritten as

$$\begin{aligned}\hat{\gamma}_{ik} &= \frac{\mu_x}{\tau\alpha_{ik}}\left(\frac{1}{\sigma_x^2}+\frac{b^2}{d^2\sigma_u^2}\right)-\frac{b\bar{P}_x}{\tau\alpha_{ik}d^2\sigma_u^2}+\frac{Y_{ik}}{\tau\alpha_{ik}\sigma_\epsilon^2}+\frac{1}{\tau\alpha_{ik}}\left(\frac{b}{d^2\sigma_u^2}-\frac{1}{\mathrm{var}(X|Y_{ik},P_x)}\right)P_x \\ &\equiv v_{ik}+r_{ik}Y_{ik}+q_{ik}P_x.\end{aligned}$$

The jth household's problem in group k is to choose the stock investment θ_{jk} and fund holdings ϕ_{ijk}, for $i = 1, \ldots, m_k$, to solve

$$\max_{\theta_{jk},\{\phi_{ijk}\}_{i=1}^{m_k}} \quad \mathbb{E}\left[-e^{-\tau W_{jk}}\middle| P_x\right]. \tag{A-5}$$

The household's wealth can be expressed as

$$\begin{aligned}W_{jk} &= \theta_{jk}(X-P_x)+\sum_{l=1}^{m_k}\left(\phi_{ljk}(1-\alpha_{lk})[P_{lk}+\hat{\gamma}_{lk}(X-P_x)]-\phi_{ljk}P_{lk}\right) \\ &= \theta_{jk}(X-P_x)+\sum_{l=1}^{m_k}\left(\phi_{ljk}(1-\alpha_{lk})\hat{\gamma}_{lk}(X-P_x)-\alpha_{lk}\phi_{ljk}P_{lk}\right).\end{aligned}$$

Recalling that the optimal strategy for fund manager i in group k can be expressed as $\hat{\gamma}_{ik} = v_{ik} + r_{ik}Y_{ik} + q_{ik}P_x$, the household's wealth can be rewritten as

$$\begin{aligned}W_{jk} &= \theta_{jk}(X-P_x)+\sum_{l=1}^{m_k}\phi_{ljk}(1-\alpha_{lk})(v_{lk}+q_{lk}P_x)(X-P_x)-\sum_{l=1}^{m_k}\alpha_{lk}\phi_{ljk}P_{lk} \\ &\quad+\sum_{l=1}^{m_k}\phi_{ljk}(1-\alpha_{lk})r_{lk}(X+\epsilon_{lk})(X-P_x) \\ &= \theta_{jk}(X-P_x)+\sum_{l=1}^{m_k}\phi_{ljk}(1-\alpha_{lk})(v_{lk}+q_{lk}P_x+r_{lk}P_x)(X-P_x)-\sum_{l=1}^{m_k}\alpha_{lk}\phi_{ljk}P_{lk} \\ &\quad+\sum_{l=1}^{m_k}\phi_{ljk}(1-\alpha_{lk})r_{lk}(X-P_x)^2+\sum_{l=1}^{m_k}\phi_{ljk}(1-\alpha_{lk})r_{lk}\epsilon_{lk}(X-P_x).\end{aligned}$$

Define the vector $\boldsymbol{\eta}_k^\top = (X - P_x, \epsilon_{1k}, \ldots, \epsilon_{m_k k})$, where $\epsilon_{1k}, \ldots, \epsilon_{m_k k}$ denotes the signal error terms for the m_k managers in group k. From the previous expression one can verify that the jth household's wealth can be expressed as

$$W_{jk} = c_{jk} + \boldsymbol{b}_{jk}^{\top}\boldsymbol{\eta}_k + \boldsymbol{\eta}_k^{\top}\boldsymbol{A}_{jk}\boldsymbol{\eta}_k,$$

where

$$\begin{aligned}
c_{jk} &= -\sum_{l=1}^{m_k} \alpha_{lk}\phi_{ljk}P_{lk}, \\
\boldsymbol{b}_{jk} &= \begin{pmatrix} \theta_{jk} + \sum_{l=1}^{m_k}\phi_{ljk}(1-\alpha_{lk})(v_{lk} + q_{lk}P_x + r_{lk}P_x) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \\
\boldsymbol{A}_{jk} &= \begin{pmatrix} \sum_{l=1}^{m_k}\phi_{ljk}(1-\alpha_{lk})r_{lk} & \frac{\phi_{1jk}(1-\alpha_{1k})r_{1k}}{2} & \dots & \frac{\phi_{m_kjk}(1-\alpha_{m_kk})r_{m_kk}}{2} \\ \frac{\phi_{1jk}(1-\alpha_{1k})r_{1k}}{2} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\phi_{m_kjk}(1-\alpha_{m_kk})r_{m_kk}}{2} & 0 & \dots & 0 \end{pmatrix}.
\end{aligned}$$

We note that $\boldsymbol{b}_{jk} \in \mathbb{R}^{m_k+1}$ and $\boldsymbol{A}_{jk} \in \mathbb{R}^{(m_k+1)\times(m_k+1)}$.

Using Lemma 1 in the Appendix of this Supplement, the jth household's objective function at the trading stage can be written as

$$\mathbb{E}\left[-e^{-\tau W_{jk}}|P_x\right] = -|\boldsymbol{B}_{jk}|^{-1/2}e^{-\tau c_{jk} - \tau\boldsymbol{b}_{jk}^{\top}\boldsymbol{\mu}_k - \tau\boldsymbol{\mu}_k{}^{\top}\boldsymbol{A}_{jk}\boldsymbol{\mu}_k + \frac{\tau^2}{2}\boldsymbol{g}_{jk}{}^{\top}\boldsymbol{C}_{jk}\boldsymbol{g}_{jk}}, \tag{A-6}$$

where $\boldsymbol{\mu}_k = \mathbb{E}[\boldsymbol{\eta}_k|P_x]$ and $\boldsymbol{V}_k = \text{var}(\boldsymbol{\eta}_k|P_x)$ denote the conditional moments of $\boldsymbol{\eta}_k$, and

$$\begin{aligned}
\boldsymbol{B}_{jk} &= \boldsymbol{I} + 2\tau\boldsymbol{V}_k\boldsymbol{A}_{jk}, \\
\boldsymbol{C}_{jk} &= \boldsymbol{B}_{jk}^{-1}\boldsymbol{V}_k, \\
\boldsymbol{g}_{jk} &= \boldsymbol{b}_{jk} + 2\boldsymbol{A}_{jk}\boldsymbol{\mu}_k,
\end{aligned}$$

where $\boldsymbol{I} \in \mathbb{R}^{(m_k+1)\times(m_k+1)}$ denotes the identity matrix, and we note $\boldsymbol{B}_{jk}, \boldsymbol{C}_{jk} \in \mathbb{R}^{(m_k+1)\times(m_k+1)}$, and $\boldsymbol{g}_{jk} \in \mathbb{R}^{m_k+1}$. Let $\boldsymbol{\phi}_{jk} \in \mathbb{R}^{m_k}$ denote the vector of mutual fund holdings for the jth household.

The next Claim characterizes the solution to the jth household's problem in (A-5).

Claim 1. *The optimal trading strategies for the jth household in group k, $\hat{\theta}_{jk}$ and $\left\{\hat{\phi}_{ijk}\right\}_{i=1}^{m_k}$, satisfy*

$$\hat{\theta}_{jk} = \frac{(\mathbb{E}[X|P_x]-P_x)}{\tau}\left[\mathrm{var}(X|P_x)^{-1}+\tau\sum_{l=1}^{m_k}\hat{\phi}_{ljk}(1-\alpha_{lk})r_{lk}\right] - \sum_{l=1}^{m_k}\hat{\phi}_{ljk}(1-\alpha_{lk})\mathbb{E}[\hat{\gamma}_{lk}|P_x], \tag{A-7}$$

$$\alpha_{ik}P_{ik} = \frac{(1-\alpha_{ik})r_{ik}(1-\tau\sigma_\epsilon^2\hat{\phi}_{ijk}(1-\alpha_{ik})r_{ik})}{\mathrm{var}(X|P_x)^{-1}+2\tau\sum_{l=1}^{m_k}\hat{\phi}_{ljk}r_{lk}(1-\alpha_{lk})-\tau^2\sigma_\epsilon^2\sum_{l=1}^{m_k}\hat{\phi}_{ljk}^2r_{lk}^2(1-\alpha_{lk})^2}. \tag{A-8}$$

Proof of Claim 1.

After multiplying (A-6) by -1, we take the natural logarithm and divide by $-\tau$ to get the certainty equivalent wealth of the jth household in group k. Thus the optimization problem of the jth household in group k can be rewritten as:

$$\max_{\theta_{jk},\boldsymbol{\phi}_{jk}} \quad M(\theta_{jk},\boldsymbol{\phi}_{jk})+\frac{1}{2\tau}\log(|\boldsymbol{B}(\boldsymbol{\phi}_{jk})|)-\frac{\tau}{2}\boldsymbol{g}(\theta_{jk},\boldsymbol{\phi}_{jk})^\top\boldsymbol{C}(\boldsymbol{\phi}_{jk})\boldsymbol{g}(\theta_{jk},\boldsymbol{\phi}_{jk}), \tag{A-9}$$

where

$$M(\theta_{jk},\boldsymbol{\phi}_{jk}) = (\mathbb{E}\left[X|P_x\right]-P_x)\left(\theta_{jk}+\sum_{l=1}^{m_k}(1-\alpha_{lk})\phi_{ljk}\mathbb{E}[\hat{\gamma}_{lk}|P_x]\right)-\sum_{l=1}^{m_k}\alpha_{lk}\phi_{ljk}P_{lk},$$

$$\boldsymbol{g}(\theta_{jk},\boldsymbol{\phi}_{jk}) = \begin{pmatrix} \theta_{jk}+\sum_{l=1}^{m_k}\phi_{ljk}(1-\alpha_{lk})\left[\mathbb{E}[\hat{\gamma}_{lk}|P_x]+r_{lk}(\mathbb{E}\left[X|P_x\right]-P_x)\right] \\ (\mathbb{E}\left[X|P_x\right]-P_x)r_{1k}(1-\alpha_{1k})\phi_{1jk} \\ \vdots \\ (\mathbb{E}\left[X|P_x\right]-P_x)r_{m_kk}(1-\alpha_{m_kk})\phi_{m_kjk} \end{pmatrix},$$

$$\mathbb{E}[\hat{\gamma}_{lk}|P_x] = v_{lk}+q_{lk}P_x+r_{lk}\mathbb{E}\left[Y_{lk}|P_x\right].$$

We note that $M(\cdot,\cdot):\mathbb{R}^{m_k+1}\to\mathbb{R}$, $\boldsymbol{g}(\cdot,\cdot):\mathbb{R}^{m_k+1}\to\mathbb{R}^{m_k+1}$, and $\boldsymbol{A}(\cdot),\boldsymbol{B}(\cdot),\boldsymbol{C}(\cdot):\mathbb{R}^{m_k}\to\mathbb{R}^{(m_k+1)\times(m_k+1)}$. For notational convenience, we have omitted in (A-9), and in the remainder of the Claim's proof, the group subscript k on the functions M, $\boldsymbol{g}$, $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$, as well as on the conditional moments $\boldsymbol{\mu}$ and $\boldsymbol{V}$.

The first order conditions for the problem in (A-9) are

$$\frac{\mathbb{E}[X|P_x] - P_x}{\tau} = \sum_{l=1}^{m_k+1} C_{1l}(\boldsymbol{\phi}_{jk}) g_l(\theta_{jk}, \boldsymbol{\phi}_{jk}), \tag{A-10}$$

$$\frac{\partial M}{\partial \phi_{ijk}} + \text{trace}\left(\boldsymbol{C}\frac{\partial \boldsymbol{A}}{\partial \phi_{ijk}}\right) = \tau\left(\frac{\partial \boldsymbol{g}}{\partial \phi_{ijk}}^{\top} - \tau \boldsymbol{g}^{\top}\boldsymbol{C}\frac{\partial \boldsymbol{A}}{\partial \phi_{ijk}}\right)\boldsymbol{C}\boldsymbol{g}, \tag{A-11}$$

where (A-11) holds for all $i = 1, \ldots, m_k$. In (A-10)-(A-11), we have used (A-65)-(A-63) together with the definitions of the matrices $\boldsymbol{B}$ and $\boldsymbol{C}$. In particular, note that

$$\begin{aligned} \frac{\partial \boldsymbol{B}}{\partial \phi_{ijk}} &= 2\tau \boldsymbol{V}\frac{\partial \boldsymbol{A}}{\partial \phi_{ijk}}, \\ \frac{\partial \boldsymbol{C}}{\partial \phi_{ijk}} &= \frac{\partial \boldsymbol{B}^{-1}}{\partial \phi_{ijk}}\boldsymbol{V} = -\boldsymbol{B}^{-1}2\tau\boldsymbol{V}\frac{\partial \boldsymbol{A}}{\partial \phi_{ijk}}\boldsymbol{B}^{-1}\boldsymbol{V} = -2\tau\boldsymbol{C}\frac{\partial \boldsymbol{A}}{\partial \phi_{ijk}}\boldsymbol{C}. \end{aligned}$$

Next we note that $\frac{\partial \boldsymbol{A}}{\partial \phi_{ijk}}$ has only three non-zero elements: the $(1,1)$ element, equal to $(1-\alpha_{ik})r_{ik}$; and the $(1, i+1)$ and $(i+1, 1)$ elements, both equal to $(1-\alpha_{ik})r_{ik}/2$. Similarly, $\frac{\partial \boldsymbol{g}}{\partial \phi_{ijk}}$ has only two non-zero elements: the first element, equal to $(1-\alpha_{ik})\mathbb{E}[\hat{\gamma}_{ik}|P_x] + (\mathbb{E}[X|P_x] - P_x)(1-\alpha_{ik})r_{ik}$, and the $(i+1)$ element, equal to $(\mathbb{E}[X|P_x] - P_x)(1-\alpha_{ik})r_{ik}$.

Next we define $\boldsymbol{f} = \boldsymbol{C}\boldsymbol{g}$. From the previous comment on $\frac{\partial \boldsymbol{g}}{\partial \phi_{ijk}}$, we have

$$\begin{aligned} \frac{\partial \boldsymbol{g}}{\partial \phi_{ijk}}^{\top}\boldsymbol{C}\boldsymbol{g} &= f_1[(1-\alpha_{ik})\mathbb{E}[\hat{\gamma}_{ik}|P_x] + (\mathbb{E}[X|P_x] - P_x)(1-\alpha_{ik})r_{ik}] \\ &\quad + f_{i+1}(\mathbb{E}[X|P_x] - P_x)(1-\alpha_{ik})r_{ik} \\ &= f_1(1-\alpha_{ik})\mathbb{E}[\hat{\gamma}_{ik}|P_x] + (\mathbb{E}[X|P_x] - P_x)(1-\alpha_{ik})r_{ik}(f_1 + f_{i+1}). \end{aligned}$$

Furthermore, from the structure of $\frac{\partial \boldsymbol{A}}{\partial \phi_{ijk}}$, we have

$$\begin{aligned} \boldsymbol{g}^{\top}\boldsymbol{C}\frac{\partial \boldsymbol{A}}{\partial \phi_{ijk}}\boldsymbol{C}\boldsymbol{g} &= \boldsymbol{f}^{\top}\frac{\partial \boldsymbol{A}}{\partial \phi_{ijk}}\boldsymbol{f} \\ &= f_1\left(r_{ik}(1-\alpha_{ik})f_1 + \frac{r_{ik}(1-\alpha_{ik})f_{i+1}}{2}\right) + \frac{f_1 f_{i+1} r_{ik}(1-\alpha_{ik})}{2} \\ &= f_1 r_{ik}(1-\alpha_{ik})\left(f_1 + f_{i+1}\right). \end{aligned}$$

Noting that (A-10) implies that $f_1 = (\mathbb{E}[X|P_x] - P_x)/\tau$, one obtains

$$\frac{\partial \boldsymbol{g}}{\partial \phi_{ijk}}^{\top} \boldsymbol{C}\boldsymbol{g} - \tau \boldsymbol{g}^{\top} \boldsymbol{C} \frac{\partial \boldsymbol{A}}{\partial \phi_{ijk}} \boldsymbol{C}\boldsymbol{g} = \frac{(\mathbb{E}[X|P_x] - P_x)}{\tau}(1-\alpha_{ik})\mathbb{E}[\hat{\gamma}_{ik}|P_x].$$

From the definition of $M(\theta_{jk}, \boldsymbol{\phi}_{jk})$ one has

$$\frac{\partial M}{\partial \phi_{ijk}} = (\mathbb{E}[X|P_x] - P_x)(1-\alpha_{ik})\mathbb{E}[\hat{\gamma}_{ik}|P_x] - \alpha_{ik}P_{ik},$$

which together with the previous equation proves that (A-11) reduces to

$$\alpha_{ik}P_{ik} \quad = \quad \operatorname{trace}\left(\boldsymbol{C}\frac{\partial \boldsymbol{A}}{\partial \phi_{ijk}}\right) = (1-\alpha_{ik})r_{ik}(C_{11} + C_{1,i+1}), \tag{A-12}$$

where the last equality follows from the structure of $\frac{\partial \boldsymbol{A}}{\partial \phi_{ijk}}$.

Next we note that $\boldsymbol{C} = (\boldsymbol{V}^{-1} + 2\tau\boldsymbol{A})^{-1}$, where $\boldsymbol{V}^{-1} + 2\tau\boldsymbol{A}$ has the block structure

$$\boldsymbol{V}^{-1} + 2\tau\boldsymbol{A} = \begin{bmatrix} D_{11} & \boldsymbol{D}_{12} \\ \boldsymbol{D}_{21} & \boldsymbol{D}_{22} \end{bmatrix},$$

where

$$\begin{aligned} D_{11} &= \operatorname{var}(X|P_x)^{-1} + 2\tau \sum_{l=1}^{m_k} \phi_{ljk} r_{lk}(1-\alpha_{lk}), \\ \boldsymbol{D}_{12}^{\top} &= \boldsymbol{D}_{21} = \tau \begin{pmatrix} \phi_{1jk} r_{1k}(1-\alpha_{1k}) \\ \vdots \\ \phi_{m_k jk} r_{m_k k}(1-\alpha_{m_k k}) \end{pmatrix}, \\ \boldsymbol{D}_{22} &= \frac{1}{\sigma_\epsilon^2} \boldsymbol{I}_{m_k}. \end{aligned}$$

Using (A-66) from Lemma 2 in the Appendix to this Supplement, we have

$$\begin{aligned} C_{11} &= \frac{1}{\operatorname{var}(X|P_x)^{-1} + 2\tau \sum_{l=1}^{m_k} \phi_{ljk} r_{lk}(1-\alpha_{lk}) - \tau^2 \sigma_\epsilon^2 \sum_{l=1}^{m_k} \phi_{ljk}^2 r_{lk}^2 (1-\alpha_{lk})^2}, \\ C_{1,i+1} &= -\frac{\tau \sigma_\epsilon^2 \phi_{ijk} r_{ik}(1-\alpha_{ik})}{\operatorname{var}(X|P_x)^{-1} + 2\tau \sum_{l=1}^{m_k} \phi_{ljk} r_{lk}(1-\alpha_{lk}) - \tau^2 \sigma_\epsilon^2 \sum_{l=1}^{m_k} \phi_{ljk}^2 r_{lk}^2 (1-\alpha_{lk})^2}. \end{aligned}$$

Noting that $C_{1,i+1} = -C_{11}\tau\sigma_\epsilon^2\phi_{ijk}r_{ik}(1-\alpha_{ik})$, we have

$$\begin{aligned}
\sum_{l=1}^{m_k+1} C_{1l}g_l &= C_{11}\left(\theta_{jk} + \sum_{l=1}^{m_k}\phi_{ljk}(1-\alpha_{lk})\mathbb{E}[\hat{\gamma}_{lk}|P_x] + (\mathbb{E}[X|P_x]-P_x)\left(\sum_{l=1}^{m_k}\phi_{ljk}(1-\alpha_{lk})r_{lk}\right)\right) \\
&\quad -C_{11}(\mathbb{E}[X|P_x]-P_x)\left(\sum_{k=1}^{m_k} r_{lk}^2(1-\alpha_{lk})^2\phi_{ljk}^2\tau\sigma_\epsilon^2\right) \\
&= C_{11}\left(\theta_{jk} + \sum_{l=1}^{m_k}\phi_{ljk}(1-\alpha_{lk})\mathbb{E}[\hat{\gamma}_{lk}|P_x]\right) \\
&\quad +C_{11}(\mathbb{E}[X|P_x]-P_x)\left[\sum_{l=1}^{m_k}\phi_{ljk}(1-\alpha_{lk})r_{lk} - \tau\sigma_\epsilon^2\sum_{l=1}^{m_k}\phi_{ljk}^2(1-\alpha_{lk})^2 r_{lk}^2\right].
\end{aligned}$$

Using the expression for C_{11}, the first-order condition for θ_{jk}, which is equation (A-10), reduces to

$$\begin{aligned}
\frac{\mathbb{E}\left[X|P_x\right]-P_x}{\tau C_{11}} &= \theta_{jk} + \sum_{l=1}^{m_k}\phi_{ljk}(1-\alpha_{lk})\mathbb{E}[\hat{\gamma}_{lk}|P_x] \\
&\quad +(\mathbb{E}[X|P_x]-P_x)\left[\sum_{l=1}^{m_k}\phi_{ljk}(1-\alpha_{lk})r_{lk} - \tau\sigma_\epsilon^2\sum_{l=1}^{m_k}\phi_{ljk}^2(1-\alpha_{lk})^2 r_{lk}^2\right].
\end{aligned}$$

After substituting C_{11} into the above expression and cancelling terms from both sides of the equation, (A-7) in the Claim is immediate. Likewise, using the expressions C_{11} and $C_{1,i+1}$ in equation (A-12) gives (A-8). The second-order conditions to the household's optimization problem hold (see the proof of Proposition 8 for details). This completes the proof of the Claim. □

Given the symmetry of the household sector in group k, equation (8) in the paper, which is the market clearing condition for the ith fund in group k, implies that $\hat{\phi}_{ijk} = 1/h_k$. Since $r_{ik} = 1/(\tau\sigma_\epsilon^2\alpha_{ik})$, (A-8) reduces to

$$P_{ik} = \frac{\left(\frac{1}{\tau\alpha_{ik}\sigma_\epsilon^2}\right)\left[\left(\frac{1-\alpha_{ik}}{\alpha_{ik}}\right) - \frac{1}{h_k}\left(\frac{1-\alpha_{ik}}{\alpha_{ik}}\right)^2\right]}{\left[\mathrm{var}(X|P_x)^{-1} + \frac{2}{h_k\,\sigma_\epsilon^2}\sum_{l=1}^{m_k}\left(\frac{1-\alpha_{lk}}{\alpha_{lk}}\right) - \frac{1}{h_k^2\,\sigma_\epsilon^2}\sum_{l=1}^{m_k}\left(\frac{1-\alpha_{lk}}{\alpha_{lk}}\right)^2\right]}, \tag{A-13}$$

which is equation (14) in the paper. Substituting the optimal mutual fund demands into

(A-7) gives, after some simple manipulation, the optimal risky asset demand in equation (16) in the paper.

We extract the price coefficients a, b, and d from the risky asset's market clearing condition, which is equation (7) in the paper. Recall that the optimal demand of the ith fund manager in group k takes the form $\hat{\gamma}_{ik} = v_{ik} + r_{ik}Y_{ik} + q_{ik}P_x$, which can be written as

$$\hat{\gamma}_{ik} = \frac{1}{\alpha_{ik}}\left(\bar{v} + Y_{ik}\bar{r} + \bar{q}P_x\right),$$

where

$$\begin{aligned}
\bar{v} &= \frac{\mu_x}{\tau}\left(\frac{1}{\sigma_x^2} + \frac{b^2}{d^2\sigma_u^2}\right) - \frac{b\bar{P}_x}{\tau d^2\sigma_u^2}, \\
\bar{r} &= \frac{1}{\tau\sigma_\epsilon^2}, \\
\bar{q} &= -\frac{1}{\tau}\left(\frac{1}{\sigma_x^2} + \frac{1}{\sigma_\epsilon^2} + \frac{b^2}{d^2\sigma_u^2} - \frac{b}{d^2\sigma_u^2}\right).
\end{aligned}$$

Similarly, from (A-7), the optimal stock market demand for the jth household in group k is of the form $\hat{\theta}_{jk} = s_k + t_k P_x$. To compute s_k and t_k explicitly, note that

$$\begin{aligned}
\mathbb{E}[\hat{\gamma}_{ik}|P_x] &= \mathbb{E}[v_{ik} + r_{ik}Y_{ik} + q_{ik}P_x|P_x] \\
&= v_{ik} + r_{ik}\mathbb{E}[Y_{ik}|P_x] + q_{ik}P_x \\
&= \frac{1}{\alpha_{ik}}\left(\bar{v} + \bar{r}\mathbb{E}[X|P_x] + \bar{q}P_x\right).
\end{aligned}$$

Using this expression and $\hat{\phi}_{ijk} = 1/h_k$, (A-7) is

$$\begin{aligned}
\theta_{jk} &= \frac{\mathbb{E}[X|P_x] - P_x}{\tau\mathrm{var}(X|P_x)} + \frac{\mathbb{E}[X|P_x] - P_x}{\tau\sigma_\epsilon^2 h_k}\sum_{l=1}^{m_k}\frac{(1-\alpha_{lk})}{\alpha_{lk}} \\
&\quad - \frac{(\bar{v} + \bar{r}\mathbb{E}[X|P_x] + \bar{q}P_x)}{h_k}\sum_{l=1}^{m_k}\frac{(1-\alpha_{lk})}{\alpha_{lk}} \\
&= \frac{\mathbb{E}[X|P_x] - P_x}{\tau\mathrm{var}(X|P_x)} - \frac{P_x}{\tau\sigma_\epsilon^2 h_k}\sum_{l=1}^{m_k}\frac{(1-\alpha_{lk})}{\alpha_{lk}} \\
&\quad - \frac{(\bar{v} + \bar{q}P_x)}{h_k}\sum_{l=1}^{m_k}\frac{(1-\alpha_{lk})}{\alpha_{lk}}.
\end{aligned}$$

From this expression it follows that

$$
\begin{aligned}
s_k &= \frac{\mu_x}{\tau}\left(\frac{1}{\sigma_x^2}+\frac{b^2}{d^2\sigma_u^2}\right)-\frac{b\mathbb{E}[P_x]}{\tau d^2\sigma_u^2}-\frac{\bar{v}}{h_k}\sum_{l=1}^{m_k}\frac{(1-\alpha_{lk})}{\alpha_{lk}},\\
t_k &= -\frac{1}{\tau}\left(\frac{1}{\sigma_x^2}+\frac{b^2}{d^2\sigma_u^2}-\frac{b}{d^2\sigma_u^2}+\frac{1}{h_k}\left(\frac{1}{\sigma_\epsilon^2}+\bar{q}\tau\right)\sum_{l=1}^{m_k}\frac{(1-\alpha_{lk})}{\alpha_{lk}}\right).
\end{aligned}
$$

Aggregating over all households in group k we have

$$
\begin{aligned}
\sum_{j=1}^{h_k}\theta_{jk} &= h_k\left[\frac{\mu_x}{\tau\mathrm{var}(X|P_x)}-\frac{b\mathbb{E}[P_x]}{\tau d^2\sigma_u^2}\right]-\bar{v}\sum_{l=1}^{m_k}\frac{(1-\alpha_{lk})}{\alpha_{lk}}\\
&\quad +h_kP_x\left[\frac{b}{\tau d^2\sigma_u^2}-\frac{1}{\tau\mathrm{var}(X|P_x)}\right]-P_x\left(\frac{1}{\tau\sigma_\epsilon^2}+\bar{q}\right)\sum_{l=1}^{m_k}\frac{(1-\alpha_{lk})}{\alpha_{lk}}.
\end{aligned}
$$

Aggregating over all fund managers in group k we have

$$
\sum_{l=1}^{m_k}\hat{\gamma}_{lk}=\bar{r}\sum_{l=1}^{m_k}\frac{Y_{lk}}{\alpha_{lk}}+(\bar{v}+\bar{q}P_x)\sum_{l=1}^{m_k}\frac{1}{\alpha_{lk}}.
$$

Thus the aggregate demand of the managers and households in group k can be expressed as

$$
\sum_{j=1}^{h_k}\theta_{jk}+\sum_{l=1}^{m_k}\hat{\gamma}_{lk}=\bar{r}\sum_{l=1}^{m_k}\frac{Y_{lk}}{\alpha_{lk}}+n\bar{v}-\frac{nP_x}{\tau}\left(\mathrm{var}(X|P_x)^{-1}-\frac{b}{d^2\sigma_u^2}\right)-\frac{P_x}{\tau\sigma_\epsilon^2}\sum_{l=1}^{m_k}\frac{1}{\alpha_{lk}}.
$$

Aggregating over all groups and equating the aggregate demand to the random supply U, we get

$$
\int_0^{\bar{k}}\left(\sum_{j=1}^{h_k}\theta_{jk}+\sum_{l=1}^{m_k}\hat{\gamma}_{lk}\right)dk=U.
$$

Using the law of large numbers, we get

$$
\bar{r}X\int_0^{\bar{k}}\bar{\alpha}_k^{-1}dk+\bar{v}-\frac{P_x}{\tau}\left(\mathrm{var}(X|P_x)^{-1}-\frac{b}{d^2\sigma_u^2}\right)-\frac{P_x}{\tau\sigma_\epsilon^2}\int_0^{\bar{k}}\bar{\alpha}_k^{-1}dk=U,
$$

where $\bar{\alpha}_k^{-1} = \sum_{l=1}^{m_k} \alpha_{lk}^{-1}$ and $\bar{k} = 1/n$. Thus it follows that

$$\bar{r}X \int_0^{\bar{k}} \bar{\alpha}_k^{-1} dk + \bar{v} - U = \frac{P_x}{\tau} \left(\mathrm{var}(X|P_x)^{-1} - \frac{b}{d^2 \sigma_u^2} + \frac{1}{\sigma_\epsilon^2} \int_0^{\bar{k}} \bar{\alpha}_k^{-1} dk \right).$$

Using the conjectured price function $P_x = a + bX - dU$, we obtain the following system of equations for the price coefficients a, b and d:

$$\begin{aligned}
\bar{v} &= \frac{a}{\tau} \left(\mathrm{var}(X|P_x)^{-1} - \frac{b}{d^2 \sigma_u^2} + \frac{1}{\sigma_\epsilon^2} \int_0^{\bar{k}} \bar{\alpha}_k^{-1} dk \right), \\
\bar{r} \int_0^{\bar{k}} \bar{\alpha}_k^{-1} dk &= \frac{b}{\tau} \left(\mathrm{var}(X|P_x)^{-1} - \frac{b}{d^2 \sigma_u^2} + \frac{1}{\sigma_\epsilon^2} \int_0^{\bar{k}} \bar{\alpha}_k^{-1} dk \right), \\
1 &= \frac{d}{\tau} \left(\mathrm{var}(X|P_x)^{-1} - \frac{b}{d^2 \sigma_u^2} + \frac{1}{\sigma_\epsilon^2} \int_0^{\bar{k}} \bar{\alpha}_k^{-1} dk \right).
\end{aligned}$$

These three conditions are solved uniquely to obtain the expressions for a, b, and d given in equations (12)-(13) in the paper. This confirms the conjectured form for P_x in equation (1) in the paper and it completes the proof of Proposition 1. $\square$

Proof of Proposition 2.

Proposition 2 characterizes our equilibrium at date 1, which is the fund formation stage of the model. At this stage we take the set $\mathcal{M} = \{m_k : k \in [0, \bar{k}]\}$ as given and we solve for the set of optimal fees, $\hat{\mathcal{A}} = \{\hat{\alpha}_{ik} : i = 1, \ldots, m_k; k \in [0, \bar{k}]\}$. In part (i) of the proposition we characterize a fee setting equilibrium in which m_k varies with k, but every manager in group k chooses the same contingent fee, $\hat{\alpha}_{ik} = \hat{\alpha}_k$ for all $i = 1, \ldots, m_k$. In part (ii) of the proposition we specialize our fee setting equilibrium to the case in which $m_k = m$. Thus all groups are identical in part (ii).

According to Definition 2 in the paper, the optimal fee of the ith manager in group k is given by

$$\hat{\alpha}_{ik} = \arg\max_{\alpha_{ik}} \quad \mathcal{U}_{ik}(m_k, \{\alpha_{lk}\}_{l=1}^{m_k}; \mathcal{M}_k, \mathcal{A}_k)\big|_{\hat{\alpha}_{-i}^k, \hat{\mathcal{A}}_k}, \tag{A-14}$$

where $\hat{\mathcal{A}}_k = \{\hat{\alpha}_{ik'} : i = 1, \ldots, m_{k'}; k' \in [0, \bar{k}], k' \neq k\}$ and $\hat{\alpha}_{-i}^k$ is the set $\{\hat{\alpha}_{lk}\}_{l=1}^{m_k}$ with the ith element omitted. Due to the form of $\mathcal{U}_{ik}$, the problem in (A-14) reduces to the following optimization problem

$$\max_{\alpha_{ik}} \quad \alpha_{ik}P_{ik} = \frac{\left(\frac{1}{\tau\sigma_\epsilon^2}\right)\left[\left(\frac{1-\alpha_{ik}}{\alpha_{ik}}\right) - \frac{1}{h_k}\left(\frac{1-\alpha_{ik}}{\alpha_{ik}}\right)^2\right]}{\left[\mathrm{var}(X|P_x)^{-1} + \frac{2}{h_k\,\sigma_\epsilon^2}\sum_{l=1}^{m_k}\left(\frac{1-\alpha_{lk}}{\alpha_{lk}}\right) - \frac{1}{h_k^2\,\sigma_\epsilon^2}\sum_{l=1}^{m_k}\left(\frac{1-\alpha_{lk}}{\alpha_{lk}}\right)^2\right]},$$

where we have used (A-13). Letting $\rho_{ik} = (1-\alpha_{ik})/(\alpha_{ik}h_k)$ and noting that α_{ik} and ρ_{ik} are monotonically related, the ith manager's problem can be written as

$$\max_{\rho_{ik}} \quad \alpha_{ik}P_{ik} = \frac{h_k\left(\rho_{ik} - \rho_{ik}^2\right)}{\tau\sigma_\epsilon^2\left[\mathcal{R} + \frac{2}{\sigma_\epsilon^2}\sum_{l=1}^{m_k}\rho_{lk} - \frac{1}{\sigma_\epsilon^2}\sum_{l=1}^{m_k}\rho_{lk}^2\right]}, \tag{A-15}$$

where $\mathcal{R} = \mathrm{var}(X|P_x)^{-1}$. The first order condition is

$$\sigma_\epsilon^2\left(1-2\hat{\rho}_{ik}\right)\left[\mathcal{R} + \frac{2}{\sigma_\epsilon^2}\sum_{l=1}^{m_k}\hat{\rho}_{lk} - \frac{1}{\sigma_\epsilon^2}\sum_{l=1}^{m_k}\hat{\rho}_{lk}^2\right] = 2\hat{\rho}_{ik}\left(1-\hat{\rho}_{ik}\right)^2, \tag{A-16}$$

where we have used the fact that $\mathcal{R}$ depends only on $\int_0^1 \bar{\alpha}_k^{-1}dk$ and not on any individual manager's contingent fee.

Substituting $\hat{\rho}_{ik} = \hat{\rho}_k$ for all i into (A-16), which imposes symmetry within group k, (A-16) becomes

$$2(m_k-1)\hat{\rho}_k^3 + (4-5m_k)\hat{\rho}_k^2 + 2(m_k - 1 - \sigma_\epsilon^2\mathcal{R})\hat{\rho}_k + \sigma_\epsilon^2\mathcal{R} = 0, \tag{A-17}$$

which coincides with the cubic equation in (17) in the paper. To be precise, we have $\hat{\rho}_k = \hat{\rho}(m_k, \mathcal{R})$, where the function $\hat{\rho}(m, r)$ is given in (17) in the paper.

Now define the function $g(\hat{\rho}_k)$ as

$$g(\hat{\rho}_k) = 2(m_k-1)\hat{\rho}_k^3 + (4-5m_k)\hat{\rho}_k^2 + 2(m_k - 1 - \sigma_\epsilon^2\mathcal{R})\hat{\rho}_k + \sigma_\epsilon^2\mathcal{R}. \tag{A-18}$$

Since $g(0) = \sigma_\epsilon^2\mathcal{R} > 0$ and $g(1/2) = -1/4 < 0$, it must be the case that either one root or all three roots of (A-17) lie in the interval $(0, 1/2)$. We show the latter case does not occur. First, suppose $m_k > 1$. In this case the leading term that multiplies $\hat{\rho}_k^3$ in (A-18) is positive so $g(\hat{\rho}_k) \to \infty$ as $\hat{\rho}_k \to \infty$. Thus there is a root in the interval $(1/2, \infty)$, which precludes more than one root in $(0, 1/2)$. Second, suppose $m_k = 1$. In this case (A-17) reduces to a quadratic equation with a positive root in $(0,1/2)$, while the other root is negative. Thus the solution of (A-17) that lies in the interval $(0,1/2)$ is unique. Since $\hat{\rho}_k = (1-\hat{\alpha}_k)/(\hat{\alpha}_k h_k)$ and $\hat{\rho}_k \in (0,1/2)$, we have $\hat{\alpha}_k = 1/(1+h_k\hat{\rho}_k)$ and thus $\hat{\alpha}_k \in (1/(1+0.5h_k),1)$.

Having established that $\hat{\rho}_{ik} \in (0,1/2)$, we return to (A-15) to examine the second order condition. The second order condition reduces to

$$\begin{aligned} & -2\sigma_\epsilon^2\left[\mathcal{R}+\frac{1}{\sigma_\epsilon^2}\sum_{l=1}^{m_k}\rho_{lk}(2-\rho_{lk})\right]+2(1-2\rho_{ik})(1-\rho_{ik})-2(1-\rho_{ik})^2+4\rho_{ik}(1-\rho_{ik}) \\ = \quad & -2\sigma_\epsilon^2\mathcal{R}-2\sum_{l=1}^{m_k}\rho_{lk}(2-\rho_{lk})+2\rho_{ik}(1-\rho_{ik})<0, \end{aligned}$$

which verifies that (A-16) characterizes the optimal solution to the manager's problem.

We have characterized the managers' optimal fees in group k in terms of $\mathcal{R}>0$. For the equilibrium in Proposition 1,

$$\begin{aligned} \mathcal{R} &= \operatorname{var}(X|P_x)^{-1} \\ &= \frac{1}{\sigma_x^2}+\left(\frac{b}{d}\right)^2\frac{1}{\sigma_u^2} \\ &= \frac{1}{\sigma_x^2}+\left(\frac{1}{\tau\sigma_\epsilon^2}\int_0^{\bar{k}}\bar{\alpha}_k^{-1}dk\right)^2\frac{1}{\sigma_u^2} \\ &= \frac{1}{\sigma_x^2}+\frac{1}{\sigma_u^2}\left(\frac{1}{\tau\sigma_\epsilon^2}\right)^2\left(\int_0^{\bar{k}}(1+h_k\hat{\rho}(m_k,\mathcal{R}))m_k dk\right)^2. \end{aligned} \tag{A-19}$$

We now identify the equilibrium level of $\mathcal{R}$ by showing that there is a fixed point to equation (A-19). Applying the implicit function theorem to (A-17), one has

$$\frac{\partial\rho_k}{\partial\mathcal{R}} = -\frac{\sigma_\epsilon^2(1-2\rho_k)}{6(m_k-1)\rho_k^2+2(4-5m_k)\rho_k+2(m_k-1-\sigma_\epsilon^2\mathcal{R})}.$$

The denominator of the above expression is negative, which can be seen by noting that

$$\begin{aligned}
& \hat{\rho}_k \left(6(m_k-1)\hat{\rho}_k^2 + 2(4-5m_k)\hat{\rho}_k + 2(m_k - 1 - \sigma_\epsilon^2 \mathcal{R})\right) \\
= \;& 6(m_k-1)\hat{\rho}_k^3 + 2(4-5m_k)\hat{\rho}_k^2 + 2(m_k-1-\sigma_\epsilon^2\mathcal{R})\hat{\rho}_k \\
= \;& 2(m_k-1)\hat{\rho}_k^3 + (4-5m_k)\hat{\rho}_k^2 + 2(m_k-1-\sigma_\epsilon^2\mathcal{R})\hat{\rho}_k + 4(m_k-1)\hat{\rho}_k^3 + (4-5m_k)\hat{\rho}_k^2 \\
= \;& -\sigma_\epsilon^2\mathcal{R} + 4(m_k-1)\hat{\rho}_k^2(\hat{\rho}_k - 1) - m_k\hat{\rho}_k^2 < 0,
\end{aligned}$$

where we used (A-17) in the last equality and the fact that $\hat{\rho}_k \in (0, 1/2)$. Thus we have established that $\partial\rho(m_k, \mathcal{R})/\partial\mathcal{R} > 0$. Since $\rho(m_k, \mathcal{R}) \in (0, 1/2)$, the right-hand side of (A-19), as a function of $\mathcal{R}$, is bounded from above, whereas the left-hand side is not. Furthermore, the right-hand side of (A-19) is bounded away from zero as $\mathcal{R} \downarrow 0$, whereas the left-hand size tends to zero. Thus there always exists a fixed point to (A-19).

To verify part (ii) of the Proposition, we note that if all groups have $m_k = m$ managers and all managers choose the same fee $\hat{\alpha}_{ik} = \hat{\alpha}(m)$ for all i and k, then

$$\begin{aligned}
\mathcal{R} &= \frac{1}{\sigma_x^2} + \frac{1}{\sigma_u^2}\left(\frac{1}{\tau\sigma_\epsilon^2}\right)^2 \left(\int_0^{\bar{k}} (1 + h_k\hat{\rho}(m_k, \mathcal{R}))m_k dk\right)^2 \\
&= \frac{1}{\sigma_x^2} + \left(\frac{m}{n\tau\sigma_\epsilon^2\hat{\alpha}(m)}\right)^2 \frac{1}{\sigma_u^2}. \qquad \text{(A-20)}
\end{aligned}$$

After substituting (A-20) for $\mathcal{R}$ in (A-17) and rewriting the cubic equation in terms of $\hat{\alpha}(m)$ by letting $\hat{\rho}_k = (1-\hat{\alpha}(m))/(h\hat{\alpha}(m))$, one obtains equation (19) in the paper. This completes the proof. □

Proof of Proposition 3.

Proposition 3 characterizes our equilibrium at date 0, which is the information acquisition stage of the model. At this stage we solve for a set $\mathcal{M} = \{m_k : k \in [0, \bar{k}]\}$ such that none of the m_k informed agents in group k is better off being uninformed and none of the $n - m_k$ uninformed agents is better off being informed. As we mention in the paper, multiple equilibria can arise at the information acquisition stage of our model. Proposition 3 focuses on the case in which there are at most two types of groups, where the two types differ only with respect to the number of informed agents m_k.

To prove Proposition 3, we need the ex ante expected utilities of the managers and the households. To get the ith manager's ex ante utility, let

$$\chi_{ik} = \frac{\mathbb{E}(X|Y_{ik}, P_x) - P_x}{\text{var}(X|Y_{ik}, P_x)}. \tag{A-21}$$

Upon substituting the manager's optimal trading strategy (A-4) into the manager's conditional expected utility (A-1) we get an expression that includes

$$\begin{aligned}
\mathbb{E}\left[-e^{-\tau\alpha_{ik}\hat{\gamma}_{ik}(X-P_x)}\middle| P_x, Y_{ik}\right] &= \mathbb{E}\left[-e^{-\chi_{ik}(X-P_x)}\middle| P_x, Y_{ik}\right] \\
&= -e^{-\chi_{ik}\mathbb{E}[X-P_x|Y_{ik},P_x]+\frac{\chi_{ik}^2}{2}\text{var}(X-P_x|Y_{ik},P_x)} \\
&= -e^{-\chi_{ik}\left(\mathbb{E}[X-P_x|Y_{ik},P_x]-\frac{\chi_{ik}}{2}\text{var}(X-P_x|Y_{ik},P_x)\right)} \\
&= -e^{-\frac{\chi_{ik}^2\text{var}(X|Y_{ik},P_x)}{2}}.
\end{aligned}$$

Next we note that

$$\begin{aligned}
\mathbb{E}[\chi_{ik}] &= \mathbb{E}\left[\frac{\mathbb{E}(X|Y_{ik}, P_x) - P_x}{\text{var}(X|Y_{ik}, P_x)}\right] \\
&= \frac{\mu_x - \mathbb{E}[P_x]}{\text{var}(X|Y_{ik}, P_x)}, \\
\text{var}(\chi_{ik}) &= \text{var}\left(\frac{\mathbb{E}(X|Y_{ik}, P_x) - P_x}{\text{var}(X|Y_{ik}, P_x)}\right) \\
&= \frac{1}{\text{var}(X|Y_{ik}, P_x)^2}\text{var}\left(\mathbb{E}(X|Y_{ik}, P_x) - P_x\right) \\
&= \frac{1}{\text{var}(X|Y_{ik}, P_x)^2}\left(\text{var}(X - P_x) - \text{var}(X - P_x|Y_{ik}, P_x)\right),
\end{aligned}$$

where we have used the variance decomposition formula $\text{var}(Z) = \text{var}(\mathbb{E}[Z|\Omega]) + \mathbb{E}[\text{var}(Z|\Omega)]$ (where Z is an arbitrary random variable and Ω is an information set) and the fact that $\text{var}(X|Y_{ik}, P_x)$ is non-random.

Applying Lemma 1 from the Appendix we get

$$\begin{aligned}
&\mathbb{E}\left[-e^{-\frac{\chi_{ik}^2\text{var}(X|Y_{ik},P_x)}{2}}\right] \\
&= -|1 + \text{var}(X|Y_{ik}, P_x)\text{var}(\chi_{ik})|^{-1/2}\, e^{-\frac{\mathbb{E}[\chi_{ik}]^2\text{var}(X|Y_{ik},P_x)}{2}+\frac{\text{var}(X|Y_{ik},P_x)^2\mathbb{E}[\chi_{ik}]^2\text{var}(\chi_{ik})}{2(1+\text{var}(X|Y_{ik},P_x)\text{var}(\chi_{ik}))}} \\
&= -\sqrt{\frac{\text{var}(X|Y_{ik}, P_x)}{\text{var}(X - P_x)}}\, e^{-\frac{(\mu_x-\mathbb{E}[P_x])^2}{2\text{var}(X-P_x)}},
\end{aligned}$$

where in the last equality we use

$$\begin{aligned} 1+\operatorname{var}(X|Y_{ik},P_x)\operatorname{var}(\chi_{ik}) &= 1+\frac{1}{\operatorname{var}(X|Y_{ik},P_x)}\left(\operatorname{var}(X-P_x)-\operatorname{var}(X-P_x|Y_{ik},P_x)\right) \\ &= \frac{\operatorname{var}(X-P_x)}{\operatorname{var}(X|Y_{ik},P_x)}. \end{aligned}$$

This gives the expression in the paper for the ith manager's ex ante utility,

$$\mathbb{E}\left[-e^{-\tau\hat{\alpha}_{ik}(P_{ik}+\hat{\gamma}_{ik}(X-P_x))+\tau c}\right] = -e^{-\tau(\hat{\alpha}_{ik}P_{ik}-c)-\frac{(\mu_x-\mathbb{E}[P_x])^2}{2\Lambda}}\sqrt{\frac{1}{F\Lambda}}, \tag{A-22}$$

where $F=\operatorname{var}(X|P_x,Y_{ik})^{-1}$ and $\Lambda=\operatorname{var}(X-P_x)=d^2\sigma_u^2+(b-1)^2\sigma_x^2$.

Since the managers in group k choose identical contingent fees, the ith manager's certainty equivalent wealth, $\mathcal{U}_k$, is given by expression (21) in the paper,

$$\mathcal{U}_k(m_k,\hat{\alpha}_k(m_k);\mathcal{M}_k,\hat{\mathcal{A}}_k)=\frac{1}{2\tau}\log(\mathcal{R}+1/\sigma_\epsilon^2)+f(m_k,\mathcal{R})-c+H, \tag{A-23}$$

where H is

$$H=\frac{(\mu_x-\mathbb{E}[P_x])^2}{2\tau\Lambda}+\frac{1}{2\tau}\log(\Lambda) \tag{A-24}$$

and $f:\mathbb{R}^2\to\mathbb{R}$ is defined in the statement of the Proposition.

We now turn to the jth household's ex ante expected utility. From Proposition 1, we know the conditional expected utility for the jth household can be expressed as in (A-6). We will rewrite (A-6) in a simpler form before calculating the jth household's ex ante expected utility. Given that each fund manager in group k chooses the same proportional fee, we have $\hat{\alpha}_{ik}=\hat{\alpha}_k$ so that $\hat{\rho}_{ik}=\hat{\rho}_k=(1-\hat{\alpha}_k)/(h_k\hat{\alpha}_k)$ and $r_{ik}=1/(\tau\hat{\alpha}_k\sigma_\epsilon^2)$. Thus we can write $\boldsymbol{A}_{jk}$ as

$$\boldsymbol{A}_{jk}=\begin{pmatrix} m_k\hat{\rho}_k/(\tau\sigma_\epsilon^2) & \hat{\rho}_k/(2\tau\sigma_\epsilon^2)\mathbf{1}^\top \\ \hat{\rho}_k/(2\tau\sigma_\epsilon^2)\mathbf{1} & \mathbf{0} \end{pmatrix},$$

where $\mathbf{1}$ is a vector of 1's and $\mathbf{0}$ is a matrix of 0's.

Applying (A-67) in the Appendix to the matrix

$$\boldsymbol{C}_{jk}^{-1}=\boldsymbol{V}_{jk}^{-1}+2\tau\boldsymbol{A}_{jk}=\begin{pmatrix} \operatorname{var}(X|P_x)^{-1}+2m_k\hat{\rho}_k/\sigma_\epsilon^2 & \hat{\rho}_k/(\sigma_\epsilon^2)\mathbf{1}^\top \\ \hat{\rho}_k/(\sigma_\epsilon^2)\mathbf{1} & (1/\sigma_\epsilon^2)\boldsymbol{I} \end{pmatrix}$$

we get

$$|\boldsymbol{V}_{jk}^{-1} + 2\tau \boldsymbol{A}_{jk}| = \frac{1}{\sigma_\epsilon^{2m_k}} \left(\text{var}(X|P_x)^{-1} + 2\frac{m_k}{\sigma_\epsilon^2}\hat{\rho}_k - \frac{m_k}{\sigma_\epsilon^2}\hat{\rho}_k^2 \right)$$
$$= \frac{1}{\sigma_\epsilon^{2m_k}} \left(\mathcal{R} + Q(m_k, \mathcal{R})\right),$$

where $Q : \mathbb{R}^2 \to \mathbb{R}$ is defined in the statement of the Proposition. This implies

$$\begin{aligned} |\boldsymbol{B}_{jk}| &= |\boldsymbol{I} + 2\tau \boldsymbol{V}_{jk}\boldsymbol{A}_{jk}| \\ &= |\boldsymbol{V}_{jk}||\boldsymbol{V}_{jk}^{-1} + 2\tau \boldsymbol{A}_{jk}| \\ &= \sigma_\epsilon^{2m_k}\text{var}(X|P_x)\frac{1}{\sigma_\epsilon^{2m_k}} \left(\text{var}(X|P_x)^{-1} + 2\frac{m_k}{\sigma_\epsilon^2}\hat{\rho}_k - \frac{m_k}{\sigma_\epsilon^2}\hat{\rho}_k^2 \right) \\ &= \text{var}(X|P_x)\left(\mathcal{R} + Q(m_k, \mathcal{R})\right). \end{aligned}$$

Using (A-66) in the Appendix, we have

$$\boldsymbol{C}_{jk} = \begin{pmatrix} c_{11} & -\hat{\rho}_k c_{11}\mathbf{1}^\top \\ -\hat{\rho}_k c_{11}\mathbf{1} & \sigma_\epsilon^2 \boldsymbol{I} + \hat{\rho}_k^2 c_{11}\mathbf{1}\mathbf{1}^\top \end{pmatrix},$$

where $\mathbf{1}$ denotes a m_k-vector of 1's and

$$c_{11} = \left(\text{var}(X|P_x)^{-1} + 2\frac{m_k}{\sigma_\epsilon^2}\hat{\rho}_k - \frac{m_k}{\sigma_\epsilon^2}\hat{\rho}_k^2 \right)^{-1}.$$

Noting that

$$\hat{\theta}_{jk} = \frac{\mathbb{E}[X|P_x] - P_x}{\tau \text{var}(X|P_x)} - \left(\sum_{l=1}^{m_k} \hat{\rho}_{lk}\right)\left(\frac{P_x}{\tau\sigma_\epsilon^2} + \bar{v} + \bar{q}P_x\right)$$

and using the definition of $\hat{\rho}_{ik}$ and the previous expressions for v_{lk}, q_{lk} and r_{lk}, we obtain

$$\hat{\theta}_{jk} + \sum_{l=1}^{m_k} \hat{\phi}_{ljk}(1 - \hat{\alpha}_{lk})(v_{lk} + q_{lk}P_x + r_{lk}P_x) = \frac{\mathbb{E}[X|P_x] - P_x}{\tau \text{var}(X|P_x)}.$$

Thus using the notation from Proposition 1,

$$\boldsymbol{g}_{jk} = \frac{\mathbb{E}[X|P_x] - P_x}{\tau} \begin{bmatrix} \text{var}(X|P_x)^{-1} + 2m_k\hat{\rho}_k/\sigma_\epsilon^2 \\ \hat{\rho}_k/\sigma_\epsilon^2\mathbf{1} \end{bmatrix}.$$

Some algebra shows that

$$
\begin{aligned}
\boldsymbol{g}_{jk}^\top \boldsymbol{C}_{jk}\boldsymbol{g}_{jk} &= \frac{(\mathbb{E}[X|P_x]-P_x)^2}{\tau^2}\left[\left(\mathrm{var}(X|P_x)^{-1}+2m_k\hat{\rho}_k/\sigma_\epsilon^2\right)^2 c_{11}\right] \\
&\quad -\frac{(\mathbb{E}[X|P_x]-P_x)^2}{\tau^2}\left[2(\mathrm{var}(X|P_x)^{-1}+2m_k\hat{\rho}_k/\sigma_\epsilon^2)\frac{\hat{\rho}_k^2 m_k}{\sigma_\epsilon^2}c_{11}\right] \\
&\quad +\frac{(\mathbb{E}[X|P_x]-P_x)^2}{\tau^2}\left[\frac{\hat{\rho}_k^2 m_k}{\sigma_\epsilon^2}\left(1+\frac{\hat{\rho}_k^2 m_k}{\sigma_\epsilon^2}c_{11}\right)\right] \\
&= \frac{(\mathbb{E}[X|P_x]-P_x)^2}{\tau^2}\left[c_{11}\left(\mathrm{var}(X|P_x)^{-1}+\frac{2m_k\hat{\rho}_k}{\sigma_\epsilon^2}-\frac{\hat{\rho}_k^2 m_k}{\sigma_\epsilon^2}\right)^2+\frac{\hat{\rho}_k^2 m_k}{\sigma_\epsilon^2}\right] \\
&= \frac{(\mathbb{E}[X|P_x]-P_x)^2}{\tau^2}\left[c_{11}^{-1}+\frac{\hat{\rho}_k^2 m_k}{\sigma_\epsilon^2}\right] \\
&= \frac{(\mathbb{E}[X|P_x]-P_x)^2}{\tau^2}\left[\mathrm{var}(X|P_x)^{-1}+\frac{2m_k\hat{\rho}_k}{\sigma_\epsilon^2}\right].
\end{aligned}
$$

Finally, we note that

$$
\begin{aligned}
& -\tau\boldsymbol{\mu}_k{}^\top \boldsymbol{A}_{jk}\boldsymbol{\mu}_k+\frac{\tau^2}{2}\boldsymbol{g}_{jk}{}^\top \boldsymbol{C}_{jk}\boldsymbol{g}_{jk} \\
= & -\frac{(\mathbb{E}[X|P_x]-P_x)^2 m_k\hat{\rho}_k}{\sigma_\epsilon^2}+\frac{(\mathbb{E}[X|P_x]-P_x)^2}{2}\left[\mathrm{var}(X|P_x)^{-1}+\frac{2m_k\hat{\rho}_k}{\sigma_\epsilon^2}\right] \\
= & \frac{(\mathbb{E}[X|P_x]-P_x)^2}{2\mathrm{var}(X|P_x)},
\end{aligned}
$$

so (A-6) reduces to the simpler expression

$$
\mathbb{E}\left[-e^{-\tau\hat{W}_{jk}}|P_x\right] = -\sqrt{\frac{1}{\mathrm{var}(X|P_x)\left(\mathcal{R}+Q(m_k,\mathcal{R})\right)}}\, e^{-\frac{(\mathbb{E}[X|P_x]-P_x)^2}{2\mathrm{var}(X|P_x)}+\frac{\tau}{h_k}\sum_{l=1}^{m_k}\hat{\alpha}_{lk}P_{lk}}. \tag{A-25}
$$

Now let

$$
\chi_h = \frac{\mathbb{E}(X|P_x)-P_x}{\mathrm{var}(X|P_x)}. \tag{A-26}
$$

As in the case of the informed agents, we have

$$
\begin{aligned}
\mathbb{E}[\chi_h] &= \mathbb{E}\left[\frac{\mathbb{E}(X|P_x) - P_x}{\mathrm{var}(X|P_x)}\right] \\
&= \frac{\mu_x - \mathbb{E}[P_x]}{\mathrm{var}(X|P_x)}, \\
\mathrm{var}(\chi_h) &= \mathrm{var}\left(\frac{\mathbb{E}(X|P_x) - P_x}{\mathrm{var}(X|P_x)}\right) \\
&= \frac{1}{\mathrm{var}(X|P_x)^2}\mathrm{var}\left(\mathbb{E}(X|P_x) - P_x\right) \\
&= \frac{1}{\mathrm{var}(X|P_x)^2}\left(\mathrm{var}(X - P_x) - \mathrm{var}(X - P_x|P_x)\right).
\end{aligned}
$$

Applying Lemma 1 in the Appendix we get

$$
\begin{aligned}
\mathbb{E}\left[-e^{-\frac{(\mathbb{E}[X|P_x]-P_x)^2}{2\mathrm{var}(X|P_x)}}\right] &= \mathbb{E}\left[-e^{-\frac{\chi_h^2 \mathrm{var}(X|P_x)}{2}}\right] \\
&= -|1 + \mathrm{var}(X|P_x)\mathrm{var}(\chi_h)|^{-1/2}\, e^{-\frac{\mathbb{E}[\chi_h]^2 \mathrm{var}(X|P_x)}{2} + \frac{\mathrm{var}(X|P_x)^2 \mathbb{E}[\chi_h]^2 \mathrm{var}(\chi_h)}{2(1+\mathrm{var}(X|P_x)\mathrm{var}(\chi_h))}} \\
&= -\sqrt{\frac{\mathrm{var}(X|P_x)}{\mathrm{var}(X - P_x)}}\, e^{-\frac{(\mu_x - \mathbb{E}[P_x])^2}{2\mathrm{var}(X-P_x)}},
\end{aligned}
$$

where in the last equality we use

$$
\begin{aligned}
1 + \mathrm{var}(X|P_x)\mathrm{var}(\chi_h) &= 1 + \frac{1}{\mathrm{var}(X|P_x)}\left(\mathrm{var}(X - P_x) - \mathrm{var}(X - P_x|P_x)\right) \\
&= \frac{\mathrm{var}(X - P_x)}{\mathrm{var}(X|P_x)}.
\end{aligned}
$$

Using this in (A-25) gives equation (35) in the paper,

$$
\mathbb{E}\left[-e^{-\tau \hat{W}_{jk}}\right] = -\sqrt{\frac{1}{D\Lambda}}\, e^{-\frac{(\mu_x - \mathbb{E}[P_x])^2}{2\Lambda} + \frac{\tau}{h_k}\sum_{l=1}^{m_k} \hat{\alpha}_{lk} P_{lk}}, \tag{A-27}
$$

where $D = \mathcal{R} + Q(m_k, \mathcal{R})$.

Since the households in group k are identical, the jth household's certainty equivalent wealth, $\mathcal{V}_k$, is given by

$$
\mathcal{V}_k(m_k, \hat{\alpha}_k(m_k); \mathcal{M}_k, \hat{\mathcal{A}}_k) = \frac{1}{2\tau}\log(\mathcal{R} + Q(m_k, \mathcal{R})) - \frac{m_k}{n - m_k} f(m_k, \mathcal{R}) + H, \tag{A-28}
$$

where H is given by (A-24).

Now suppose that $m_k = m$ for all k. In this case, $\mathcal{R}$ is given by (20) in the paper. After substituting this expression for $\mathcal{R}$ into (A-23) and (A-28), the equilibrium conditions (10)-(11) in the paper can be written as $c \leq \bar{c}(m)$ and $c \geq \underline{c}(m)$, where $\bar{c}(m)$ and $\underline{c}(m)$ are defined in the statement of the Proposition.

Recall from Definition 3 in the paper that only condition (10) in the paper must hold if there are n informed agents in a group. Thus if $c \leq \bar{c}(n)$, there exists an equilibrium with identical groups where $m_k = n$ for all k. This proves part (iii) of the proposition. Likewise, recall from Definition 3 in the paper that only condition (11) in the paper must hold if there are no informed agents in a group. Thus if $c \geq \underline{c}(0)$, there exists an equilibrium with identical groups where $m_k = 0$ for all k. This proves part (iv) of the proposition.

To prove part (i), let $1 \leq m \leq n-1$. We first show that $\bar{c}(m) \geq \underline{c}(m)$, i.e., the interval $[\underline{c}(m), \bar{c}(m)]$ is non-empty. Using the definitions of $\bar{c}(m)$ and $\underline{c}(m)$, we have

$$\begin{aligned} \bar{c}(m) - \underline{c}(m) &= \frac{1}{2\tau}\log\left(\frac{\mathcal{R} + Q(m,\mathcal{R})}{\mathcal{R} + Q(m-1,\mathcal{R})}\right) + f(m,\mathcal{R}) - f(m+1,\mathcal{R}) \\ &\quad + \left(\frac{m-1}{n-m+1}\right) f(m-1,\mathcal{R}) - \left(\frac{m}{n-m}\right) f(m,\mathcal{R}), \end{aligned} \tag{A-29}$$

where $\mathcal{R} = \mathcal{R}(m)$. Using the definition of Q in the statement of the Proposition, we have

$$\frac{\partial Q(m_k,\mathcal{R})}{\partial m_k} = \frac{1}{\sigma_\epsilon^2}\left[\hat{\rho}_k(2-\hat{\rho}_k) + 2m_k(1-\hat{\rho}_k)\frac{\partial \hat{\rho}_k}{\partial m_k}\right],$$

where $\hat{\rho}_k = \hat{\rho}(m_k,\mathcal{R})$. Applying the implicit function theorem to (A-17), we have

$$\frac{\partial \hat{\rho}_k}{\partial m_k} = -\frac{\hat{\rho}_k(1-2\hat{\rho}_k)(2-\hat{\rho}_k)}{6(m_k-1)\hat{\rho}_k^2 + 2(4-5m_k)\hat{\rho}_k + 2(m_k - 1 - \sigma_\epsilon^2\mathcal{R})}.$$

Using the same argument as when signing $\partial\hat{\rho}_k/\partial\mathcal{R}$ (see the proof of Proposition 2), we conclude that $\partial\hat{\rho}_k/\partial m_k > 0$. This shows that $\partial Q/\partial m_k > 0$ and thus the log term on the right-hand side of (A-29) is positive. To show that the terms involving f on the right-hand side of (A-29) are positive, let $g(x) = (1-2x)/(2\tau(1-x))$. Note that $g(x) > 0$ and $g'(x) < 0$ if $x \in (0, 1/2)$. Using (33) in the paper, $f(m,\mathcal{R})$ can be written as

$$f(m,\mathcal{R}) = \frac{(n-m)}{2\tau}\left(\frac{1-2\hat{\rho}(m,\mathcal{R})}{1-\hat{\rho}(m,\mathcal{R})}\right) = (n-m)g(\hat{\rho}(m,\mathcal{R})).$$

Thus the terms involving f in (A-29) can be written as

$$(n-m-1)\left[g\left(\hat{\rho}(m,\mathcal{R})\right)-g\left(\hat{\rho}(m+1,\mathcal{R})\right)\right]+(m-1)\left[g\left(\hat{\rho}(m-1,\mathcal{R})\right)-g\left(\hat{\rho}(m,\mathcal{R})\right)\right].$$

Since $n \geq 2$, $1 \leq m \leq n-1$, $g'(x) < 0$, and $\partial\hat{\rho}/\partial m_k > 0$, this expression is greater than or equal to zero. This shows $[\underline{c}(m), \bar{c}(m)]$ is non-empty. Thus if $c \in [\underline{c}(\hat{m}), \bar{c}(\hat{m})]$ for some $\hat{m} \in \{1, 2, \ldots, n-1\}$, the equilibrium conditions (10)-(11) in the paper are satisfied. In this case there exists an equilibrium with identical groups where $m_k = \hat{m}$ for all k. This proves part (i) of the proposition.

To prove part (ii), define $\bar{c}(m, \lambda)$ as

$$\begin{aligned}\bar{c}(m,\lambda) &= \frac{1}{2\tau}\log\left(\frac{\mathcal{R}(m,\lambda)+1/\sigma_\epsilon^2}{\mathcal{R}(m,\lambda)+Q(m-1,\mathcal{R}(m,\lambda))}\right)+f(m,\mathcal{R}(m,\lambda))\\ &\quad+\frac{(m-1)}{(n-m+1)}f(m-1,\mathcal{R}(m,\lambda)),\end{aligned}$$

where $\mathcal{R}(m, \lambda)$ is given by

$$\mathcal{R}(m,\lambda)=\frac{1}{\sigma_x^2}+\frac{1}{\sigma_u^2}\left(\frac{1}{n\tau\sigma_\epsilon^2}\right)^2\left(\frac{\lambda m}{\hat{\alpha}_k(m)}+\frac{(1-\lambda)(m-1)}{\hat{\alpha}_k(m-1)}\right)^2. \tag{A-30}$$

In (A-30), $\hat{\alpha}_k(m) = 1/(1+(n-m)\hat{\rho}(m,\mathcal{R}))$ and $\hat{\alpha}_k(m-1) = 1/(1+(n-m+1)\hat{\rho}(m-1,\mathcal{R}))$, where $\hat{\rho}(m,\mathcal{R})$ and $\hat{\rho}(m-1,\mathcal{R})$ solve the pair of cubic equations that arise from part (i) of Proposition 2 in the paper. Note that (A-30) is the price informativeness when there are two types of groups; a fraction λ of the groups has m fund managers and a fraction $1-\lambda$ has $m-1$ fund managers.

If $c \in (\bar{c}(m), \underline{c}(m-1))$ for some m, an equilibrium with identical groups does not exist since conditions (10)-(11) in the paper fail to hold. However, in this case an equilibrium with two types of groups does exist; a fraction λ of the groups has m managers and a fraction $1-\lambda$ of the groups has $m-1$ managers. To see this, note that $\bar{c}(m,\lambda)$ is continuous in λ, where $\bar{c}(m,1) = \bar{c}(m)$ and $\bar{c}(m,0) = \underline{c}(m-1)$. Thus if $c \in (\bar{c}(\hat{m}), \underline{c}(\hat{m}-1))$ for some $\hat{m} \in \{1, 2, \ldots, n\}$, there exists $\lambda \in (0,1)$ that solves $c = \bar{c}(\hat{m}, \lambda)$. For the groups with $\hat{m}$ managers, condition (10) in the paper holds as an equality while condition (11) in the paper holds as an inequality. Likewise, for the groups with $\hat{m}-1$ managers, condition (10) in the paper holds as an inequality while condition (11) in the paper holds as an equality. Thus conditions (10)-(11) in the paper are satisfied for both types of groups, which proves part (ii). This completes the proof of the proposition. $\square$

Proof of Proposition 4.

Proposition 4 characterizes our model when the group size n is large. For large n, we show that the number of fund managers in each group grows at a rate that is proportional to $\sqrt{n}$. We let ξ_n and $\mathcal{R}_n$ denote the equity risk premium and the price informativeness, respectively, when the group size is n. For large n, we show $\xi_n < \xi_1$ and $\mathcal{R}_n > \mathcal{R}_1$, where ξ_1 and $\mathcal{R}_1$ correspond to the standard rational expectations model without mutual funds. Thus Proposition 4 gives a sufficient condition (namely, large n) such that our model with mutual funds produces a lower equity risk premium and a higher price informativeness relative to the standard model without mutual funds.

Fix a finite cost $c > 0$. We first argue that for n sufficiently large, cases (iii) and (iv) from Proposition 3 cannot arise. First note that

$$\underline{c}(0) = \frac{1}{2\tau}\log\left(\frac{1/\sigma_x^2 + 1/\sigma_\epsilon^2}{1/\sigma_x^2}\right) + \frac{(n-1)\zeta(1-\zeta)}{\tau(\sigma_\epsilon^2/\sigma_x^2 + \zeta(2-\zeta))},$$

where $\zeta = \hat{\rho}(1, 1/\sigma_x^2) = (\sigma_\epsilon^2/\sigma_x^2)(\sqrt{1+\sigma_x^2/\sigma_\epsilon^2} - 1)$ is obtained from Proposition 2. It is immediate from the above expression that $\lim_{n\uparrow\infty} \underline{c}(0) > c$, which rules out case (iv) of Proposition 3. Second, note that

$$\begin{aligned}\bar{c}(n) &= \frac{1}{2\tau}\log\left(\frac{\mathcal{R}(n) + 1/\sigma_\epsilon^2}{\mathcal{R}(n) + Q(n-1, \mathcal{R}(n))}\right) + f(n, \mathcal{R}(n)) + (n-1)f(n-1, \mathcal{R}(n)) \\ &= \frac{1}{2\tau}\log\left(\frac{\mathcal{R}(n) + 1/\sigma_\epsilon^2}{\mathcal{R}(n) + Q(n-1, \mathcal{R}(n))}\right) + (n-1)\frac{\hat{\rho}(1-\hat{\rho})}{\tau\sigma_\epsilon^2(\mathcal{R}(n) + Q(n-1, \mathcal{R}(n)))}.\end{aligned}$$

Since $Q(n-1, \mathcal{R}(n))$ grows with n but $\mathcal{R}(n)$ is bounded, it follows that $\lim_{n\uparrow\infty} \bar{c}(n) < c$. This rules out case (iii) from Proposition 3. Thus either case (i) or case (ii) of Proposition 3 must hold for sufficiently large n.

Now suppose that $\lim_{n\uparrow\infty} \hat{m}n^{-\beta} = \nu$ for some $\nu > 0$ and $\beta \in (0,1)$ which are constants to be determined. Using this conjecture in (A-17) we have

$$\lim_{n\uparrow\infty} \hat{\rho}_k = \frac{1}{2}.$$

Furthermore,

$$\begin{aligned}
\lim_{n\uparrow\infty} Q(\hat{m},\mathcal{R}(\hat{m}))n^{-\beta} &= \lim_{n\uparrow\infty} \frac{\hat{m}\hat{\rho}_k(2-\hat{\rho}_k)}{\sigma_\epsilon^2} n^{-\beta} = \frac{3\nu}{4\sigma_\epsilon^2}, \\
\lim_{n\uparrow\infty} \frac{\hat{m}}{n^{1+\beta}\hat{\alpha}(\hat{m})} &= \frac{\nu}{2}, \\
\lim_{n\uparrow\infty} \mathcal{R}(\hat{m})n^{-2\beta} &= \frac{\nu^2}{4\tau^2\sigma_u^2\sigma_\epsilon^4},
\end{aligned}$$

where the last limit applies to both of the price informativeness functions in (A-20) and (A-30). Given the asymptotic behavior of $Q(\hat{m},\mathcal{R})$ and $\mathcal{R}$, we conclude that

$$\lim_{n\uparrow\infty} \log\left[\frac{\mathcal{R}(\hat{m})+1/\sigma_\epsilon^2}{\mathcal{R}(\hat{m})+Q(\hat{m},\mathcal{R}(\hat{m}))}\right] = 0.$$

Thus $\underline{c}(\hat{m})$ and $\bar{c}(\hat{m})$, which are given in the statement of Proposition 3, depend only on the total fee f when n is large.

We now consider three cases for β. In each case, we evaluate the total fee f when n is large, where the total fee is given in Proposition 3 in the paper. First suppose $\beta > 1/2$. In this case, using the above limits, $\lim_{n\uparrow\infty} f(\hat{m},\mathcal{R}(\hat{m})) = 0$, which implies $\underline{c}(\hat{m}) \to 0$ and $\bar{c}(\hat{m}) \to 0$ as $n \to \infty$. Since $c > 0$, $\beta > 1/2$ cannot be an equilibrium growth rate for $\hat{m}$.

Next suppose $\beta < 1/2$. In this case, $\lim_{n\uparrow\infty} f(\hat{m},\mathcal{R}(\hat{m})) = \infty$, which implies $\underline{c}(\hat{m}) \to \infty$ and $\bar{c}(\hat{m}) \to \infty$ as $n \to \infty$. Since c is finite, $\beta < 1/2$ cannot be an equilibrium growth rate for $\hat{m}$.

Lastly, suppose $\beta = 1/2$. In this case,

$$\lim_{n\uparrow\infty} f(\hat{m},\mathcal{R}(\hat{m})) = \frac{\tau\sigma_u^2\sigma_\epsilon^2}{\nu^2},$$

which implies $\underline{c}(\hat{m})$ and $\bar{c}(\hat{m})$ both converge to $\tau\sigma_u^2\sigma_\epsilon^2/\nu^2$ as $n \to \infty$. Since an equilibrium always exists in our model, $\underline{c}(\hat{m})$ and $\bar{c}(\hat{m})$ must be in the neighborhood of c for large n, so $\nu = \sqrt{(\tau\sigma_u^2\sigma_\epsilon^2)/c}$.

The equity premium in the model is given by equation (24) in the paper, namely

$$\mu_x - \mathbb{E}[P_x] = \xi_n = \frac{\tau\mu_u}{\mathcal{R}_n + \tau b_n/d_n}, \tag{A-31}$$

where b_n and d_n are the equilibrium price coefficients and $\mathcal{R}_n$ is the price informativeness when the group size is n. The asymptotic behavior of $\mathcal{R}_n$, discussed above, shows that $\lim_{n\uparrow\infty}\mathcal{R}_n(\hat{m})/n$ is finite when $\beta = 1/2$. Furthermore, $\lim_{n\uparrow\infty}(b_n/d_n)n^{-1/2}$ is also finite. We conclude that $\lim_{n\uparrow\infty}\xi_n = 0$. Thus the equity risk premium is lower and the price informativeness is higher in an economy with mutual funds relative to the standard economy without mutual funds. This completes the proof. □

Proof of Proposition 5.

Proposition 5 extends our model to the case in which fund managers are allowed to use affine compensation contracts. The ith fund manager in group k charges a fixed account fee δ_{ik} in addition to the proportional fee α_{ik}. Taking the set $\mathcal{M} = \left\{m_k : k \in \left[0, \bar{k}\right]\right\}$ as given, Proposition 5 characterizes the optimal fees $\hat{\alpha}_{ik}$ and $\hat{\delta}_{ik}$ for the ith manager in group k. Our results in Proposition 5 show that efficient risk sharing is attained, $\hat{\alpha}_{ik} = 1/(1+h_k)$. This is because the ith manager uses the fixed fee $\hat{\delta}_{ik}$ to extract consumer surplus without distorting risk sharing when choosing $\hat{\alpha}_{ik}$.

We proceed as follows. First note that the ith informed agent in group k is always better off choosing α_{ik} and δ_{ik} such that $\phi_{ijk} > 0$ for all $j = 1, \ldots, h_k$. Since the households are symmetric within each group, if α_{ik} and δ_{ik} are chosen such that $\phi_{ijk} = 0$ for some j, then all households in group k hold zero units of the fund. In this case, the ith manager's profit is zero. But by choosing $\alpha_{ik} < 1$ and $\delta_{ik} > 0$ small enough such that $\phi_{ijk} > 0$ for all $j = 1, \ldots, h_k$, the ith manager can achieve a positive profit and is better off.

By the same arguments that we used in Propositions 1-3, the ex ante certainty equivalent wealth for the ith fund manager in group k is

$$\mathcal{U}_{ik}(m_k, \{\alpha_{lk}, \delta_{lk}\}_{l=1}^{m_k}; \mathcal{M}_k, \mathcal{A}_k) = \frac{1}{2\tau}\log\left(\mathcal{R} + 1/\sigma_\epsilon^2\right) + \alpha_{ik}P_{ik} + h_k\delta_{ik} - c + H, \tag{A-32}$$

where $\mathcal{A}_k = \left\{\alpha_{ik'}, \delta_{ik'} : i = 1, \ldots, m_{k'}; k' \in \left[0, \bar{k}\right], k' \neq k\right\}$ and where H is given by (A-24). Likewise, the certainty equivalent wealth for the jth household in group k is

$$\begin{aligned}\mathcal{V}_{jk}(m_k, \{\alpha_{lk}, \delta_{lk}\}_{l=1}^{m_k}; \mathcal{M}_k, \mathcal{A}_k) \quad &= \quad \frac{1}{2\tau}\log\left(\mathcal{R} + \frac{1}{\sigma_\epsilon^2}\sum_{l=1}^{m_k}\left(2\rho_{lk} - \rho_{lk}^2\right)\right) \\ &\quad -\sum_{l=1}^{m_k}\left(\frac{\alpha_{lk}P_{lk}}{h_k} + \delta_{lk}\right) + H,\end{aligned} \tag{A-33}$$

where $\rho_{ik} = (1-\alpha_{ik})/(\alpha_{ik}h_k)$.

Taking the optimal actions of the other agents as given, the ith manager chooses α_{ik}

and δ_{ik} to maximize (A-32). The optimal δ_{ik} is given by the fee that makes each household indifferent between buying and not buying the ith fund. If the jth household buys all m_k funds except the ith fund, the household's certainty equivalent wealth is

$$\mathcal{W}_{jk}(m_k, \{\alpha_{lk}, \delta_{lk}\}_{l=1}^{m_k}; \mathcal{M}_k, \mathcal{A}_k) = \frac{1}{2\tau} \log\left(\mathcal{R} + \frac{1}{\sigma_\epsilon^2} \sum_{\substack{l=1 \\ l \neq i}}^{m_k} \left(2\rho_{lk} - \rho_{lk}^2\right)\right) - \sum_{\substack{l=1 \\ l \neq i}}^{m_k} \left(\frac{\alpha_{lk} P_{lk}}{h_k} + \delta_{lk}\right) + H. \tag{A-34}$$

Comparing (A-33) and (A-34), the optimal fixed fee $\hat{\delta}_{ik}$ that makes the jth household indifferent is

$$\hat{\delta}_{ik} = \frac{1}{2\tau} \log\left(\frac{\mathcal{R} + \frac{1}{\sigma_\epsilon^2} \sum_{l=1}^{m_k} \left(2\rho_{lk} - \rho_{lk}^2\right)}{\mathcal{R} + \frac{1}{\sigma_\epsilon^2} \sum_{l=1, l\neq i}^{m_k} \left(2\rho_{lk} - \rho_{lk}^2\right)}\right) - \frac{\alpha_{ik} P_{ik}}{h_k}. \tag{A-35}$$

We note that (A-35) is a function of the contingent fee α_{ik}, which shows up in the last term and in the variable ρ_{ik}. Thus we substitute (A-35) back into (A-32) and maximize over ρ_{ik} to get $\hat{\rho}_{ik} = 1$, which reduces to (25) in the paper. The second order condition for a maximum is easily verified. We note that when $\hat{\rho}_{ik} = 1$, (A-15) implies the fund price is $P_{ik} = 0$. We then substitute the optimal contingent fee $\hat{\alpha}_{ik}$ into (A-35) to get equation (26) in the paper. Lastly, we construct the equilibrium price informativeness $\mathcal{R}$ by substituting the equilibrium contingent fees into (A-19), which yields the expression for $\mathcal{R}$ in the Proposition. This completes the proof. □

Proof of Proposition 6.

Proposition 6 characterizes our equilibrium at the information acquisition stage (i.e., at date 0) for the case in which fund managers are allowed to use affine compensation contracts. The optimal affine contract is given in Proposition 5, which characterizes the optimal fixed fee $\hat{\delta}_{ik}$ and the optimal proportional fee $\hat{\alpha}_{ik}$ for the ith manager in group k. Taking these optimal fees as given, we solve for a set $\mathcal{M} = \left\{m_k : k \in \left[0, \bar{k}\right]\right\}$ in Proposition 6 such that none of the m_k informed agents in group k is better off being uninformed and none of the $n - m_k$ uninformed agents is better off being informed. As we mention in the paper, multiple equilibria can arise at the information acquisition stage of our model. Thus we focus on the case in which there are at most two types of groups, where the two types differ only with respect to the number of informed agents m_k. This mirrors our approach in Proposition 3.

Our proof is constructive and follows the arguments used in Proposition 3. Upon substituting $\hat{\alpha}_{ik}$ and $\hat{\delta}_{ik}$ from Proposition 5 into expressions (A-32) and (A-33), and using the fact that Proposition 5 implies $P_{ik} = 0$, we get

$$\begin{aligned}
\mathcal{U}_k(m_k, \hat{\beta}_k(m_k); \mathcal{M}_k, \hat{\mathcal{A}}_k) &= \frac{\log\left(\mathcal{R} + 1/\sigma_\epsilon^2\right)}{2\tau} + \frac{n - m_k}{2\tau} \log\left[\frac{\mathcal{R} + m_k/\sigma_\epsilon^2}{\mathcal{R} + (m_k - 1)/\sigma_\epsilon^2}\right] - c + H, \\
\mathcal{V}_k(m_k, \hat{\beta}_k(m_k); \mathcal{M}_k, \hat{\mathcal{A}}_k) &= \frac{\log\left(\mathcal{R} + m_k/\sigma_\epsilon^2\right)}{2\tau} - \frac{m_k}{2\tau} \log\left[\frac{\mathcal{R} + m_k/\sigma_\epsilon^2}{\mathcal{R} + (m_k - 1)/\sigma_\epsilon^2}\right] + H,
\end{aligned}$$

where $\hat{\beta}_k(m_k) = \left\{\hat{\alpha}_k(m_k), \hat{\delta}_k(m_k)\right\}$ are the optimal fees for each manager in group k. Note that the above expressions for $\mathcal{U}_k$ and $\mathcal{V}_k$ coincide with those in the paper that immediately precede the statement of Proposition 6.

Now suppose $m_k = m$ for all $k \in [0, \bar{k}]$. In this case it is easy to verify that $\mathcal{R} = \text{var}(X|P_x)^{-1}$ is given in the Proposition statement. As before, the inequalities that define an equilibrium at the information acquisition stage, (10)-(11) in the paper, can be written as $c \leq \bar{c}(m)$ and $c \geq \underline{c}(m)$, where the functions $\bar{c}(m)$ and $\underline{c}(m)$ are given in the statement of Proposition 6. The two corner solutions $m_k = n$ and $m_k = 0$ provide parts (iii) and (iv) of the proposition.

To prove part (i), we need to show that $[\underline{c}(m), \bar{c}(m)]$ is a non-empty interval for $m \in \{1, \ldots, n-1\}$. Define

$$\begin{aligned}
J(r, m) &= \frac{1}{2\tau} \log\left(\frac{r + 1/\sigma_\epsilon^2}{r + (m-1)/\sigma_\epsilon^2}\right) + \frac{(m-1)}{2\tau} \log\left(\frac{r + (m-1)/\sigma_\epsilon^2}{r + (m-2)/\sigma_\epsilon^2}\right) \\
&\quad + \frac{(n-m)}{2\tau} \log\left(\frac{r + m/\sigma_\epsilon^2}{r + (m-1)/\sigma_\epsilon^2}\right)
\end{aligned}$$

where $r > 0$. By construction, $J(\mathcal{R}(m), m) = \bar{c}(m)$ and $J(\mathcal{R}(m), m+1) = \underline{c}(m)$. To show $[\underline{c}(m), \bar{c}(m)]$ is non-empty, it is sufficient to show $\partial J(r, m)/\partial m < 0$. Direct computation shows that

$$\begin{aligned}\frac{\partial J(r,m)}{\partial m} &= \frac{1}{2\tau}\log\left(\frac{r+(m-1)/\sigma_\epsilon^2}{r+(m-2)/\sigma_\epsilon^2}\right)-\frac{1}{2\tau}\log\left(\frac{r+m/\sigma_\epsilon^2}{r+(m-1)/\sigma_\epsilon^2}\right)\\ &\quad -\frac{1}{2\tau\sigma_\epsilon^2}\left(\frac{1}{r+(m-1)/\sigma_\epsilon^2}\right)+\frac{(m-1)}{2\tau\sigma_\epsilon^2}\left(\frac{1}{r+(m-1)/\sigma_\epsilon^2}-\frac{1}{r+(m-2)/\sigma_\epsilon^2}\right)\\ &\quad +\frac{(n-m)}{2\tau\sigma_\epsilon^2}\left(\frac{1}{r+m/\sigma_\epsilon^2}-\frac{1}{r+(m-1)/\sigma_\epsilon^2}\right).\end{aligned}$$

Clearly the sum of the non-logarithmic terms above are negative, and further

$$-\log\left(\mathcal{R}+m/\sigma_\epsilon^2\right)+2\log\left(\mathcal{R}+(m-1)/\sigma_\epsilon^2\right)-\log\left(\mathcal{R}+(m-2)/\sigma_\epsilon^2\right)<0$$

due to the concavity of the log function. This shows $[\underline{c}(m),\bar{c}(m)]$ is a non-empty interval. Thus if $c\in[\underline{c}(\hat{m}),\bar{c}(\hat{m})]$ for some $\hat{m}\in\{1,\ldots,n-1\}$, conditions (10)-(11) in the paper are satisfied and there exists an equilibrium where all groups have $m_k=\hat{m}$ managers.

To see part (ii) of Proposition 6, let

$$\begin{aligned}\bar{c}(m,\lambda) &= \frac{1}{2\tau}\log\left(\frac{\mathcal{R}(m,\lambda)+1/\sigma_\epsilon^2}{\mathcal{R}(m,\lambda)+(m-1)/\sigma_\epsilon^2}\right)\\ &\quad +\frac{(m-1)}{2\tau}\log\left(\frac{\mathcal{R}(m,\lambda)+(m-1)/\sigma_\epsilon^2}{\mathcal{R}(m,\lambda)+(m-2)/\sigma_\epsilon^2}\right)\\ &\quad +\frac{(n-m)}{2\tau}\log\left(\frac{\mathcal{R}(m,\lambda)+m/\sigma_\epsilon^2}{\mathcal{R}(m,\lambda)+(m-1)/\sigma_\epsilon^2}\right),\end{aligned}\tag{A-36}$$

where

$$\mathcal{R}(m,\lambda)=\frac{1}{\sigma_x^2}+\frac{1}{\sigma_u^2}\left(\frac{\lambda m(1+n-m)+(1-\lambda)(m-1)(1+n-(m-1))}{n\tau\sigma_\epsilon^2}\right)^2.\tag{A-37}$$

Equation (A-37) corresponds to the price informativeness when a fraction λ of the groups has m managers and a fraction $1-\lambda$ has $m-1$ managers. Note that $\bar{c}(m,\lambda)$ in (A-36) is continuous in λ and satisfies $\bar{c}(m,0)=\underline{c}(m-1)$ and $\bar{c}(m,1)=\bar{c}(m)$. Thus if $c\in(\bar{c}(\hat{m}),\underline{c}(\hat{m}-1))$ for some $\hat{m}\in\{1,\ldots,n\}$, there exists $\lambda\in(0,1)$ such that $c=\bar{c}(\hat{m},\lambda)$. The equilibrium conditions (10)-(11) from Definition 3 in the paper are satisfied for both the groups with $m_k=\hat{m}$ managers and the groups with $m_k=\hat{m}-1$ managers. In particular, for the groups with $\hat{m}$ managers, condition (10) in the paper holds as an equality while condition (11) in the paper holds as an inequality. Likewise, for the groups with $\hat{m}-1$

managers, condition (10) in the paper holds as an inequality while condition (11) in the paper holds as an equality. This completes the proof of the proposition. □

Proof of Proposition 7.

Proposition 7 characterizes our model when the group size n is large and fund managers are allowed to use affine compensation contracts. For large n, we show that the number of fund managers in each group grows at a rate that is proportional to $\sqrt{n}$. Proposition 7 gives a sufficient condition (namely, large n) such that our mutual fund model with affine compensation contracts produces a lower equity risk premium and a higher price informativeness relative to the standard model without mutual funds. This mirrors our result in Proposition 4 where only a proportional fee is allowed.

The proof is similar to that of Proposition 4, with the exception that $\underline{c}(m)$ and $\bar{c}(m)$ are now given in Proposition 6. Fix a finite cost $c > 0$. Using Proposition 6 we have

$$\underline{c}(0) = \frac{n}{2\tau} \log\left(\frac{\sigma_x^2 + \sigma_\epsilon^2}{\sigma_\epsilon^2}\right)$$

and thus $\lim_{n\uparrow\infty} \underline{c}(0) > c$, which rules out part (iv) of Proposition 6 for large n. Similarly, using Proposition 6, note that

$$\bar{c}(n) = \frac{1}{2\tau} \log\left(\frac{\mathcal{R}(n) + 1/\sigma_\epsilon^2}{\mathcal{R}(n) + (n-1)/\sigma_\epsilon^2}\right) + \frac{(n-1)}{2\tau} \log\left(\frac{\mathcal{R}(n) + (n-1)/\sigma_\epsilon^2}{\mathcal{R}(n) + (n-2)/\sigma_\epsilon^2}\right)$$

and thus $\lim_{n\uparrow\infty} \bar{c}(n) < c$, which rules out part (iii) of Proposition 6 for large n. Thus it must be the case that part (i) or part (ii) holds for sufficiently large n.

Now suppose $\lim_{n\uparrow\infty} \hat{m}n^{-\beta} = \nu$ for some $\nu > 0$ and $\beta \in (0,1)$ to be determined. If $\beta > 1/2$, $\lim_{n\uparrow\infty} \underline{c}(\hat{m}) = 0$ and $\lim_{n\uparrow\infty} \bar{c}(\hat{m}) = 0$. Since $c > 0$, $\beta > 1/2$ cannot be an equilibrium growth rate for $\hat{m}$. Next suppose $\beta < 1/2$. In this case, $\lim_{n\uparrow\infty} \underline{c}(\hat{m}) = \infty$ and $\lim_{n\uparrow\infty} \bar{c}(\hat{m}) = \infty$. Since c is finite, $\beta < 1/2$ cannot be an equilibrium growth rate for $\hat{m}$. Lastly, suppose $\beta = 1/2$. In this case, one can verify that

$$\lim_{n\uparrow\infty} \underline{c}(\hat{m}) = \lim_{n\uparrow\infty} \bar{c}(\hat{m}) = \frac{\tau\sigma_u^2\sigma_\epsilon^2}{2\nu^2}.$$

Since an equilibrium always exists in our model, $\underline{c}(\hat{m})$ and $\bar{c}(\hat{m})$ must be in the neighborhood of c for large n, so $\nu = \sqrt{(\tau\sigma_u^2\sigma_\epsilon^2)/2c}$. Lastly, note that $\mathcal{R}_n(\hat{m})$, which is the price informativeness when the group size is n, satisfies

$$\lim_{n\uparrow\infty}\frac{\mathcal{R}_n(\hat{m})}{n}=\frac{1}{2\tau\sigma_\epsilon^2 c}$$

when $\beta=1/2$. Also note that $\lim_{n\uparrow\infty}(b_n/d_n)n^{-1/2}$ is finite. Using (A-31) and the asymptotic behavior for $\mathcal{R}_n$ and b_n/d_n, we conclude that for sufficiently large n, the equity risk premium is lower and the price informativeness is higher in our economy with mutual funds relative to the standard economy without mutual funds. This completes the proof. □

Proof of Proposition 8.

Proposition 8 characterizes some features of our equilibrium at dates 1 and 2 when agents and groups are heterogeneous. We allow for different group sizes, where n_k is the size of group k. We also allow the households to have heterogeneous risk aversion parameters, where τ_{jk} is the risk aversion parameter of the jth household in group k. Lastly, we allow the fund managers to have heterogeneous risk aversion parameters and heterogeneous signal variances; the ith fund manager in group k has risk aversion parameter τ_{ik} and signal variance σ_{ik}^2. Taking the set $\{n_k, m_k : k\in[0,\bar{k}]\}$ as given, Proposition 8 extends Propositions 1 and 2 to the heterogeneous case.

The optimal demand of the ith manager in group k is given by the usual mean-variance form

$$\hat{\gamma}_{ik}=\frac{\mathbb{E}\left[X|P_x,Y_{ik}\right]-P_x}{\alpha_{ik}\tau_{ik}\,\mathrm{var}(X|P_x,Y_{ik})}, \tag{A-38}$$

where τ_{ik} is the manager's risk aversion parameter. For the jth household in group k, we evaluate the household's conditional expected utility using Lemma 1 to get

$$U_{jk}\equiv\mathbb{E}\left[-e^{-\tau_{jk}W_{jk}}\,\middle|\,P_x\right]=-\sqrt{\frac{1}{B_{jk}\mathrm{var}(X|P_x)}}\,e^{C_{jk}+\frac{D_{jk}^2}{2B_{jk}}}, \tag{A-39}$$

where

$$B_{jk} = \mathrm{var}(X|P_x)^{-1} + 2\tau_{jk}\sum_{l=1}^{m_k}\phi_{ljk}\omega_{lk} - \tau_{jk}^2\sum_{l=1}^{m_k}\phi_{ljk}^2\sigma_{lk}^2\omega_{lk}^2, \tag{A-40}$$

$$\begin{aligned} C_{jk} &= -\frac{(\mathbb{E}[X|\,P_x])^2}{2\mathrm{var}(X|P_x)} + \tau_{jk}\theta_{jk}P_x + \tau_{jk}\sum_{l=1}^{m_k}\phi_{ljk}\alpha_{lk}P_{lk} \\ &\quad +\tau_{jk}P_x\sum_{l=1}^{m_k}\phi_{ljk}A_{lk}(1-\alpha_{lk}) + \frac{P_x^2\tau_{jk}^2}{2}\sum_{l=1}^{m_k}\phi_{ljk}^2\sigma_{lk}^2\omega_{lk}^2, \end{aligned} \tag{A-41}$$

$$\begin{aligned} D_{jk} &= \frac{\mathbb{E}[X|\,P_x]}{\mathrm{var}(X|P_x)} - \tau_{jk}\theta_{jk} - \tau_{jk}\sum_{l=1}^{m_k}\phi_{ljk}A_{lk}(1-\alpha_{lk}) \\ &\quad +P_x\tau_{jk}\sum_{l=1}^{m_k}\phi_{ljk}\omega_{lk} - P_x\tau_{jk}^2\sum_{l=1}^{m_k}\phi_{ljk}^2\sigma_{lk}^2\omega_{lk}^2, \end{aligned} \tag{A-42}$$

$$A_{ik} = \frac{1}{\alpha_{ik}\tau_{ik}}\left[(\mu_x - P_x)\left(\frac{1}{\sigma_x^2} + \frac{1}{\sigma_{ik}^2} + \frac{b^2}{d^2\sigma_u^2}\right) + \frac{b\left(P_x - \bar{P}_x\right)}{d^2\sigma_u^2} - \frac{\mu_x}{\sigma_{ik}^2}\right],$$

$$\omega_{ik} = \frac{1-\alpha_{ik}}{\alpha_{ik}\tau_{ik}\sigma_{ik}^2}. \tag{A-43}$$

We remark that (A-39)-(A-43) is the counterpart to (A-6) for the case in which households and fund managers are heterogeneous. It is straightforward, but somewhat tedious, to verify that (A-39)-(A-43) coincides with (A-6) in the special case of $\tau_{ik} = \tau_{jk} = \tau$ and $\sigma_{ik}^2 = \sigma_\epsilon^2$ for all i, j, k.

The first order condition for the household's investment in the risky asset is

$$\frac{\hat{D}_{jk}}{\mathrm{var}(X|P_x)^{-1} + 2\tau_{jk}\sum_{l=1}^{m_k}\hat{\phi}_{ljk}\omega_{lk} - \tau_{jk}^2\sum_{l=1}^{m_k}\hat{\phi}_{ljk}^2\sigma_{lk}^2\omega_{lk}^2} = P_x, \tag{A-44}$$

where

$$\begin{aligned} \hat{D}_{jk} &= \frac{\mathbb{E}[X|\,P_x]}{\mathrm{var}(X|P_x)} - \tau_{jk}\hat{\theta}_{jk} - \tau_{jk}\sum_{l=1}^{m_k}\hat{\phi}_{ljk}A_{lk}(1-\alpha_{lk}) \\ &\quad +P_x\tau_{jk}\sum_{l=1}^{m_k}\hat{\phi}_{ljk}\omega_{lk} - P_x\tau_{jk}^2\sum_{l=1}^{m_k}\hat{\phi}_{ljk}^2\sigma_{lk}^2\omega_{lk}^2, \end{aligned}$$

while the first order condition for the household's investment in the ith fund is

$$\frac{\omega_{ik} - \tau_{jk}\hat{\phi}_{ijk}\sigma_{ik}^2\omega_{ik}^2}{\text{var}(X|P_x)^{-1} + 2\tau_{jk}\sum_{l=1}^{m_k}\hat{\phi}_{ljk}\omega_{lk} - \tau_{jk}^2\sum_{l=1}^{m_k}\hat{\phi}_{ljk}^2\sigma_{lk}^2\omega_{lk}^2} = \alpha_{ik}P_{ik}, \tag{A-45}$$

for $i = 1, \ldots, m_k$.

In Proposition 1 we used the fact that the households were homogeneous within a group to obtain $\hat{\phi}_{ijk} = 1/h_k$. Since now the households have heterogeneous risk aversion parameters, this argument no longer works. Thus we solve the first order conditions directly to obtain the optimal mutual fund demands. Using (A-45), the first order condition for ϕ_{ijk} can be rewritten as

$$\alpha_{ik}P_{ik} = \hat{B}_{jk}^{-1}\left[\omega_{ik} - \tau_{jk}\hat{\phi}_{ijk}\sigma_{ik}^2\omega_{ik}^2\right], \tag{A-46}$$

where $\hat{B}_{jk}$ is

$$\hat{B}_{jk} = \text{var}(X|P_x)^{-1} + 2\tau_{jk}\sum_{l=1}^{m_k}\hat{\phi}_{ljk}\omega_{lk} - \tau_{jk}^2\sum_{l=1}^{m_k}\hat{\phi}_{ljk}^2\sigma_{lk}^2\omega_{lk}^2.$$

If we sum (A-46) over all funds in group k, we get

$$\sum_{l=1}^{m_k}\alpha_{lk}P_{lk} = \hat{B}_{jk}^{-1}\left[\sum_{l=1}^{m_k}\omega_{lk} - \tau_{jk}\sum_{l=1}^{m_k}\hat{\phi}_{ljk}\sigma_{lk}^2\omega_{lk}^2\right]. \tag{A-47}$$

Solving for $\hat{B}_{jk}$ and substituting into (A-46) gives

$$\frac{\alpha_{ik}P_{ik}}{\sum_{l=1}^{m_k}\alpha_{lk}P_{lk}} = \frac{\omega_{ik} - \tau_{jk}\hat{\phi}_{ijk}\sigma_{ik}^2\omega_{ik}^2}{\sum_{l=1}^{m_k}\omega_{lk} - \tau_{jk}\sum_{l=1}^{m_k}\hat{\phi}_{ljk}\sigma_{lk}^2\omega_{lk}^2}. \tag{A-48}$$

Next we solve (A-48) for $\hat{\phi}_{ijk}$ to get

$$\hat{\phi}_{ijk} = \frac{1}{\sigma_{ik}^2\omega_{ik}^2}\left\{\frac{\omega_{ik}}{\tau_{jk}} - \frac{1}{\tau_{jk}}\left[\frac{\alpha_{ik}P_{ik}}{\sum_{l=1}^{m_k}\alpha_{lk}P_{lk}}\right]\sum_{l=1}^{m_k}\omega_{lk} + \left[\frac{\alpha_{ik}P_{ik}}{\sum_{l=1}^{m_k}\alpha_{lk}P_{lk}}\right]\sum_{l=1}^{m_k}\hat{\phi}_{ljk}\sigma_{lk}^2\omega_{lk}^2\right\}.$$

Now apply the market clearing condition $\sum_{j=1}^{h_k} \hat{\phi}_{ijk} = 1$ to get

$$\begin{aligned}
\sigma_{ik}^2 \omega_{ik}^2 &= \omega_{ik} \left(\sum_{j=1}^{h_k} \frac{1}{\tau_{jk}} \right) - \left(\sum_{j=1}^{h_k} \frac{1}{\tau_{jk}} \right) \left[\frac{\alpha_{ik} P_{ik}}{\sum_{l=1}^{m_k} \alpha_{lk} P_{lk}} \right] \sum_{l=1}^{m_k} \omega_{lk} \\
&\quad + \left[\frac{\alpha_{ik} P_{ik}}{\sum_{l=1}^{m_k} \alpha_{lk} P_{lk}} \right] \sum_{l=1}^{m_k} \left(\sum_{j=1}^{h_k} \hat{\phi}_{ljk} \right) \sigma_{lk}^2 \omega_{lk}^2 \\
&= \omega_{ik} \bar{\tau}_k^{-1} - \bar{\tau}_k^{-1} \left[\frac{\alpha_{ik} P_{ik}}{\sum_{l=1}^{m_k} \alpha_{lk} P_{lk}} \right] \sum_{l=1}^{m_k} \omega_{lk} + \left[\frac{\alpha_{ik} P_{ik}}{\sum_{l=1}^{m_k} \alpha_{lk} P_{lk}} \right] \sum_{l=1}^{m_k} \sigma_{lk}^2 \omega_{lk}^2,
\end{aligned}$$

where $\sum_{j=1}^{h_k} \frac{1}{\tau_{jk}} = \bar{\tau}_k^{-1}$. Solving the last expression for $\alpha_{ik} P_{ik}$ gives

$$\alpha_{ik} P_{ik} = \frac{\omega_{ik} - \bar{\tau}_k \sigma_{ik}^2 \omega_{ik}^2}{\left[\sum_{l=1}^{m_k} \omega_{lk} - \bar{\tau}_k \sum_{l=1}^{m_k} \sigma_{lk}^2 \omega_{lk}^2 \right]} \left[\sum_{l=1}^{m_k} \alpha_{lk} P_{lk} \right]. \tag{A-49}$$

Now use (A-47) to substitute for $\sum_{l=1}^{m_k} \alpha_{lk} P_{lk}$ in (A-49) to get

$$\alpha_{ik} P_{ik} = \frac{\omega_{ik} - \bar{\tau}_k \sigma_{ik}^2 \omega_{ik}^2}{\left[\sum_{l=1}^{m_k} \omega_{lk} - \bar{\tau}_k \sum_{l=1}^{m_k} \sigma_{lk}^2 \omega_{lk}^2 \right]} \left[\sum_{l=1}^{m_k} \omega_{lk} - \tau_{jk} \sum_{l=1}^{m_k} \hat{\phi}_{ljk} \sigma_{lk}^2 \omega_{lk}^2 \right] \hat{B}_{jk}^{-1}. \tag{A-50}$$

Comparing (A-46) and (A-50) shows that $\tau_{jk} \hat{\phi}_{ijk} = \bar{\tau}_k$ for all i. We can rearrange this to get $\hat{\phi}_{ijk} = \bar{\tau}_k / \tau_{jk}$, which verifies part (ii) of Proposition 8 in the paper. Substituting $\hat{\phi}_{ijk}$ back into (A-50) gives the equilibrium price of fund i in group k

$$P_{ik} = \frac{\frac{1}{\alpha_{ik}} \left\{ \left[\frac{1-\alpha_{ik}}{\tau_{ik} \alpha_{ik} \sigma_{ik}^2} \right] - \bar{\tau}_k \left[\frac{1-\alpha_{ik}}{\tau_{ik} \alpha_{ik} \sigma_{ik}} \right]^2 \right\}}{\mathrm{var}(X|P_x)^{-1} + 2\bar{\tau}_k \sum_{l=1}^{m_k} \left[\frac{1-\alpha_{lk}}{\tau_{lk} \alpha_{lk} \sigma_{lk}^2} \right] - \bar{\tau}_k^2 \sum_{l=1}^{m_k} \left[\frac{1-\alpha_{lk}}{\tau_{lk} \alpha_{lk} \sigma_{lk}} \right]^2}. \tag{A-51}$$

To get part (i) of Proposition 8, we use (A-38) and (A-44) along with the risky asset's market clearing condition, which is given by equation (7) in the paper. Note that the market clearing condition gives three equations: one for the constant terms, one for the terms that multiply X, and one for the terms that multiply U. The equation for the terms that multiply X gives

$$b=\frac{\int_0^{\bar{k}}\sum_{l=1}^{m_k}\left(\frac{1}{\tau_{lk}\alpha_{lk}\sigma_{lk}^2}\right)dk+\frac{b^2}{d^2\sigma_u^2}\left(\int_0^{\bar{k}}\bar{\tau}_k^{-1}dk+\int_0^{\bar{k}}\sum_{l=1}^{m_k}\tau_{lk}^{-1}dk\right)}{\int_0^{\bar{k}}\sum_{l=1}^{m_k}\left(\frac{1}{\tau_{lk}\alpha_{lk}\sigma_{lk}^2}\right)dk+\left(\frac{1}{\sigma_x^2}+\frac{b^2}{d^2\sigma_u^2}\right)\left(\int_0^{\bar{k}}\bar{\tau}_k^{-1}dk+\int_0^{\bar{k}}\sum_{l=1}^{m_k}\tau_{lk}^{-1}dk\right)},$$

while the equation for the terms that multiply U gives

$$d=\frac{1+\frac{b}{d\sigma_u^2}\left(\int_0^{\bar{k}}\bar{\tau}_k^{-1}dk+\int_0^{\bar{k}}\sum_{l=1}^{m_k}\tau_{lk}^{-1}dk\right)}{\int_0^{\bar{k}}\sum_{l=1}^{m_k}\left(\frac{1}{\tau_{lk}\alpha_{lk}\sigma_{lk}^2}\right)dk+\left(\frac{1}{\sigma_x^2}+\frac{b^2}{d^2\sigma_u^2}\right)\left(\int_0^{\bar{k}}\bar{\tau}_k^{-1}dk+\int_0^{\bar{k}}\sum_{l=1}^{m_k}\tau_{lk}^{-1}dk\right)}.$$

Thus the ratio b/d is

$$\frac{b}{d}=\frac{\int_0^{\bar{k}}\sum_{l=1}^{m_k}\left(\frac{1}{\tau_{lk}\alpha_{lk}\sigma_{lk}^2}\right)dk+\frac{b^2}{d^2\sigma_u^2}\left(\int_0^{\bar{k}}\bar{\tau}_k^{-1}dk+\int_0^{\bar{k}}\sum_{l=1}^{m_k}\tau_{lk}^{-1}dk\right)}{1+\frac{b}{d\sigma_u^2}\left(\int_0^{\bar{k}}\bar{\tau}_k^{-1}dk+\int_0^{\bar{k}}\sum_{l=1}^{m_k}\tau_{lk}^{-1}dk\right)},$$

which can be simplified to give

$$\frac{b}{d}=\int_0^{\bar{k}}\sum_{l=1}^{m_k}\left(\frac{1}{\tau_{lk}\alpha_{lk}\sigma_{lk}^2}\right)dk.$$

This proves part (i) of Proposition 8 in the paper. To get part (iii) of the proposition, the ith fund manager in group k solves

$$\max_{\alpha_{ik}}\alpha_{ik}P_{ik},$$

where P_{ik} is given by (A-51). The first order condition can be written as

$$1-2\bar{\tau}_k\sigma_{ik}^2\hat{\omega}_{ik}=\frac{2\bar{\tau}_k\hat{\omega}_{ik}\left(1-\bar{\tau}_k\sigma_{ik}^2\hat{\omega}_{ik}\right)^2}{\text{var}(X|P_x)^{-1}+2\bar{\tau}_k\sum_{l=1}^{m_k}\hat{\omega}_{lk}-\bar{\tau}_k^2\sum_{l=1}^{m_k}\sigma_{lk}^2\hat{\omega}_{lk}^2},\tag{A-52}$$

where $\hat{\omega}_{ik}=(1-\hat{\alpha}_{ik})/\left(\hat{\alpha}_{ik}\tau_{ik}\sigma_{ik}^2\right)$. Expressing $\hat{\alpha}_{ik}$ in terms of $\hat{\omega}_{ik}$ gives the expression $\hat{\alpha}_{ik}=1/\left(1+\tau_{ik}\sigma_{ik}^2\hat{\omega}_{ik}\right)$.

Since there are m_k managers in group k, the first order condition in (A-52) holds for each manager $i=1,\ldots,m_k$. Thus there are m_k equations that must be solved jointly to obtain the optimal fees $\hat{\alpha}_{1k},\hat{\alpha}_{2k},\ldots,\hat{\alpha}_{m_kk}$. This system of equations for group k is given by equation (28) in the paper, which coincides with (A-52).

To see that $\hat{\alpha}_{ik} \in \left((1+0.5(\tau_{ik}/\bar{\tau}_k))^{-1}, 1\right)$, we analyze the ith manager's first order condition in (A-52) by taking the other $m_k - 1$ managers' optimal fees as given. Let

$$F_{-i}^k = \mathrm{var}(X|P_x)^{-1} + 2\bar{\tau}_k \sum_{\substack{l=1 \\ l\neq i}}^{m_k} \hat{\omega}_{lk} - \bar{\tau}_k^2 \sum_{\substack{l=1 \\ l\neq i}}^{m_k} \sigma_{lk}^2 \hat{\omega}_{lk}^2. \qquad \text{(A-53)}$$

Then (A-52) can be written as

$$1 - 2\bar{\tau}_k \sigma_{ik}^2 \hat{\omega}_{ik} = \frac{2\bar{\tau}_k \hat{\omega}_{ik} \left(1 - \bar{\tau}_k \sigma_{ik}^2 \hat{\omega}_{ik}\right)^2}{F_{-i}^k + 2\bar{\tau}_k \hat{\omega}_{ik} - \bar{\tau}_k^2 \sigma_{ik}^2 \hat{\omega}_{ik}^2},$$

and some algebra gives

$$-\left(\bar{\tau}_k^2 \sigma_{ik}^2\right) \hat{\omega}_{ik}^2 - 2\left(\bar{\tau}_k \sigma_{ik}^2 F_{-i}^k\right) \hat{\omega}_{ik} + F_{-i}^k = 0, \qquad \text{(A-54)}$$

which is a quadratic function of $\hat{\omega}_{ik}$. Assuming $F_{-i}^k > 0$, which we verify to be true in equilibrium, (A-54) has one negative root, which is not meaningful economically, and one positive root. Defining the function $g(x)$ as

$$g(x) = -\left(\bar{\tau}_k^2 \sigma_{ik}^2\right) x^2 - 2\left(\bar{\tau}_k \sigma_{ik}^2 F_{-i}^k\right) x + F_{-i}^k\,,$$

note that $g(0) = F_{-i}^k > 0$ and $g\left(1/\left(2\bar{\tau}_k \sigma_{ik}^2\right)\right) < 0$. Thus the positive root of (A-54) lies in the interval $\left(0, 1/\left(2\bar{\tau}_k \sigma_{ik}^2\right)\right)$. Since $\hat{\omega}_{ik} = \left(1 - \hat{\alpha}_{ik}\right)/(\hat{\alpha}_{ik}\tau_{ik}\sigma_{ik}^2)$ and $\hat{\omega}_{ik} \in \left(0, 1/\left(2\bar{\tau}_k \sigma_{ik}^2\right)\right)$, it is clear that $\hat{\alpha}_{ik} \in \left((1+0.5(\tau_{ik}/\bar{\tau}_k))^{-1}, 1\right)$, as claimed.

To verify that $F_{-i}^k > 0$ in equilibrium, note that our argument above implies that $\hat{\omega}_{lk} \in \left(0, 1/\left(2\bar{\tau}_k \sigma_{lk}^2\right)\right)$ for all $l = 1, \ldots, m_k$. From (A-53), a sufficient condition for $F_{-i}^k > 0$ is

$$2\bar{\tau}_k \hat{\omega}_{lk} - \bar{\tau}_k^2 \sigma_{lk}^2 \hat{\omega}_{lk}^2 > 0$$

for all l. This sufficient condition is satisfied when $\hat{\omega}_{lk} \in \left(0, 1/\left(2\bar{\tau}_k \sigma_{lk}^2\right)\right)$ for all l.

Given the above results, we now go back and verify the second order condition to the ith fund manager's problem $\max_{\alpha_{ik}} \alpha_{ik} P_{ik}$. Using the first order condition in (A-52), the second order condition can be written as

$$-2\bar{\tau}_k\sigma_{ik}^2\frac{\left[\text{var}(X|P_x)^{-1}+2\bar{\tau}_k\sum_{l=1}^{m_k}\hat{\omega}_{lk}-\bar{\tau}_k^2\sum_{l=1}^{m_k}\sigma_{lk}^2\hat{\omega}_{lk}^2-\bar{\tau}_k\left(\hat{\omega}_{ik}-\bar{\tau}_k\sigma_{ik}^2\hat{\omega}_{ik}^2\right)\right]}{\left[\text{var}(X|P_x)^{-1}+2\bar{\tau}_k\sum_{l=1}^{m_k}\hat{\omega}_{lk}-\bar{\tau}_k^2\sum_{l=1}^{m_k}\sigma_{lk}^2\hat{\omega}_{lk}^2\right]^2}.$$

The expression in the numerator can be rewritten as

$$\text{var}(X|P_x)^{-1}+2\bar{\tau}_k\sum_{\substack{l=1\\l\neq i}}^{m_k}\hat{\omega}_{lk}-\bar{\tau}_k^2\sum_{\substack{l=1\\l\neq i}}^{m_k}\sigma_{lk}^2\hat{\omega}_{lk}^2+\bar{\tau}_k\hat{\omega}_{ik}=F_{-i}^k+\bar{\tau}_k\hat{\omega}_{ik}>0.$$

Thus the second order condition for a maximum is verified.

We have shown that in the case of heterogeneous managers and households, $\alpha_{ik}=1$ is never optimal for the ith manager. To understand this result, note that if $\alpha_{ik}=1$, the fund's price is $P_{ik}=0$. But given $\hat{\alpha}_{lk}\in(0,1)$ for $l\neq i$, there always exist $0<\hat{\alpha}_{ik}<1$ so that $P_{ik}>0$ and the ith manager has positive profits.

Now we return to the household's optimization problem and we verify the second order conditions for a maximum. The expected utility of the jth household in group k is given by (A-39). The first order condition for the household's investment in the risky asset can be written as

$$\frac{\partial U_{jk}}{\partial\theta_{jk}}=U_{jk}\left[\frac{\partial C_{jk}}{\partial\theta_{jk}}+\frac{D_{jk}}{B_{jk}}\frac{\partial D_{jk}}{\partial\theta_{jk}}\right]=0, \tag{A-55}$$

where U_{jk} is from (A-39) and B_{jk}, C_{jk}, and D_{jk} are from (A-40)-(A-42). Likewise, the first order condition for the household's investment in the ith fund can be written as

$$\frac{\partial U_{jk}}{\partial\phi_{ijk}}=U_{jk}\left[\frac{\partial C_{jk}}{\partial\phi_{ijk}}+\frac{D_{jk}}{B_{jk}}\frac{\partial D_{jk}}{\partial\phi_{ijk}}-\frac{1}{2}\left(\frac{1}{B_{jk}}+\frac{D_{jk}^2}{B_{jk}^2}\right)\frac{\partial B_{jk}}{\partial\phi_{ijk}}\right]=0, \tag{A-56}$$

for $i=1,\ldots,m_k$.

To check the second order conditions for a (local) maximum, we need to show that the Hessian matrix, which is a symmetric matrix of second derivatives, is negative definite. First note that

$$\frac{\partial^2 U_{jk}}{\partial\theta_{jk}^2}=\frac{U_{jk}}{B_{jk}}\left(\frac{\partial D_{jk}}{\partial\theta_{jk}}\right)^2=\frac{U_{jk}\tau_{jk}^2}{B_{jk}}, \tag{A-57}$$

where we have used the first order condition in (A-55) and the fact that the second derivatives of C_{jk} and D_{jk} with respect to θ_{jk} are zero. Since $U_{jk}<0$ and $B_{jk}>0$, the expression in (A-57) is negative. Next, using (A-56), we have

$$\frac{\partial^2 U_{jk}}{\partial \phi_{ijk}^2} = U_{jk}\tau_{jk}^2 \left[\frac{\sigma_{ik}^2\omega_{ik}^2}{B_{jk}} + 2\left(\alpha_{ik}P_{ik}\right)^2 + \frac{\sigma_{ik}^4\omega_{ik}^2}{B_{jk}}\left[\frac{\mathbb{E}\left[X|\,P_x\right] - P_x}{\mathrm{var}(X|P_x)}\right]^2\right], \tag{A-58}$$

for $i = 1, \ldots, m_k$. Furthermore, using (A-55) and (A-56), we have

$$\frac{\partial^2 U_{jk}}{\partial \theta_{jk}\partial \phi_{ijk}} = U_{jk}\tau_{jk}^2 \frac{\sigma_{ik}^2\omega_{ik}}{B_{jk}}\left[\frac{\mathbb{E}\left[X|\,P_x\right] - P_x}{\mathrm{var}(X|P_x)}\right] \tag{A-59}$$

and

$$\frac{\partial^2 U_{jk}}{\partial \phi_{ijk}\partial \phi_{ljk}} = U_{jk}\tau_{jk}^2 \left[2\alpha_{ik}P_{ik}\alpha_{lk}P_{lk} + \frac{\sigma_{ik}^2\omega_{ik}\sigma_{lk}^2\omega_{lk}}{B_{jk}}\left[\frac{\mathbb{E}\left[X|\,P_x\right] - P_x}{\mathrm{var}(X|P_x)}\right]^2\right]. \tag{A-60}$$

The $(m_k+1)\times(m_k+1)$ Hessian matrix is

$$\boldsymbol{H} = \begin{bmatrix} \frac{\partial^2 U_{jk}}{\partial \theta_{jk}^2} & \frac{\partial^2 U_{jk}}{\partial \theta_{jk}\partial \phi_{1jk}} & \cdots & \frac{\partial^2 U_{jk}}{\partial \theta_{jk}\partial \phi_{m_k jk}} \\ \frac{\partial^2 U_{jk}}{\partial \theta_{jk}\partial \phi_{1jk}} & \frac{\partial^2 U_{jk}}{\partial \phi_{1jk}^2} & & \vdots \\ \vdots & & \ddots & \\ \frac{\partial^2 U_{jk}}{\partial \theta_{jk}\partial \phi_{m_k jk}} & \cdots & & \frac{\partial^2 U_{jk}}{\partial \phi_{m_k jk}^2} \end{bmatrix}. \tag{A-61}$$

Let $\boldsymbol{H}_r$ denote the upper left $r \times r$ square sub-matrix of $\boldsymbol{H}$. It is well known that $\boldsymbol{H}$ is negative definite if and only if $(-1)^r|\boldsymbol{H}_r| > 0$ for $r = 1, 2, \ldots, m_k + 1$, where $|\boldsymbol{H}_r|$ is the determinant of $\boldsymbol{H}_r$. For $r = 1$ we have

$$-|H_1| = -\frac{\partial^2 U_{jk}}{\partial \theta_{jk}^2} = -\frac{U_{jk}\tau_{jk}^2}{B_{jk}} > 0,$$

where we have used the fact that (A-57) is negative. For $r = 2$ we have

$$(-1)^2|\boldsymbol{H}_2| = \begin{vmatrix} \frac{\partial^2 U_{jk}}{\partial \theta_{jk}^2} & \frac{\partial^2 U_{jk}}{\partial \theta_{jk}\partial \phi_{1jk}} \\ \frac{\partial^2 U_{jk}}{\partial \theta_{jk}\partial \phi_{1jk}} & \frac{\partial^2 U_{jk}}{\partial \phi_{1jk}^2} \end{vmatrix} = \frac{\partial^2 U_{jk}}{\partial \theta_{jk}^2}\frac{\partial^2 U_{jk}}{\partial \phi_{1jk}^2} - \left(\frac{\partial^2 U_{jk}}{\partial \theta_{jk}\partial \phi_{1jk}}\right)^2. \tag{A-62}$$

Using (A-57)-(A-59), we can evaluate the right-hand side of (A-62) to get

$$(-1)^2 |\boldsymbol{H}_2| = \frac{U_{jk}^2 \tau_{jk}^4}{B_{jk}} \left[\frac{\sigma_{1k}^2 \omega_{1k}^2}{B_{jk}} + 2 (\alpha_{1k} P_{1k})^2 \right] > 0.$$

For $r > 2$, we apply (A-68) to $|H_r|$ by letting

$$\begin{aligned}
\boldsymbol{D}_{11} &= \frac{\partial^2 U_{jk}}{\partial \theta_{jk}^2}, \\
\boldsymbol{D}_{12} = \boldsymbol{D}_{21}^\top &= \begin{bmatrix} \frac{\partial^2 U_{jk}}{\partial \theta_{jk} \partial \phi_{1jk}} & \frac{\partial^2 U_{jk}}{\partial \theta_{jk} \partial \phi_{2jk}} & \cdots & \frac{\partial^2 U_{jk}}{\partial \theta_{jk} \partial \phi_{r-1,jk}} \end{bmatrix}, \\
\boldsymbol{D}_{22} &= \begin{bmatrix} \frac{\partial^2 U_{jk}}{\partial \phi_{1jk}^2} & \frac{\partial^2 U_{jk}}{\partial \phi_{1jk} \partial \phi_{2jk}} & \cdots & \frac{\partial^2 U_{jk}}{\partial \phi_{1jk} \partial \phi_{r-1,jk}} \\ \frac{\partial^2 U_{jk}}{\partial \phi_{1jk} \partial \phi_{2jk}} & \frac{\partial^2 U_{jk}}{\partial \phi_{2jk}^2} & & \vdots \\ \vdots & & \ddots & \\ \frac{\partial^2 U_{jk}}{\partial \phi_{1jk} \partial \phi_{r-1,jk}} & \cdots & & \frac{\partial^2 U_{jk}}{\partial \phi_{r-1,jk}^2} \end{bmatrix}.
\end{aligned}$$

We note that the matrix $\boldsymbol{D}_{22} - \boldsymbol{D}_{21} \boldsymbol{D}_{11}^{-1} \boldsymbol{D}_{12}$ takes the form

$$\begin{bmatrix} \frac{\partial^2 U_{jk}}{\partial \phi_{1jk}^2} - \left(\frac{\partial^2 U_{jk}}{\partial \theta_{jk}^2} \right)^{-1} \left(\frac{\partial^2 U_{jk}}{\partial \theta_{jk} \partial \phi_{1jk}} \right)^2 & \frac{\partial^2 U_{jk}}{\partial \phi_{1jk} \partial \phi_{2jk}} - \left(\frac{\partial^2 U_{jk}}{\partial \theta_{jk}^2} \right)^{-1} \left(\frac{\partial^2 U_{jk}}{\partial \theta_{jk} \partial \phi_{1jk}} \right) \left(\frac{\partial^2 U_{jk}}{\partial \theta_{jk} \partial \phi_{2jk}} \right) & \cdots \\ \frac{\partial^2 U_{jk}}{\partial \phi_{1jk} \partial \phi_{2jk}} - \left(\frac{\partial^2 U_{jk}}{\partial \theta_{jk}^2} \right)^{-1} \left(\frac{\partial^2 U_{jk}}{\partial \theta_{jk} \partial \phi_{1jk}} \right) \left(\frac{\partial^2 U_{jk}}{\partial \theta_{jk} \partial \phi_{2jk}} \right) & \frac{\partial^2 U_{jk}}{\partial \phi_{2jk}^2} - \left(\frac{\partial^2 U_{jk}}{\partial \theta_{jk}^2} \right)^{-1} \left(\frac{\partial^2 U_{jk}}{\partial \theta_{jk} \partial \phi_{2jk}} \right)^2 & \\ \vdots & \vdots & \ddots \end{bmatrix}.$$

Using (A-58)-(A-60), we can rewrite the above as

$$\boldsymbol{D}_{22} - \boldsymbol{D}_{21} \boldsymbol{D}_{11}^{-1} \boldsymbol{D}_{12} = \tau_{jk}^2 U_{jk} S + 2 \tau_{jk}^2 U_{jk} T,$$

where S is the diagonal matrix

$$S = \begin{bmatrix} \frac{\sigma_{1k}^2 \omega_{1k}^2}{B_{jk}} & 0 & \cdots & 0 \\ 0 & \frac{\sigma_{2k}^2 \omega_{2k}^2}{B_{jk}} & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \frac{\sigma_{r-1,k}^2 \omega_{r-1,k}^2}{B_{jk}} \end{bmatrix}$$

and T is the matrix

$$T = \begin{bmatrix} (\alpha_{1k}P_{1k})^2 & (\alpha_{1k}P_{1k})(\alpha_{2k}P_{2k}) & \cdots & (\alpha_{1k}P_{1k})(\alpha_{r-1,k}P_{r-1,k}) \\ (\alpha_{1k}P_{1k})(\alpha_{2k}P_{2k}) & (\alpha_{2k}P_{2k})^2 & & \vdots \\ \vdots & & \ddots & \\ (\alpha_{1k}P_{1k})(\alpha_{r-1,k}P_{r-1,k}) & \cdots & & (\alpha_{r-1,k}P_{r-1,k})^2 \end{bmatrix}.$$

Thus we have

$$\left|\boldsymbol{D}_{22} - \boldsymbol{D}_{21}\boldsymbol{D}_{11}^{-1}\boldsymbol{D}_{12}\right| = \left|\tau_{jk}^2 U_{jk} S + 2\tau_{jk}^2 U_{jk} T\right| = U_{jk}^{r-1}\tau_{jk}^{2(r-1)} \left|S + 2T\right|.$$

Noting that S is a positive definite matrix and T is a positive semi-definite matrix, we have $|S + 2T| > 0$. Thus for $r > 2$ we have

$$\begin{aligned} (-1)^r |\boldsymbol{H}_r| &= (-1)^r |\boldsymbol{D}_{11}| \left|\boldsymbol{D}_{22} - \boldsymbol{D}_{21}\boldsymbol{D}_{11}^{-1}\boldsymbol{D}_{12}\right| \\ &= (-1)^r \frac{\partial^2 U_{jk}}{\partial \theta_{jk}^2} U_{jk}^{r-1}\tau_{jk}^{2(r-1)} \left|S + 2T\right|. \end{aligned}$$

Substituting (A-57) we get

$$(-1)^r |\boldsymbol{H}_r| = \frac{(-1)^r}{B_{jk}} U_{jk}^r \tau_{jk}^{2r} \left|S + 2T\right| > 0,$$

which follows since $U_{jk} < 0$. We have shown that $(-1)^r |H_r| > 0$ for $r = 1, 2, \ldots, m_k + 1$, which proves that $\boldsymbol{H}$ in (A-61) is a negative definite matrix. This verifies the household's second order condition for a (local) maximum. This completes the proof of Proposition 8. □

Appendix

The following result is well known in the probability literature and will be used in the proofs. For a proof of Lemma 1 in the special case of $\boldsymbol{\mu} = \boldsymbol{0}$, see Marín and Rahi, 1999, Speculative Securities, *Economic Theory*, 14, p. 653–668.

Lemma 1. *Suppose* $\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ *is an* n*-dimensional Gaussian random vector,* $\boldsymbol{\Sigma}$ *is positive definite,* $\boldsymbol{b} \in \mathbb{R}^n$ *is a vector, and* $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ *is a symmetric matrix. Then*

$\mathbb{E}\left[e^{\boldsymbol{b}^\top \boldsymbol{X}+\boldsymbol{X}^\top \boldsymbol{A}\boldsymbol{X}}\right]$ *is well defined if and only if* $\boldsymbol{I}-2\boldsymbol{\Sigma}\boldsymbol{A}$ *is positive definite. In this case,*

$$\mathbb{E}\left[e^{\boldsymbol{b}^\top \boldsymbol{X}+\boldsymbol{X}^\top \boldsymbol{A}\boldsymbol{X}}\right] = |\boldsymbol{I}-2\boldsymbol{\Sigma}\boldsymbol{A}|^{-1/2}\, e^{\boldsymbol{b}^\top\boldsymbol{\mu}+\boldsymbol{\mu}^\top\boldsymbol{A}\boldsymbol{\mu}+\frac{1}{2}(\boldsymbol{b}+2\boldsymbol{A}\mu)^\top(\boldsymbol{I}-2\boldsymbol{\Sigma}\boldsymbol{A})^{-1}\boldsymbol{\Sigma}(\boldsymbol{b}+2\boldsymbol{A}\boldsymbol{\mu})}.$$

The facts stated in the following lemma are well known results from matrix algebra and the calculus of matrix-valued functions. See, for example, the book *Mathematics for Econometrics*, by Phoebus J. Dhrymes, Springer-Verlag, 1978 (in particular Propositions 30 and 32, as well as Corollaries 28, 30 and 38).

Lemma 2.

(a) If $\boldsymbol{f}(x)$ *is a vector,* x *a scalar, and* $\boldsymbol{D}$ *a constant symmetric matrix, then*

$$\frac{\partial \boldsymbol{f}(x)^\top \boldsymbol{D}\boldsymbol{f}(x)}{\partial x} = 2\frac{\partial \boldsymbol{f}(x)}{\partial x}^\top \boldsymbol{D}\boldsymbol{f}(x). \tag{A-63}$$

For a matrix $\boldsymbol{D}(x)$*, where* x *is a scalar variable:*

$$\frac{\partial \boldsymbol{D}(x)^{-1}}{\partial x} = -\boldsymbol{D}(x)^{-1}\frac{\partial \boldsymbol{D}(x)}{\partial x}\boldsymbol{D}(x)^{-1}, \tag{A-64}$$

$$\frac{\partial \log(|\boldsymbol{D}(x)|)}{\partial x} = \text{trace}\left(\boldsymbol{D}(x)^{-1}\frac{\partial \boldsymbol{D}(x)}{\partial x}\right). \tag{A-65}$$

(b) If $\boldsymbol{D}$ *is a conformably partitioned block matrix, then*

$$\begin{bmatrix} \boldsymbol{D}_{11} & \boldsymbol{D}_{12} \\ \boldsymbol{D}_{21} & \boldsymbol{D}_{22} \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{F}_{11}^{-1} & -\boldsymbol{F}_{11}^{-1}\boldsymbol{D}_{12}\boldsymbol{D}_{22}^{-1} \\ -\boldsymbol{D}_{22}^{-1}\boldsymbol{D}_{21}\boldsymbol{F}_{11}^{-1} & \boldsymbol{F}_{22}^{-1} \end{bmatrix} \tag{A-66}$$

where

$$\begin{aligned} \boldsymbol{F}_{11} &= \boldsymbol{D}_{11}-\boldsymbol{D}_{12}\boldsymbol{D}_{22}^{-1}\boldsymbol{D}_{21} = (\boldsymbol{I}+\boldsymbol{D}_{12}\boldsymbol{F}_{22}^{-1}\boldsymbol{D}_{21}\boldsymbol{D}_{11}^{-1})^{-1}\boldsymbol{D}_{11} \\ \boldsymbol{F}_{22} &= \boldsymbol{D}_{22}-\boldsymbol{D}_{21}\boldsymbol{D}_{11}^{-1}\boldsymbol{D}_{12} = (\boldsymbol{I}+\boldsymbol{D}_{21}\boldsymbol{F}_{11}^{-1}\boldsymbol{D}_{12}\boldsymbol{D}_{22}^{-1})^{-1}\boldsymbol{D}_{22}. \end{aligned}$$

Moreover,

$$\begin{vmatrix} \boldsymbol{D}_{11} & \boldsymbol{D}_{12} \\ \boldsymbol{D}_{21} & \boldsymbol{D}_{22} \end{vmatrix} = |\boldsymbol{D}_{22}| \left|\boldsymbol{D}_{11} - \boldsymbol{D}_{12}\boldsymbol{D}_{22}^{-1}\boldsymbol{D}_{21}\right| \tag{A-67}$$

$$= |\boldsymbol{D}_{11}| \left|\boldsymbol{D}_{22} - \boldsymbol{D}_{21}\boldsymbol{D}_{11}^{-1}\boldsymbol{D}_{12}\right| \tag{A-68}$$