

# Quality, Upgrades and Equilibrium in a Dynamic Monopoly Market

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## Abstract

We examine an infinite horizon model of quality growth for a durable goods monopoly. Quality improvements may be sold in any desired bundles. Consumers are identical and for a quality improvement to have value the buyer must possess previous qualities: goods are upgrades. Subgame perfect seller payoffs range from capturing the full social surplus down to only the initial flow value of each good. For any discount factor, each of these payoffs is realized in a Markov perfect equilibrium that follows the socially efficient path. However, inefficient delay equilibria, with bundling, exist for innovation rates above a threshold.

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In this paper, we examine the commercialization process - pricing and adoption - of an upgrade good in a dynamic monopoly market. Prominent examples are provided by technology markets, such as those for software, where cycles of upgrades to existing products have become the norm.<sup>1</sup> Ongoing innovation implies that buyers face a sequence

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<sup>1</sup>Quality that improves over time is a significant feature of many durable goods markets, as emphasized by Waldman (2003). In addition to software, upgrades to cellular networks often allow vendors to offer,

of purchasing decisions. Thus, rather than timing a single purchase and then exiting the market, buyers have an incentive to return to the market and ‘upgrade’ to a higher quality level. In equilibrium, the price(s) for the commercialization of upgrades reflects the ability of the seller to tempt a buyer to purchase when others are not, the cost to a buyer of falling behind the market, and the resulting structure of credible buyer threats to refuse seller offers. Buyer expectations are pivotal for these decisions and, given the recurrent aspect of upgrading, bundling by the seller emerges as a critical aspect of the upgrade offers.

The Microsoft antitrust cases highlight a fundamental question with regard to how prices are determined in an upgrade market. Fudenberg and Tirole (2000) observed that the expert witnesses all appeared to agree that Microsoft was pricing the Windows operating system well below the static monopoly price. There was, however, wide disagreement as to why. Prominent arguments included network formation with low prices spurring adoption, limit pricing where a low price deters rivals, and leverage to gain sales in markets for application programs. Implicit in all of these arguments is the presumption that prices would be higher in the absence of these forces. There is, however, no model of dynamic monopoly that provides a basis for this claim. We provide a game theoretic analysis of dynamic monopoly pricing for an upgrade good and establish that, in equilibrium, high prices are not a necessary outcome. Significantly, low prices, as measured by a seller who captures a small share of the social surplus, emerge in equilibrium. The key to this result involves buyer adoption incentives in an upgrade market and the existence of a credible buyer threat to reject an upgrade offer from the seller.

Upgrade markets, by definition, regularly confront buyers with the choice of adopting a new higher-quality version or remaining with their current version. Microsoft’s recent introduction of Vista was an adoption failure as buyers overwhelmingly chose to stay with their existing XP version, echoing a previous episode with Windows Millennium in 2000. Microsoft moved quickly to introduce a new version. Windows 7 was launched in late October 2009 to a much more favorable buyer response. As early as May 2009, Microsoft CEO Steve Ballmer acknowledged that “If people want to wait [for Windows 7], they certainly can.” This simple observation, which implicitly takes the failure of Vista as a given, leads to a more subtle set of questions regarding the full range of individual and collective adoption incentives.

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for an added charge, new or improved services such as web browsing, e-mail access and text messaging. Many capital goods are regularly upgraded, including airports (terminals and runways) and oil refineries, among others.

Consider the initial offer of Vista. An individual buyer has the option of remaining with XP. If, however, most other buyers had purchased Vista then we can expect a concern about “falling behind” the market to be pivotal for an individual buyer’s willingness to pay. Given that others did not, an individual buyer who stayed with XP would instead be in the position Ballmer described. Alternatively, by purchasing Vista a buyer would “jump ahead” of the market. This buyer would then be confronted with the choice of purchasing Windows 7 to “keep up” with market, assuming that Windows 7 is widely adopted. How does this recurrent interplay of individual and collective decisions with respect to incentives to “fall behind” or “jump ahead” of the market work to determine prices and adoption in an upgrade market? We argue that the ability of the seller to tempt an individual buyer to “jump ahead” is the critical factor and that this incentive forms the basis of a credible threat for buyers to reject an upgrade offer. Moreover, low prices can emerge in equilibrium even when buyers have a very strong incentive not to “fall behind” the market.

We examine an infinite horizon model of an upgrade market with a very simple core economic structure. Innovation is exogenous but ongoing and in each period it is feasible for the seller to offer an additional quality increment. Buyers are homogeneous and have a fixed valuation per unit of quality; this corresponds to a horizontal demand curve in a static setting. Building on the recent literature, we capture the notion that these are ‘upgrade’ goods by assuming that the sequence of quality increments satisfies a downward complementarity property: for an additional quality increment to be valuable, a buyer must also hold all previous quality increments. The seller is unconstrained with respect to bundling options and any combination of quality increments (a single bundle or a set of bundles) may be offered in each period. Bundling is thus endogenous.

This basic structure is arguably a very attractive setting for the seller - homogenous buyers with a fixed valuation per quality unit, unrestricted bundling, and a costless exogenous flow of upgrade innovations. It is natural to expect a “perfect” monopoly outcome in which the seller captures all of the social surplus, and this is exactly what happens in several benchmark cases. For example, when the seller can only offer a single quality good and with homogeneous buyers, the seller can make an offer that inevitably tempts buyers to purchase now relative to any expectations for how surplus will be shared in the future. This “speed-up” argument, for which an elegant version was developed by Fudenberg, Levine, and Tirole (1985) for a sequential offer game, is quite powerful and it undermines

the credibility of a buyer's threat to reject offers with high prices.<sup>2</sup> In sharp contrast, we find that surplus growth due to rising quality in an upgrade market provides buyers with an option to return to the market for future purchases, rather than exiting permanently after a single purchase, and that this option supports a very powerful credible threat on the buyer side.

The primary intuition for the credible buyer threat is as follows. Suppose that buyers expect to receive a positive share of the surplus on future quality improvements. Further, imagine that the seller offers a price above the candidate equilibrium for today's upgrade. Is it credible for buyers to refuse the offer? Consider the willingness to pay of an individual buyer when other buyers are expected to refuse the offer. When others refuse, we have delay and the next period will have the larger surplus due to quality growth as the market position involves buyers who lack the previous upgrade. When the typical buyer's share of this surplus is significant, a solitary individual buyer who purchased the high priced upgrade in the last period will wish to purchase again; despite the fact that this may require the buyer to purchase a bundle that includes quality increments already held, the assumed positive buyer share of future surplus makes it attractive to acquire the new upgrade and keep up with the market. But, then the initial upgrade purchase of a buyer who "jumps ahead" of the market reduces to a one-period flow of value since such a buyer expects to acquire this upgrade next period. As a result, willingness to pay is limited to the one-period flow value of the upgrade. This is a credible threat for buyers to reject prices above the candidate equilibrium price. Moreover, we can apply the logic behind this threat - a willingness to pay that has been pushed to the one period flow value - at any stage of the game and for any given discount factor. The credible threat to reject a seller offer, given that other buyers also reject, leads to an implicit form of coordination among buyers and, in turn, to multiple equilibria. This intuition also proves to be robust in the presence of network externalities, compatibility issues and adoption costs. Because each of these effects weakens the incentive for an individual buyer to jump ahead of the market, the basic logic that supports a credible threat in our analysis is reinforced.

We construct Markov perfect equilibria for this dynamic game and show that every

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<sup>2</sup>The standard incentive (Coase (1972)) to cut price over time and move down the demand curve is not present when buyers are identical. Papers on the Coase conjecture with a single good and a set of heterogeneous buyers include Stokey (1981), Bulow (1982), and Gul, Sonnenschein, and Wilson (1986). Ausubel and Deneckere (1989), Fehr and Kuhn (1995) and Sobel (1991) provide folk theorems for the durable goods model. Bond and Samuelson (1984) examine a rational expectations equilibrium with depreciation and replacement sales. Methodologically, our paper is closest to Sobel (1991) where there is entry of new consumers over time. In both cases, the market never closes, due to new demand in the case of Sobel and to quality growth in our case.

subgame perfect equilibrium payoff is achieved by some Markov perfect equilibrium. Two classes of equilibria are identified: efficient and generational. Efficient equilibria have buyers acquiring a new upgrade each period. Equilibrium payoffs span a significant economic range. At one extreme, there is an efficient equilibrium in which the seller captures all of the social surplus and each quality increment sells immediately at a price equal to the full present discounted value of the flow of benefits to a buyer (extraction of all buyer surplus). At the other extreme, there is an efficient equilibrium in which each quality increment sells only for the one period flow value, leaving a buyer with the entire residual surplus. These equilibrium payoff results contrast sharply with a traditional folk theorem in two important respects. First, a high discount factor is not necessary; the result applies for any discount factor (innovation rate) between zero and one. Second, severe punishments are not required; after any deviation, the market returns to the equilibrium path after one period.

In contrast, and despite the complete information setting, inefficient equilibria do exist. These “generational” equilibria exhibit cyclical delay in which multiple quality increments go unsold until they are bundled together for sale and, necessarily, the market returns to the “state of the art” with a new generation (an upgrade to the quality frontier). The cycle length reflects a second type of equilibrium coordination in an upgrade market. The seller could profitably “speed up” buyer acquisition were there to be any persistent quality gap between buyers and feasible quality, but this is not so when inefficient delay is periodically eliminated. An important property of a generational equilibrium with respect to this and other delay incentives is that the division of surplus is necessarily compressed relative to the set of efficient equilibria. Intuitively, delay requires that deviations to make early trades are unattractive, and these incentives imply that the seller and the buyers share the joint surplus more equally. We show that there is a critical threshold such that a longer generational cycle requires a higher discount factor (innovation rate).

The seller is free to offer any feasible collection of quality units. We find that, in equilibrium, it is sufficient to consider only upgrade offers in which a contiguous set of quality units is bundled. Equivalently, we show how to interpret these upgrade offers in terms of a full bundle (new version of the product) with pricing contingent on a buyer’s current product holding, much as the owner of an existing product faces an upgrade price to acquire a new version. Furthermore, we find no role for the commitment period (time between seller offers), in contrast to the literature on the Coase Conjecture. Instead, what matters for equilibrium outcomes is the frequency at which quality improves and allowing

the seller to make offers more frequently has no impact with homogenous buyers.

There is a relatively small literature on upgrade models, with most of the work involving a finite horizon. Waldman (1996) and Nahm (2004) each examine a two period model, focusing on the incentive to invest in quality growth and R&D time inconsistency. Fudenberg and Tirole (1998) examine a two-period model where consumers are heterogeneous and the period two (new) good renders the period one (old) good obsolete for a buyer. Ellison and Fudenberg (2000) analyze a series of static and two period models that feature network externalities and a cost to consumers of upgrading the good. In the finite horizon version of our model, the monopolist captures all the surplus because the basis for a credible buyer threat is undermined by the terminal period. Fishman and Rob (2000) examine an infinite horizon upgrade model, focusing on innovation incentives, and analyze a rational expectations equilibrium in which the seller is assumed to offer only a single bundle consisting of all prior quality levels. We focus on pricing and adoption, taking innovation as exogenously given, and provide a game-theoretic analysis in which the seller choice of which bundles (and prices) to offer is endogenous. We discuss this and other aspects of bundling, as well as the extensions to include complementary issues such as compatibility, network effects and adoption costs, in relation to the literature in Section 6.

In the next section, we present the model. Benchmark cases are generated in Section 2 to help differentiate our work from the literature and to understand the implications of the model assumptions. We provide basic results in Section 3, where we show that all equilibria are cyclical. In Section 4 we examine efficient equilibria and in Section 5 we examine generational equilibria. We discuss the upgrade structure of our model in Section 6 and consider directions for future research in Section 7. All proofs are in the Appendix.

## 1 The Model

We first describe the basic elements of the game. We next turn to strategies and payoffs, and then we define and discuss Markov perfect equilibrium. We allow for complete freedom with respect to the bundling of units. The formal game theoretic framework for bundling is presented in Appendix A; this is essential for the construction of supporting continuation equilibria.

## 1.1 Basic Elements

We examine an infinite horizon, discrete time model. Let  $\tau = 1, 2, \dots$  index periods. There is a continuum of identical buyers with a measure of 1 represented by the unit interval and a single seller. A new perfectly durable good, unit  $\tau$ , becomes available in each period  $\tau$ . All seller costs are 0. Within each period  $\tau$ , feasible offers for the seller consist of any collection of subsets of  $\{1, 2, \dots, \tau\}$  and associated prices. For example, the seller can offer the bundle of all feasible qualities  $\{1, 2, \dots, \tau\}$  for a price  $p$ , so that the new unit is made available only as part of a larger bundle. Alternatively, the seller can offer a collection of individual unit bundles,  $\{1\}$  at price  $p_1$ , quality  $\{2\}$  at a price  $p_2$ , and so on; a buyer would then have the option to purchase every feasible quality or any subset of the available unit bundles. Of course, the seller can also withhold some qualities or even make no offer.

Given a seller offer, the buyers respond simultaneously with each buyer choosing which bundle(s) to accept in period  $\tau$ .<sup>3</sup>

Any bundle that consists only of a set of contiguous qualities is defined as an upgrade. For example, an upgrade to the “state of the art” from a status quo of 0 is the bundle  $\{1, \dots, \tau\}$ ; we also refer to this as a version. A partial upgrade is a bundle  $\{\sigma, \dots, \sigma + k\}$ , where  $1 \leq \sigma \leq \sigma + k \leq \tau$ . We will show that, in equilibrium, a seller need only make upgrade offers.

A buyer receives a flow utility of  $vq$  in period  $\tau$  when contiguous units  $1, \dots, q$  but not unit  $q + 1$  are held by the buyer. Thus, we are imposing the condition that a buyer must have all lower quality units for quality  $q$  to have value. This “downward complementarity” assumption is the upgrade payoff structure in our model. For example, if a buyer holds quality units 1 and 3 but not 2 in a given period, then the flow utility is  $v$ . Thus, quality unit  $\tau$  is always a complementary good with respect to prior quality units (the downward direction) while the prior quality units do not require unit  $\tau$  in order to provide value to a buyer.

Players are all risk neutral and have a common discount factor  $\delta < 1$ . Because a new unit of quality becomes available in each period, the discount factor reflects the rate of innovation as well as the rate of time preference for the players. Thus, a large  $\delta$  can be

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<sup>3</sup>We do not impose any arbitrage structure across bundles. For example, if the seller offers a bundle with only good 1, and a bundle with only good 2, then there is no restriction on the price of a bundle that includes both goods 1 and 2. Rather, buyer choices determine which of these bundles will be purchased. Also, since buyers are identical, there are no possible gains in equilibrium for buyers from the possibility of resale.

interpreted in terms of a rapid rate of innovation while a small  $\delta$  means that innovations are infrequent. Also, for later interpretation, we can employ the familiar relationship of  $\delta = e^{-r\Delta}$ , where  $\Delta$  is the length of a time period (the innovation rate) and  $r$  is the interest rate, together with an appropriate measure of flow utility, to assess limiting behavior as  $\Delta \rightarrow 0$  and  $\Delta \rightarrow \infty$ .

Consider the payoff for a buyer. In each period, a buyer holds some subset of the feasible qualities. Let  $q_\tau$  denote the maximal contiguous quality held by a buyer after any purchase in period  $\tau$ . That is, a buyer holds units 1 up through  $q_\tau$  but does not hold unit  $q_\tau + 1$ . From any point in the game, the payoff of the buyer is the present discounted value from quality flows net of payments. From the start of the game this is given by

$$\sum_{\tau=1}^{\infty} \delta^{\tau-1} (vq_\tau - p_\tau),$$

where  $p_\tau$  is the payment made by the buyer in period  $\tau$ . Similarly, the payoff of the seller from any point onward is the present discounted value of revenues,  $r_\tau$ , from sales to buyers. From the start of the game, this is given by

$$\sum_{\tau=1}^{\infty} \delta^{\tau-1} r_\tau.$$

Consider efficient allocations, where joint surplus is maximized. First, note that the payments and revenues are transfers that do not affect total surplus. Thus, for any path of quality holdings and payments, the sum of surplus for any given buyer and the seller from any period  $\tau_0$  is

$$\sum_{\tau=\tau_0}^{\infty} \delta^{\tau-\tau_0} vq_\tau.$$

Thus, the realized joint surplus is fully determined by the quality path. Since  $q_\tau \leq \tau$  for any feasible path and  $q_0 \equiv 0$ , the joint surplus is maximized when each buyer holds the maximal quality,  $q_\tau = \tau$ .

The total available surplus in an efficient allocation from the start of the game is therefore

$$S_1 = v + \delta 2v + \delta^2 3v + \dots = \frac{v}{(1-\delta)^2}.$$

Intuitively,  $S_1$  is the surplus created when buyers acquire one new unit in each period, where each new unit has a present discounted value of  $\frac{v}{1-\delta}$ . Starting from any period  $\tau$ ,

the maximal available surplus is

$$S_\tau = v\tau + \delta v(\tau + 1) + \delta^2 v(\tau + 2) + \dots = \frac{v(\tau - 1)}{1 - \delta} + S_1.$$

Intuitively, the difference between  $S_\tau$  and  $S_{\tau+1}$  is the flow value of  $\tau$  units in period  $\tau$ . Thus, we always have  $S_\tau > \delta S_{\tau+1}$ , as delay necessarily involves lost surplus and hence inefficiency. However, because each unit generates surplus, we also have  $S_\tau < S_{\tau+1}$ .

The formal game-theoretic definitions for buyer and seller strategies are developed in Appendix A along with the requirements for Nash and subgame perfect equilibrium.

We employ a continuum, the unit interval, for the set of buyers in order to capture the idea that individual buyers are insignificant with respect to market outcomes. Thus, we follow Gul, Sonnenschein, and Wilson (1986), Ausubel and Deneckere (1989), and Sobel (1991), among others, and restrict attention to equilibria that satisfy a zero-measure property: for any two histories (past seller offers and buyer acceptances) that differ only with respect to the actions of a set of buyers of measure zero, the strategies of the seller and all other buyers are the same across the two histories. As a result no individual buyer expects that their own acceptance/rejection decision will have any impact on subsequent play, such as affecting the set of bundles that will be available for purchase in the future. Thus, buyers act as price takers.

## 1.2 Markov Perfect Equilibrium

In this paper, we examine Markov perfect equilibria (MPE) as defined by Maskin and Tirole (2001), with the natural modification for a continuum of agents. By definition, Markov strategies depend only on the payoff relevant aspects of a history of the game. In our model, the seller's flow payoff depends only on revenues and each buyer's flow payoff depends only on the maximal contiguous unit held and the payments in a period. Thus, past prices and the timing of buyer acquisitions do not influence current period payoffs.

The simplest form of Markovian behavior is to focus on the distribution of maximal contiguous units across buyers and the gap relative to the current period  $\tau$ , which indexes the seller's feasible units. This allows us to generate all subgame perfect equilibrium seller payoffs. More complex Markovian behavior, as noted in Appendix A, supports no additional payoffs. To proceed, we need to define a state of the game. Consider any history in which all buyers enter period  $\tau$  with the same maximal quality level  $Q$  (units 1 through  $Q$ ). We define this to be state  $(\tau, Q)$  and refer to  $\tau - Q$  as the quality gap. We then define

Markovian behavior by the condition that players' strategies depend only on the size of the quality gap. This means, for instance, that if the seller offers an upgrade of  $\sigma$  units at a price  $p$  in state  $(\tau, 0)$ , then an upgrade from  $Q$  to  $Q + \sigma$  at the same price  $p$  must be offered in state  $(\tau', Q)$ , provided that the gaps coincide,  $\tau' - Q = \tau$ . Furthermore, except for a translation of the index number on quality units, buyers' accept/reject decisions are the same in states  $(\tau, 0)$  and  $(\tau', Q)$ . This implies that the seller's profits and buyers' utilities satisfy

$$\pi_\tau \equiv \pi(\tau, 0) = \pi(\tau', Q)$$

and

$$u(\tau', Q) = \frac{vQ}{1 - \delta} + u(\tau, 0) \quad \text{and} \quad u_\tau \equiv u(\tau, 0)$$

for  $\tau' - Q = \tau$ .<sup>4</sup> Thus, buyer payoffs are always the sum of the PDV of current holdings,  $vQ/(1 - \delta)$ , and the incremental utility,  $u_\tau$ .

This definition of Markovian behavior implies that the same number of units are included in an upgrade bundle whenever the quality gaps coincide. In particular, the seller is not required to offer the full bundle of all feasible units. While buyers who all hold the same maximal contiguous quality will have the same value for the full bundle and the upgrade to the state of the art, the offer of a full bundle leads to buyer deviation incentives that are not present with upgrades. We return to this point in section 6 when we discuss contingent pricing and show how to interpret an upgrade offer in terms of a version (full bundle) with a price contingent on a buyer's current holding.

We show that every equilibrium payoff can be implemented by an upgrade offer structure, where at each state  $(\tau, Q)$ , the seller either delays by making no offer or offers one upgrade level  $Q' \in \{Q + 1, \dots, \tau\}$  and an associated price. Thus, on the equilibrium path, all buyers have the same quality holdings.<sup>5</sup> Whenever a sale is supposed to occur, the equilibrium is supported with the following simple Markovian behavior. Buyers reject an attempted price increase by the seller (a deviation) based on the expectation that the seller will return to the market next period with a better version. Thus, our continuation

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<sup>4</sup>Feasible payoffs in our model have a simple stationary structure. Any subgame perfect equilibrium from state  $(\tau + 1 - Q, 0)$  is also subgame perfect from state  $(\tau + 1, Q)$  once we relabel units and translate the buyer's payoff.

<sup>5</sup>States where buyers have asymmetric holdings are off-the-equilibrium path as are histories where the seller makes multiple upgrade offers. Note that mixing by buyers in response to a seller offer would lead to asymmetric holdings. This is often required for continuation equilibria in the durable goods literature. In our case, because buyers never exit the market, we are able to construct pure strategy continuation equilibria in all states. The Appendix provides a detailed analysis of continuation equilibria for any distribution of buyer holdings and for any bundles offered by the seller.

strategies follow a ‘cash-in’ support at any state (distribution of buyers across quality levels) that is off of the equilibrium path: the seller offers buyers a bundle(s) that is sufficiently attractive to move the market up to the current state of the art.

We will provide an explicit construction of the strategies and show that the equilibrium behavior of buyers and sellers necessarily follows a simple cyclical structure when strategies only depend on the quality gap. Furthermore, the definition of Markov perfect equilibrium is flexible enough to allow for both efficient and inefficient equilibria. Henceforth, we use equilibrium to refer to a pure strategy buyer symmetric Markov perfect equilibrium in the quality gap.

## 2 Benchmarks for the Upgrade Model

We begin our analysis by identifying the equilibrium outcomes for several simplified versions of our model. These benchmarks strongly suggest that the seller is in a strong position because nothing will support a credible threat for buyers. We find, however, that a strong credible threat is supported when we have an open ended market (infinite horizon), buyers with no individual market power (a continuum), and ongoing growth (upgrade innovation). Removing any one of these elements will result in the seller capturing the full social surplus. Thus, it is the interaction of these three features in an upgrade market that supports an endogenous level for the credible threat by buyers and, hence, multiple equilibria.

### 2.1 Finite Horizon

An infinite horizon with quality growth implies that buyers will always seek to acquire higher quality units. We consider, then, a finite horizon of  $T$  after which the prospect of acquiring higher quality units is eliminated. For simplicity, let  $v/(1-\delta)$  be the continuation value after the final period for each contiguous quality unit held by a buyer.<sup>6</sup>

Consider the final period and suppose the state is  $(T, q)$ , where  $q \leq T - 1$  is any feasible quality held by buyers at the start of period  $T$ .<sup>7</sup> Then there exists a unique

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<sup>6</sup>More generally, any ‘scrap value’ of  $w \in [0, v/(1-\delta)]$  for each (contiguous) quality unit leads to the full capture of the social surplus by the seller, with  $w$  replacing  $v/(1-\delta)$  in all payoffs.

<sup>7</sup>If buyers are distributed across holdings from 0 to  $T - 1$ , then the seller fully extracts the surplus of each segment via a set of upgrade offers. A buyer selects the intended offer, since the offer to a lower segment is more expensive and the offer to a higher segment lacks necessary contiguous units relative to the buyer’s current position. Alternatively, as discussed in section 6, the seller could offer

outcome (subgame perfect) in which the seller offers an upgrade from  $q$  to  $T$  units, and prices the upgrade at an extraction level. Subgame perfection is being employed to rule out non-credible threats in which buyers do not accept positive surplus offers. All buyers will accept the offer. Thus,  $u_T = [v + \delta v / (1 - \delta)] q$  and the buyers are held to their status quo utility as of the start of period  $T$ . In the final period, with no future upgrades and, hence, no prospect of future utility increments, the incentive to jump ahead of the market (purchase even if no others do so) is the dominant incentive for an individual buyer.

Now consider period  $T - 1$  and suppose the state is  $(T - 1, q)$ , where  $q \leq T - 2$ . Since buyers know that they will not receive any incremental surplus in period  $T$ , they will necessarily accept any offer that provides a positive utility increment in period  $T - 1$ . Even if other buyers were to refuse such an offer, an individual buyer would choose to jump ahead of the market in this case. The seller clearly prefers to sell a unit in period  $T - 1$  rather than period  $T$ . Thus, there exists a unique outcome (subgame perfect) in which the seller offers an upgrade of  $(T - 1 - q)$  units at the extraction price. Working backwards to period 1, the seller always offers an upgrade to the current state of the art at the extraction price. This outcome does not depend on whether we have a single buyer or a continuum. It also prevails if the quality units are independent goods (no upgrade payoff structure). In another familiar benchmark, leasing on a period-by-period would break all of the dynamic payoff links. The model then collapses to a sequence of independent one-period markets in which all surplus accrues to the seller. To summarize, when buyers make a final purchase and exit the market, the absence of future transactions implies that the seller captures all of the social surplus.

## 2.2 Infinite Horizon, Single Buyer

Now, suppose we have a single buyer instead of a continuum. Under the Markovian hypothesis, individual buyers in a continuum have no effect on the state. By contrast, with a single buyer, the state (quality gap) necessarily depends on the buyer's purchasing decision. We find that any Markov perfect equilibrium (MPE) necessarily has the properties that the seller will follow the efficient path, selling the new unit in each period, and price each unit at extraction,  $v / (1 - \delta)$ , so that all surplus accrues as profits to the seller. Let us start with a simple example to see why sales must occur without delay. Suppose

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versions contingent on a buyer's holdings.

that there is delay and two units are sold in period 2 at price  $p$ . MPE then implies

$$\pi_1 = \delta p + \delta^2 \pi_1$$

and

$$u_1 = \delta(2v - p) + \delta^2 \left( \frac{2v}{1 - \delta} + u_1 \right).$$

We can now apply a modified version of the familiar argument of Fudenberg, Levine, and Tirole (1985) to obtain a profitable speed up deviation by the seller. Suppose the seller offered one unit at a price  $\hat{p}$  in period 1. If the buyer accepts then the quality gap is 1 starting next period rather than 2 (note that by accepting the single buyer changes the continuation state), and the seller earns  $\hat{\pi} = \hat{p} + \delta\pi_1$ . The buyer accepts provided that  $\hat{u} = v - \hat{p} + \delta v/(1 - \delta) + \delta u_1 > u_1$ . Thus, the deviation is profitable for the seller,  $\hat{\pi} > \pi_1$ , and acceptable to the buyer,  $\hat{u} > u_1$ , provided

$$\frac{v}{1 - \delta} - (1 - \delta)u_1 > \hat{p} > (1 - \delta)\pi_1,$$

as follows from the above expressions for  $u_1$  and  $\pi_1$ . Such a  $\hat{p}$  exists if and only if

$$S_1 = \frac{v}{(1 - \delta)^2} > u_1 + \pi_1.$$

Efficiency guarantees that this always holds since  $S_1$  is the maximal surplus while  $u_1 + \pi_1$  is necessarily smaller due to the assumed delay of a sale.

Thus, the seller can profitably speed up the candidate equilibrium. Intuitively, the buyer and seller can share the larger surplus of  $S_1$  by selling a unit in period 1 and it is simple to find a mutually beneficial price for that transaction. More generally, we always have  $S_\tau > \delta S_{\tau+1}$ , and the extra surplus allows us to apply a similar speed up argument to any state  $(\tau + 1, q)$  with a sale that is preceded by a delay. Hence, with a single buyer, the equilibrium path from the start of the game follows the efficient path with a sale every period and the continuation path from any state must involve an immediate upgrade to the state of the art.

We now argue that this implies extraction of the buyer. For each state  $(\tau, 0)$  we know that the continuation is an upgrade offer to the state of the art at price  $p_\tau$  for payoffs of  $\pi_\tau = p_\tau + \delta\pi_1$  and  $u_\tau = v\tau/(1 - \delta) - p_\tau + \delta u_1$ . Then,

$$\pi_\tau + u_\tau = \frac{v\tau}{1 - \delta} + \delta(\pi_1 + u_1)$$

is the joint payoff. We must have  $u_\tau = \delta u_{\tau+1}$ : if  $u_\tau < \delta u_{\tau+1}$ , the buyer would reject  $p_\tau$ , since the  $\tau + 1$  offer is more attractive; if  $u_\tau > \delta u_{\tau+1}$ , then the seller could raise the price and the buyer would still accept. This implies that  $u_1 = \delta^{\tau-1} u_\tau$ . Substituting for  $u_\tau$  in the equation for the joint payoff and simplifying, we find

$$\pi_\tau = \frac{v\tau}{1-\delta} + \delta(\pi_1 + u_1) - \frac{u_1}{\delta^{\tau-1}} = \frac{v\tau}{1-\delta} + \delta \frac{v}{(1-\delta)^2} - \frac{\delta u_1}{\delta^\tau}.$$

Suppose  $u_1$  is positive. Then, despite the growth in quality, as  $\tau$  goes to infinity the required exponential growth in the buyer's utility will eventually push the seller's profit below zero. Obviously, this cannot happen in equilibrium. Thus, the buyer lacks a credible threat and is necessarily extracted.

The above argument does not extend to a continuum of buyers: an individual buyer cannot change the state, either by delaying or accepting the seller's offer. For example, in state  $(\tau - 1, q)$  if a single buyer accepts an offer to move to the state of the art, but no other buyer accepts, then in the next period the state is  $(\tau, q)$ . The seller can only earn a profit by making an offer that targets a positive mass of buyers with quality  $q$ .

### 2.3 Infinite Horizon, Continuum of Buyers and No Growth

With no growth, the model reduces to the case of a single good: the seller has one unit to offer to buyers. Thus, when all buyers are identical we essentially have a special case of the problem studied by Fudenberg, Levine, and Tirole (1985), who allow for buyer valuation heterogeneity. Using simpler versions of the arguments employed above, we then find that there is never delay and buyers are always extracted in any MPE when there is no quality growth.

We defer to Section 6 a discussion of a fourth factor, the upgrade structure, including the independent goods benchmark, as this is best done after our results are in place.

## 3 Preliminary Results

We begin with basic results on the necessary structure of equilibria. These serve as building blocks for the main analysis. First, we show that by pricing at a very low level the seller can always induce buyers to make a purchase.

**Lemma 1** (*Flow Dominance*) *Consider any history such that, at the start of period  $\tau$ , all buyers hold the first  $Q$  quality units and no buyer holds unit  $Q + 1$ , where  $\tau > Q$ . Suppose*

the seller makes an upgrade offer for units  $\{Q + 1, \dots, \tau\}$  at price  $p$ , where  $p < v(\tau - Q)$ . Then, in any continuation equilibrium, every buyer accepts the upgrade offer.

The intuition for “flow dominance” is simple. The upgrade from  $Q$  to  $\tau$  is priced sufficiently low that it pays for itself in the current period, since  $v\tau - p > vQ$ . Moreover, even if all other buyers were to reject the offer, an individual buyer who accepts is always weakly better off in the future. This follows from (1) the upgrade payoff structure, since an accepting buyer has a flow surplus of at least  $v\tau$  in future periods, and (2) all buyers have the same opportunities for purchasing from the seller, so an accepting buyer always has the option of making the same choices in the future as other buyers. Essentially, a buyer who holds all of the first  $\tau$  units in period  $\tau + 1$  is never at a disadvantage relative to any other buyer.

It then follows directly that the seller must have a positive payoff both at the start of the game and at any point in the future. This is due to quality growth and flow dominance. At any point in time, the seller always has the option of offering a bundle that includes the new quality unit at a (flow dominant) upgrade price.

**Lemma 2** *In any equilibrium, the payoff of the seller is at least  $v/(1-\delta)$ . For any history in which all buyers hold quality units  $\{1, \dots, Q\}$  and no buyer holds unit  $Q + 1$  at the start of period  $\tau$ , the continuation payoff of the seller is at least  $v(\tau - Q) + \delta v/(1 - \delta)$ .*

The above results are very basic and, as the proofs demonstrate, they do not depend on Markovian behavior or symmetric buyer strategies. Rather, these two lemmas rely only on buyers acting as price takers. The lower bound on the seller payoff provides a reference point and we will show that in a buyer symmetric MPE, every payoff ranging from the full surplus,  $S_1 = v/(1 - \delta)^2$ , all the way down to the flow dominance lower bound,  $v/(1 - \delta)$ , can be supported.

A simple consequence of a positive seller payoff in any continuation is that the quality gap never grows without bound. That is, all new quality units are eventually sold within some fixed number of periods.

**Lemma 3** *In any equilibrium, for any state  $(\tau, Q)$ , the continuation path has a bounded quality gap.*

Now, we show that equilibria must have a simple cyclical structure. To see this, we introduce the notion of a  $t$  – cycle equilibrium. In a  $t$  – cycle equilibrium a sale occurs

every  $t$  periods, and  $t$  units are sold in each sale period. Thus, states  $(1, 0)$  through  $(t - 1, 0)$  are delay states with no sales, and state  $(t, 0)$  has a sale of units 1 through  $t$ . Hence, once a sale occurs in state  $(t, 0)$ , the quality gap falls back to 1 at the start of the next period and, effectively, the state returns to  $(1, 0)$ . Thus, the equilibrium path will cycle and the buyers will increase their quality holdings every  $t$  periods, each time buying an upgrade to go from  $t$  to  $2t$  and so on. Note that, as a special case, this includes the possibility that  $t = 1$ , where the current quality unit is sold to buyers in every period (the efficient path).

**Proposition 1** *Every equilibrium follows a  $t$ -cycle equilibrium path: the buyers purchase quality units  $\{1, \dots, t\}$  from the seller in state  $(t, 0)$ , all payments to the seller occur in state  $(t, 0)$ , and the maximal buyer quality is zero until period  $t$ .*

What makes this argument work is flow dominance and the fact that the seller can profitably deviate by speeding up a cycle that does not have buyers moving to the state of the art in  $(t, 0)$ . Thus, if the sale to buyers only involves  $\tau < t$  units, the seller can feasibly offer these units in state  $(t - 1, 0)$ . By pricing these units at  $\hat{p} = v\tau + \delta p - \varepsilon$ , where  $p$  is the price for  $\tau$  units in state  $(t, 0)$ , a seller improves his payoff if all the buyers accept since

$$\hat{p} + \delta\pi(t, \tau) = (v\tau + \delta p - \varepsilon) + \delta^2\pi(t + 1, \tau) > \delta p + \delta^2\pi(t + 1, \tau)$$

and, upon substituting for  $\hat{p}$  and noting that  $(t, \tau)$  is a delay state, this reduces to  $v\tau > \varepsilon$ .

The candidate equilibrium cannot have buyers rejecting this offer. If other buyers reject, an individual will always find it optimal to purchase the deviation offer (for small  $\varepsilon > 0$ ). By accepting, an individual buyer receives  $\delta u(t, 0) + \varepsilon$ . To see this, note that the deviating buyer does not change the state, so  $\tau$  units will be offered next period. Since the buyer already has these units, the purchase in period  $t$  can be skipped and the buyer will have the same holdings as all other buyers as of  $t + 1$ . Thus, her payoff is improved relative to waiting whenever  $\varepsilon > 0$ . Hence, all buyers rejecting the offer is not an equilibrium continuation. But, as we showed above, when all buyers accept the offer the seller can profit by making the deviation offer. Thus, an equilibrium with sales of  $\tau$  less than  $t$  cannot be supported.<sup>8</sup>

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<sup>8</sup>While the intuition is clear, the proof is more involved. This is because the intuition relies on several properties, which are established in the appendix, regarding when payments occur and when non-contiguous units are obtained.

By contrast, the speed up argument does not apply to a  $t$  – *cycle* equilibrium, where the seller offers an upgrade for all units  $\{1, \dots, t\}$ , for two reasons. The first is feasibility. The seller does not have  $t$  units to sell in period  $t - 1$ . Second, an individual buyer who accepts the deviation offer in  $t - 1$  is not in an analogous position. By acquiring  $\tau$  units when no other buyers accept, an individual buyer can no longer safely skip all purchases in state  $(t, 0)$ , since other buyers will be acquiring units 1 through  $t$ . For example, if the seller only offers the bundle of units 1 through  $t$ , then the deviating buyer will either have to buy the same bundle as the other buyers and pay for the  $\tau$  units that were previously purchased or else fall behind the market. We refer to a  $t$  – *cycle* equilibrium with  $t \geq 2$  as a generational equilibrium, as the upgrade can be viewed as a new generation of the product which incorporates a variety of new features and improvements and moves buyers to quality frontier. The length of the cycle reflects an implicit form of equilibrium coordination and, as the above argument implies, the seller cannot profitably induce buyers to purchase at an earlier date.

To summarize, a seller must either sell units as soon as they are feasible, thus following the efficient path, or delay to a maximal set of units periodically, inducing an inefficient path.

Finally, to streamline the equilibrium analysis, we specify strategies such that an individual buyer who deviates by not following other buyers in a purchase that increases the maximal buyer quality will obtain no future additional surplus. Thus, if an individual buyer has the first  $k$  units of the good, when all other buyers also have additional contiguous units, then this buyer’s continuation payoff is  $vk/(1 - \delta)$ .<sup>9</sup> We can easily allow for higher buyer catch-up continuation values as long as they do not exceed the equilibrium payoff. For the analysis, however, it is helpful to follow the above specification of a zero increment in utility for a buyer who falls behind the market. This will highlight the critical role played by the incentive for a buyer to jump ahead of the market.

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<sup>9</sup>One can interpret this as (i) the seller (optimally) ignores individual buyers (measure zero) who differ from the market path and the missing units necessary to catch up are never offered, or (ii) the seller offers the necessary units but sets the upgrade price so as to extract all of the continuation surplus. Because the seller’s strategy does not depend on a deviation by a set of measure zero buyers, the seller must either completely refrain from making “catch-up” offers, or always make such offers. Either of (i) or (ii) is sufficient for the equilibrium construction.

## 4 Efficient Equilibria

In an efficient equilibrium, a good is sold in each period when it first becomes available. At the start of the game, the seller offers the first unit at price  $p_1$  and all buyers accept. In the second period, the state is then  $(2, 1)$  and the quality gap is again 1. Under Markovian behavior, the seller offers the second unit at price  $p_1$  and all buyers accept, leading to state  $(3, 2)$ , and so on. Thus, the firm earns a profit of  $\pi_1 = p_1/(1 - \delta)$  and buyers receive a payoff of  $u_1 = \frac{1}{1-\delta} \left[ \frac{v}{1-\delta} - p_1 \right]$ . In an efficient equilibrium, the firm and buyers divide the maximal social surplus,  $S_1 = v/(1 - \delta)^2 = \pi_1 + u_1$ .

What is required for  $p_1$  to be an equilibrium price? For buyers it is natural to start with the idea of a cut-off rule: accept any price up to  $p_1$  and reject all higher prices. Given this, the seller effectively faces a binary choice of either making a sale at  $p_1$  or inducing delay. Delay leads to state  $(2, 0)$  and we need payoffs for this state in order to assess whether the seller prefers a sale or delay in state  $(1, 0)$ . Similarly, an assessment of buyer and seller choices in  $(2, 0)$  must account for a larger set of bundling options as well the delay option and, hence, payoffs in state  $(3, 0)$ , and so on.

We will show that any buyer utility level  $u_1 \in [0, \delta S_1]$  can be supported as an equilibrium payoff for any  $\delta$ . That is, the seller may be limited to only the flow payoff of  $v$  per period, which has a present discounted value of  $v/(1 - \delta)$ . Recall that this is the minimum possible payoff for the seller (flow dominance and Lemma 1). To show this result, we first derive support constraints for all (off equilibrium) states where the quality gap is above 1. These reduce to “cash-in” constraints in which the seller optimally offers a full upgrade to move the buyers up to the state of the art. Next, we generate a set of supporting utilities, the continuation payoffs for buyers in the event that there is no current transaction. Essentially, these payoffs constitute a credible threat that induces the seller to make the equilibrium offer.

### 4.1 Buyer and Seller Support Constraints

Equilibrium requires that no player can benefit from a deviation.<sup>10</sup> To rule out deviations, we must specify the continuation payoffs from state  $(2, 0)$  and other off-equilibrium states; the continuation payoff for any state is pinned down (Markov) once we specify those for states of the form  $(\tau, 0)$ . Suppose that in state  $(\tau, 0)$ , the seller offers  $\tau$  units at

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<sup>10</sup>We apply the one-stage-deviation principle to verify the proposed strategies constitute an equilibrium; our model conforms to the necessary requirement of “continuity at infinity,” since the limit of  $\tau\delta^\tau$  is 0 as  $\tau$  goes to infinity (see Fudenberg and Tirole (1991) pp. 108-110).

a price  $p_\tau$  and this is accepted by all buyers. Thus, the next state is  $(\tau + 1, \tau)$ , where the quality gap has returned to 1, and the players are back on the (incremental) equilibrium path. Then, continuation payoffs with this cash-in support at  $(\tau, 0)$  are  $\pi_\tau = p_\tau + \delta\pi_1$  for the seller and  $u_\tau = \frac{v\tau}{1-\delta} - p_\tau + \delta u_1$  for the buyers. Note that this is the efficient path from  $(\tau, 0)$ , and we have  $S_\tau = \frac{v\tau}{1-\delta} + \delta S_1 = \pi_\tau + u_\tau$ .

For a continuation equilibrium to follow this “cash-in” support, we must specify the accompanying buyer and seller strategies. The seller has three ways of deviating. The first option is to make no offer, a “delay,” which necessarily leads to state  $(\tau + 1, 0)$  and buyers make no decision. The second option is to offer an upgrade of less than  $\tau$  units, a “partial cash-in.” The final option is to offer an upgrade of  $\tau$  units but at a price different from  $p_\tau$ . It must be optimal for the seller to follow the strategy of offering  $\tau$  units at the price  $p_\tau$  in state  $(\tau, 0)$ .

For buyer strategies in state  $(\tau, 0)$  we specify a simple cut-off rule: a buyer accepts the seller offer of price  $p$  for  $\sigma$  units in state  $(\tau, 0)$  if and only if  $p \leq p(\sigma, \tau)$ . On the acceptance side, it must be optimal for an individual buyer to accept any offer  $p \leq p(\sigma, \tau)$ , given that all other buyers are accepting (symmetric strategies) and the quality gap moves to  $\tau + 1 - \sigma$ . Rejecting when others all accept yields 0 by construction, as the buyer falls behind the market. Accepting yields a current flow of  $v\sigma - p$  plus a future value of  $\delta u(\tau + 1, \sigma)$ ; recall that  $u(\tau + 1, \sigma)$  is the sum of the PDV of  $\sigma$  units and the increment  $u_{\tau+1-\sigma}$ . Thus, it is an equilibrium for all buyers to accept  $p$  for  $\sigma$  units in state  $(\tau, 0)$  if  $\frac{v\sigma}{1-\delta} + \delta u_{\tau+1-\sigma} \geq p$ . This condition reflects the incentive of a buyer to “keep up” with the market.

The rejection side of the cut-off rule reflects the incentive not to “jump ahead” of the market, and an offer of  $p > p(\sigma, \tau)$  must be rejected by all buyers. Rejecting when others reject yields a payoff of  $\delta u_{\tau+1}$ . More subtly, accepting when all other buyers reject yields a flow of  $v\sigma - p$  plus the option of purchasing the cash-in offer for  $\tau + 1$  units next period. Thus, an individual buyer optimally rejects when others reject if

$$p > v\sigma + \delta \max \left\{ \frac{v\sigma}{1-\delta}, u_{\tau+1} \right\} - \delta u_{\tau+1} \equiv g(\sigma, u_{\tau+1}).$$

To understand this condition, consider the position of a deviating buyer in the next period. When the other buyers purchase the cash-in offer in period  $\tau + 1$ , this buyer has two options. If  $u_{\tau+1} > \frac{v\sigma}{1-\delta}$ , it will be optimal to purchase when these other buyers do. Thus, the deviating buyer is initially willing to pay at most the flow value of the units,

$v\sigma$ , in period  $\tau$ . Otherwise, the buyer will not make the purchase in  $\tau + 1$  and is willing to pay up to  $\frac{v\sigma}{1-\delta} - \delta u_{\tau+1}$ .

Thus, combining the acceptance and rejection sides of the cut-off strategy we have

$$g(\sigma, u_{\tau+1}) \leq p(\sigma, \tau) \leq \frac{v\sigma}{1-\delta} + \delta u_{\tau+1-\sigma} \quad (1)$$

for all  $0 < \sigma \leq \tau$  and all  $\tau \geq 1$ . The cut-off strategies apply to full ( $\sigma = \tau$ ) and partial ( $\sigma < \tau$ ) cash-in offers. Since  $g$  is bounded above by  $\frac{v\sigma}{1-\delta}$ , cut-off strategies exist for any non-negative utility sequence. The upper bound in (1) says that prices must be sufficiently low that falling behind the market is not optimal for an individual buyer, while the lower bound says that it is optimal to jump ahead and accept offers below this level. Since  $g$  is bounded below by  $v\sigma$ , flow dominance implies that a buyer is always willing to pay at least  $v\sigma$ .

The distinct upper and lower bounds on the cut-off rule show that an individual buyer's willingness to pay depends on the actions of other buyers. It is important to recall that there are no network externalities in our model, which is a standard reason for why buyers make their purchasing decisions based on expectations of other buyers' choices. Instead, the linkage of decisions in our model arises from i) quality growth and the resulting incentive for a buyer to return to the market for another upgrade, ii) the seller's offer in the future depends on "the state of the market" and, iii) an individual buyer will be affected by his position relative to the market when making future purchasing decisions.

Given these buyer cut-off strategies, the seller must find it optimal to offer  $\tau$  units at price  $p_\tau$  in state  $(\tau, 0)$ . Beginning with partial cash-ins, note that  $p(\sigma, \tau)$  is the optimal price choice for any such offer and it generates a payoff of  $p(\sigma, \tau) + \delta\pi_{\tau+1-\sigma}$ . This implies that for an equilibrium

$$\pi_\tau - \delta\pi_{\tau+1-\sigma} \geq p(\sigma, \tau) \quad (2)$$

for  $\sigma = 1, \dots, \tau - 1$ .

The other two deviations are delay and offering  $\tau$  units at a price different than  $p_\tau$ . Delay,  $\sigma = 0$ , is not optimal if  $\pi_\tau \geq \delta\pi_{\tau+1}$ . Defining  $p(0, \tau) \equiv 0$ , (2) applies. Finally, consider a cash-in offer of  $\tau$  units. Buyers will accept any price below  $p(\tau, \tau)$ , so we must have  $p_\tau = p(\tau, \tau)$  or else the seller could successfully offer a price above  $p_\tau$ . In other words, buyers must reject any price above  $p_\tau$  for  $\tau$  units. Note that (2) holds with equality at  $\sigma = \tau$  by construction of the equilibrium continuation. Similarly, note that the buyer condition (1) also applies in the event of delay ( $\sigma = 0$ ) and on the equilibrium

path ( $\tau = \sigma = 1$ ).

## 4.2 Sufficient Conditions

We can now combine the buyer and seller support conditions, (1) and (2), to identify when there exist supporting prices  $p(\sigma, \tau)$  such that an efficient equilibrium is supported by the cash-in outcome. Combining the seller profit from (2) with the buyer lower bound on prices from (1), we must have

$$\pi_\tau - \delta\pi_{\tau+1-\sigma} \geq p(\sigma, \tau) \geq g(\sigma, u_{\tau+1}).$$

Recalling that  $S_\tau = \pi_\tau + u_\tau$ , we see that the above condition is equivalent to

$$S_\tau - \delta S_{\tau+1-\sigma} \geq u_\tau - \delta u_{\tau+1-\sigma} + g(\sigma, u_{\tau+1}), \quad (3)$$

for  $0 \leq \sigma \leq \tau$  and  $\tau \geq 1$ . The surplus difference on the left hand side is an exogenous sequence that is increasing in  $\tau$ . So, as  $\tau$  grows and more units are “on the table,” a larger set of payoff utilities can be supported.<sup>11</sup> Given a sequence of utilities that satisfies (3), we can clearly construct the supporting prices  $p(\sigma, \tau)$  for conditions (1) and (2). When (3) holds, the optimal upgrade offer for the seller is to offer  $\tau$  units for the price  $p_\tau$ . Each buyer then finds it optimal to accept the upgrade offer, given that all other buyers also accept. We then have

**Lemma 4** *Suppose the sequence of buyer utilities  $u_\tau$  satisfies (3). Then there exists an efficient equilibrium with supporting prices  $p(\sigma, \tau)$ .*

The proof that a sequence of utilities satisfying (3) is sufficient for the existence of an equilibrium outcome with payoff  $u_1$  is by construction. Taking a given  $u_1$ , the rest of the utility sequence is specified in the next section. For this sequence, we must show that it is not profitable for the seller to deviate by offering multiple upgrade options (as well as options with non-contiguous units); note that (3) only rules out seller deviations involving a single upgrade offer. This requires that we specify buyer strategies in response to any such offer from the seller. In addition, we must specify strategies for continuation equilibria in the event that buyer holdings are distributed asymmetrically across units in  $\{0, \dots, \tau - 1\}$  in any period  $\tau$ , even though such states do not arise on the equilibrium path.

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<sup>11</sup>In the benchmark case of a single good, where a buyer exits the market after a purchase, (3) reduces to  $0 \geq u_1$  and all surplus accrues as profit.

In all cases, the support returns to the equilibrium path after 1 period. See Appendix C for the support construction.

We now turn to the task of finding buyer utilities that support equilibria.

### 4.3 Support Utilities for Frequent Innovation

For each payoff  $u_1 \in [0, \delta S_1]$ , we will construct an associated supporting path of  $u_2, u_3, \dots$  such that the seller finds it optimal to make an acceptable offer to achieve a cash-in outcome in every state. A period in our model corresponds to the length of time before the next unit of quality can feasibly be offered by the seller. As noted above, frequent innovation corresponds to a large discount factor while relatively infrequent upgrades correspond to a small discount factor. First, we take up the main case of frequent innovation,  $\delta > 1/2$ , in which future units are relatively more valuable. We conclude the section with the simpler case of infrequent innovation,  $\delta \leq 1/2$ .

To gain intuition, we begin with two special cases of support utilities. First, suppose we have a constant sequence,  $u_1 = u_2 = \dots = u$ . Thus, in the event of delay, all of the gains from the growing surplus will accrue to the seller. We claim that this gives the seller an incentive to delay whenever the buyers' utility exceeds  $v/(1 - \delta)$ , which is less than  $\delta S_1$  for  $\delta > 1/2$ . Since the seller is the residual claimant of surplus, the loss from delay is just  $v$ . On the other hand, the gain from delay is the saving in utility of  $(1 - \delta)u$ . If  $u > v/(1 - \delta)$ , then the seller will prefer to delay and earn  $\delta(S_2 - u)$  rather than  $S_1 - u$  from selling today. Intuitively, the discounted share of a larger residual will always dominate once  $u$  becomes large enough. This is a direct consequence of the growth in surplus that is generated when quality improves over time.

What this example shows is that buyer utility must rise with delay for a higher  $u_1$  equilibrium payoff. Suppose, then, that the buyers' utilities are increasing and always push the seller to indifference between delay and a cash-in. This reverses the residual claimant role from the first example. The efficiency gains from early cash-in are  $v\tau$ ; this is the one period flow value of  $\tau$  units and is precisely  $S_\tau - \delta S_{\tau+1}$ . Thus, by (3),

$$u_{\tau+1} = \frac{u_\tau - v\tau}{\delta}. \quad (4)$$

Since the difference between  $u_\tau$  and  $\delta u_{\tau+1}$  is exactly  $v\tau$ , the difference between  $\pi_\tau$  and  $\delta\pi_{\tau+1}$  must be 0. The payoff to the seller rises over time, but this is exactly offset by the

discount factor  $\delta$ . The support condition (3) for a cash-in at  $\tau$  is

$$S_\tau - \delta S_1 \geq u_\tau - \delta u_1 + g(\tau, u_{\tau+1}).$$

In this case, we claim that once a sufficient number of units are available, the seller can successfully raise prices and (3) will necessarily fail. For  $\tau$  sufficiently large, it is straightforward to show that  $g(\tau, u_{\tau+1}) = v\tau/(1 - \delta) - \delta u_{\tau+1}$ ; a buyer who jumps ahead of the market will never purchase again. The support condition for a cash-in then reduces to

$$\delta u_1 \geq v\tau$$

which is violated for  $\tau$  sufficiently large.

Intuitively, the flow value of  $\tau$  units must eventually dominate any prospects. Relative to the future, buyers always receive an extra  $v\tau$  today, since  $u_\tau = \delta u_{\tau+1} + v\tau$ . For  $\tau$  sufficiently large, this utility growth cannot be maintained. Eventually, a flow-dominance offer at price  $v\tau$ , which will attract all buyers, allows the seller to charge a higher price.

Thus, we seek a support that is increasing, but not too slowly or for too long, in order to generate the full set of possible subgame perfect payoffs,  $u_1 \in [0, \delta S_1]$ . We do this by combining aspects of the two special cases. When the quality gap is  $T$  or smaller, states  $(1, 0)$  to  $(T, 0)$ , the support makes the seller indifferent between a cash-in and delay; for states with a larger quality gap the support keeps buyer utility constant at  $u_T$ . Thus, for  $T \geq 2$  we define a  $T$ -stage support utility sequence by

$$u_\tau = \begin{cases} v\tau + \delta u_{\tau+1} & \text{for } \tau = 1, \dots, T-1 \\ u_T & \text{for } \tau \geq T. \end{cases} \quad (5)$$

For any given  $u_1$  and length  $T$ , the sequence  $(u_2, \dots, u_T)$  is determined. Clearly, a larger  $u_1$  implies an upward shift in the rest of the sequence. For the special case of  $T = 1$ , we specify a constant sequence  $u_\tau = u_1$  for all  $\tau$ . For another perspective, starting from  $u_T$  and working backwards to  $u_1$ , it is clear that  $u_1$  includes all of the interim surplus generated by quality growth over the next  $T$  periods:  $u_1 = v + \delta 2v + \dots + \delta^{T-1} u_T$ . As we will see, a higher  $u_1$  or a larger  $\delta$  will require a larger length  $T$ .

A direct consequence of a  $T$ -stage support is that we only need to satisfy the support constraints, equations (3), over the range  $\tau = 1, \dots, T$ ; see Lemma A1 in Appendix C. This is because, when (3) holds at  $\tau = T$ , then it necessarily holds at all larger  $\tau$  whenever the buyer utility remains constant and the seller is the residual claimant of surplus growth.

Thus, an advantage of a  $T - stage$  support is that we only have to check a finite set of conditions. We now turn to finding the appropriate length for the  $T - stage$  support.

#### 4.4 Frequent Innovations: Existence and Payoffs

The following algorithm determines the  $T - stage$  support for any given utility level  $u_1 \in [0, \delta S_1]$ :

- Pick a utility level  $u_1$  between 0 and  $\delta S_1$ .
- If  $u_1 \leq (1 - \delta) S_1$ , then set  $u_\tau = u_1$  for all  $\tau > 1$ .
- If  $u_1 \in [(1 - \delta) S_1, \delta S_1]$ , set  $u_2 = \left(\frac{u_1 - v}{\delta}\right)$ .
- If  $u_1 < (1 - \delta^2) S_1$ , set  $u_\tau = u_2$  for all  $\tau > 2$ . If not, set  $u_3 = \left(\frac{u_2 - 2v}{\delta}\right)$ .
- Keep following this logic until  $T$  such that  $u_1 \leq (1 - \delta^T) S_1$ .

We know the algorithm always has a final step, since  $\delta S_1 \leq (1 - \delta^T) S_1$  holds for  $T$  sufficiently large. The discount factor  $\delta$  determines how  $T$  must be set in order to cover the entire range of buyer payoffs  $[0, \delta S_1]$ . To see this, we need to define a set of critical  $\delta$  cutoffs: let  $\delta_\tau$  be the root of  $\delta^\tau + \delta = 1$  for  $\delta \in (0, 1)$ . Figure 1 then illustrates the relationship between  $u_1, \delta$ , and  $T$ . For example, when  $1/2 < \delta < \delta_2$ , we use a 1-stage support for the area A in Figure 1, where  $u_1 < (1 - \delta) S_1$ , and then a 2-stage support for larger  $u_1$  in the area B. Because the  $\delta S_1$  curve is below the  $(1 - \delta^2) S_1$  surplus curve, we have covered all possible buyer payoffs for this range of  $\delta$ . In the next range,  $\delta \in (\delta_2, \delta_3)$ , after areas C and D, we must also use a 3-stage support to cover the highest buyer payoffs (in the area E). As  $\delta$  continues to rise, we use the critical  $\delta$  cutoffs to identify the appropriate (maximal) length for the  $T - stage$  support. Referring back to Figure 1, for each  $\delta$  range we rise vertically with a 1 - stage, then a 2 - stage, and so on up to the appropriate  $T - stage$  to support each payoff interval on the  $u_1$  axis. It is also helpful to note that for each  $u_1$  in a given payoff interval, the region between the surplus curves in Figure 1 shows that a given length  $T - stage$  support will support the  $u_1$  payoff across all discount factors. Thus, a 1 - stage support covers the areas A, C and F which includes all payoffs below the  $(1 - \delta) S_1$  surplus curve. Since  $(1 - \delta) S_1 = v/(1 - \delta)$ , this is exactly what we found with our first example for support utilities (constant utility). Similarly, with  $T = 2$  we cover all payoffs in the areas B, D and G, the region between the surplus curves  $(1 - \delta^2) S_1$  and  $(1 - \delta) S_1$ .

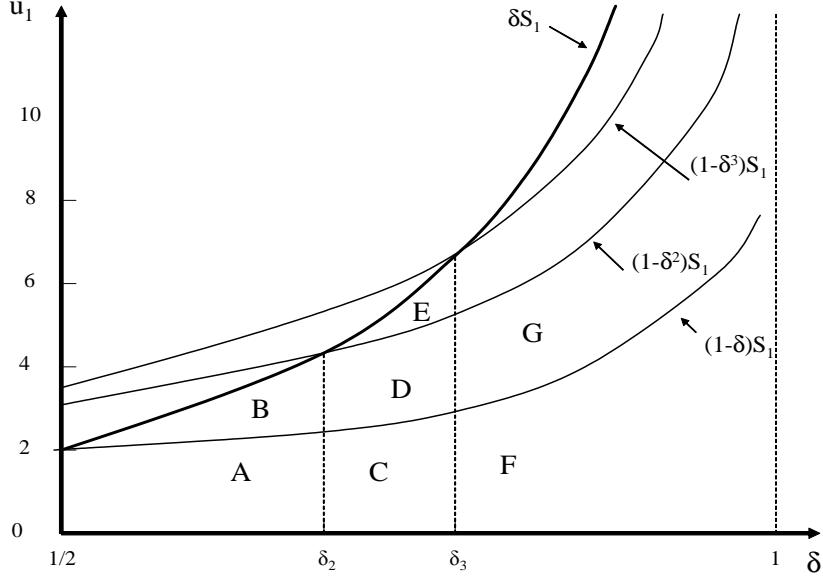


Figure 1: Boundary for T-Stage Support ( $v = 1$ )

Consider the incentive structure for the seller and the buyers under a  $T$ -stage support. The algorithm identifies a set of intervals that covers the entire range of buyer payoffs, 0 to  $\delta S_1$ . Thus, for each feasible buyer payoff,  $u_1$ , there is a unique  $T$  such that  $u_1$  lies in the interval from  $(1 - \delta^{T-1})S_1$  to  $(1 - \delta^T)S_1$ . The right end point is determined by incentives on the seller side of the market, while the left end point is determined by incentives on the buyer side.

For the seller, the initial phase of indifference between cash-in and delay is followed by a strict preference for a cash-in once the quality gap reaches  $T$ . For this second phase,  $\tau \geq T$ , we have  $\pi_\tau > \delta\pi_{\tau+1}$  as long as  $u_1 < (1 - \delta^T)S_1$ . Any  $u_1$  above this upper bound (given  $T$ ) would provide the seller with a strict incentive to delay selling in period  $T$  and the support would unravel. To understand better the economic link between buyer and seller payoffs in periods 1 and  $T$ , consider the division of surplus. In period 1, a buyer utility of  $u_1 = (1 - \delta^T)S_1$  together with the efficient outcome imply that the seller receives  $\pi_1 = \delta^T S_1$  and therefore  $\pi_T = \delta S_1$ , by seller indifference. In period  $T$ , the cash-in outcome has the parties divide the surplus of  $S_T$  which consists of the value of  $T$  units plus the discounted efficient surplus of  $S_1$ . Thus, the buyer payoff must be  $u_T = vT/(1 - \delta)$ , since the seller's share is  $\delta S_1$ . This reflects the fact that the  $T$ -stage support is constructed so that  $u_1$  capitalizes the surplus growth over  $T$  periods (see Lemma A2 in Appendix C).

The economic link between the seller's delay incentive and the buyer payoff  $u_T$  operates as follows. Since the  $T$  - *stage* support holds the buyer payoff constant for any period  $\tau \geq T$ , the seller can give the buyers  $u_T$  now or wait and give them  $u_T$  next period. Hence, the cost of selling today is  $(1 - \delta)u_T$ . The benefit of selling today is the efficiency gain  $v\tau = S_\tau - \delta S_{\tau+1}$ , since the seller is now the residual claimant for surplus growth. So, selling today is more profitable than waiting to cash-in tomorrow when

$$v\tau \geq (1 - \delta)u_T. \tag{6}$$

Otherwise, waiting is better. By choosing  $T$  so that  $u_T$  is less than  $vT/(1 - \delta)$  whenever  $u_1$  is below  $(1 - \delta^T)S_1$ , the seller has the necessary incentive to avoid delay along the support path. Of course, longer initial support phases will also be sufficient for the seller incentive. Thus, the upper bound  $(1 - \delta^T)S_1$  on the equilibrium payoff  $u_1$  is tied directly to when the seller will first have a strict preference to cash-in. In other words,  $T$  is the first time the flow dominance payoff exceeds the savings from delay.

The left end point, the lower bound on  $u_1$  of  $(1 - \delta^{T-1})S_1$ , is determined by buyer incentives. Suppose that the seller offered  $\tau$  units at a price above the supporting price,  $p_\tau$ . Given that all other buyers reject this offer, an individual buyer who deviates must assess the option value of having  $\tau$  units in later periods. In a  $T$  - *stage* support, the seller is expected to make a cash-in offer of  $\tau + 1$  units next period and a deviating buyer will have to decide whether to accept or reject this offer. A rejection means that the deviating buyer will lack unit  $\tau + 1$ . Due to the upgrade structure, a buyer who falls behind the market will not be able to make any future purchases that yield positive surplus. Thus, accepting the  $\tau + 1$  offer is best if  $u_{\tau+1}$  exceeds the status quo position of  $v\tau/(1 - \delta)$ . Clearly, when  $\tau \geq T$ , a deviating buyer rejects the offer in favor of the status quo holding because  $u_{\tau+1} = u_T$  is now below the value  $v\tau/(1 - \delta)$ . This follows directly because  $vT/(1 - \delta)$  exceeds  $u_T$  when  $u_1$  is below  $(1 - \delta^T)S_1$ , the same property that drove the seller's incentive.

A  $T$  - *stage* support has more subtle buyer incentives in the initial phase, where  $\tau < T$ , since both  $u_{\tau+1}$  and  $v\tau/(1 - \delta)$  vary with  $\tau$ . Because the  $T$  - *stage* support capitalizes the surplus growth up through period  $T$ , by taking  $u_1$  above the left end point,  $(1 - \delta^{T-1})S_1$ , we generate utilities such that (i)  $u_\tau > v\tau/(1 - \delta)$  for  $\tau = 1, \dots, T - 1$  and (ii)  $v(T - 1)/(1 - \delta) < u_T$  (see Lemma A3 in Appendix C).<sup>12</sup> From (1), a deviating buyer in

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<sup>12</sup>When  $u_1$  is at an endpoint of one the intervals in the algorithm, (i) and (ii) typically involve a weak inequality.

period  $\tau \leq T - 1$ , expects to purchase again in period  $\tau + 1$ . Thus, the most the deviating buyer would be willing to pay in  $\tau$  is just the flow value of the units,  $v\tau$ . In this way, the  $T - stage$  support provides buyers with a credible threat when presented with a higher than expected price from the seller.

The key underlying forces behind the credible buyer threat are surplus growth and implicit buyer coordination. With respect to surplus growth, in our finite horizon and no quality growth benchmarks, there is always a successful speed up deviation by the seller for any candidate equilibrium with a positive buyer payoff. By contrast, we have equilibria with positive buyer payoffs. Surplus growth allows us to construct the  $T - stage$  support that maintains incentives by making buyers and then the seller the residual claimant of quality growth. With respect to coordination, in our single buyer benchmark case a buyer has a zero payoff. With multiple buyers, no single buyer can influence the state of the market. An individual buyer's decision focuses on his position relative to the market, which matters because of the upgrade payoff structure (we return to this point in Section 6, where we consider variations on the upgrade structure). Because quality grows, buyers are forced to return to the market to acquire new units, and we find that implicit coordination is supported when a buyer's willingness to pay depends on whether they hold more or less than other buyers. For example, when other buyers are expected to refuse the current offer, a buyer who purchases today and also expects to purchase tomorrow is then only willing to pay the flow value for the units today.

Implicit coordination does not, of course, guarantee that buyers pay only the flow value for units in equilibrium. The price  $p_\tau$  is determined by  $u_\tau$ , and this is the highest price that a seller can set before the buyers reject. This price must be at least  $v\tau$ , as implied by flow dominance. The price  $p_\tau$  can be strictly above  $v\tau$ , but a seller cannot raise the price any further without losing all current sales.

We can now prove our main result for this section.

**Proposition 2** *Let  $\delta > 1/2$ . Then, for every  $u_1 \in [0, \delta S_1]$ , there exists an efficient equilibrium with buyer payoff  $u_1$ .*

Significantly, the minimum possible equilibrium payoff for the seller is the flow dominance lower bound from Lemma 1. Thus, there is an equilibrium in which the seller's market power is reduced to the static flow value of each unit with all of the future surplus from a unit accruing to buyers.

In this section we examined the situation where innovation is relatively frequent,  $\delta \geq 1/2$ . Consider the range of equilibrium payoffs as innovation occurs at an increasingly

rapid rate. Thus, we let the time between innovations,  $\Delta$  (period length), converge to 0. Recall that  $\delta = e^{-r\Delta}$ . Taking care to adjust the flow value of buyer surplus,  $v$ , for the rate of innovation via  $v = \int_0^\Delta \lambda e^{-r\tau} d\tau = \lambda(1 - e^{-r\Delta})/r$ , where  $\lambda$  is the instantaneous flow value of quality to buyers, limiting outcomes can be calculated directly. First, we observe that the flow dominance lower bound on the seller payoff of  $v/(1 - \delta)$  (Lemma 2) converges to  $\lambda/r$ . Thus, in the limit flow dominance reduces to the seller collecting a payment of  $\lambda$  at each instant. This is, however, a vanishingly small fraction of the total surplus, since  $v/(1 - \delta)^2$  grows without bound. Given this, it follows directly that in an efficient equilibrium the maximum buyer share of the surplus converges to 1. Thus, we have

**Corollary 1** *In the limit, as upgrades become increasingly frequent ( $\Delta \rightarrow 0$ ), the seller's minimum share of the surplus goes to zero and the buyers' maximum share goes to one.*

Intuitively, as innovation becomes increasingly frequent, the flow value becomes a smaller and smaller portion of the total surplus. This reflects a necessary limitation on the extent of the seller's market power. Despite the fact that the surplus grows without bound, flow dominance only ensures a finite profit for the seller. Thus, there is no guarantee of market power, as measured by profit as a share of the total surplus, when innovations arrive very frequently. Stated a bit differently, a market may allow a seller to earn a high level of absolute profit while capturing only a small share of the total surplus.<sup>13</sup>

## 4.5 Infrequent Upgrades: Existence and Payoffs

We now take up the case when innovations are infrequent,  $\delta \leq 1/2$ . Flow dominance will play a stronger role, since the flow value of a unit of quality,  $v$ , is now larger than the future discounted value of a unit,  $\delta v/(1 - \delta)$ . When  $\delta \leq 1/2$ , a buyer necessarily values one unit today more than two units tomorrow, other things equal. Stated a bit

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<sup>13</sup>In our model,  $\delta$  reflects the time between innovations as well the commitment period of the seller (time between offers). This is without loss of generality. We can allow the seller to make offers more frequently, holding the innovation rate constant, and follow the same principles for constructing the support utilities. In contrast to Coasian settings, these offers will be off-the-equilibrium path. It is straightforward to check that an efficient equilibrium with a  $T$  – stage support is robust to allowing the seller to make an interim offer of any feasible bundle at any time between innovations. Specifically, in the constant utility part of the  $T$  – Stage support we can specify a cash-in at the same utility level for the interim offer. When utility is rising it is simplest to specify a delay outcome for the support. Where as allowing interim offers in the Coasian setting with heterogenous buyers and a single good will speed up sales, we find that the original equilibrium path where the arrival of quality increments coincides with sales does not change with the commitment period of the seller.

differently, given a prospect of receiving two units tomorrow free of charge, there is always a positive price that a buyer would be willing to pay to acquire, instead, one unit today. This immediately implies that the seller will necessarily be able to induce a ‘speed up’ in a number of situations where it is not possible to do so when the discount factor is larger than  $1/2$ . As we show in the next section, it rules out the possibility of delay in equilibrium when  $\delta \leq 1/2$ .

The increased power of flow dominance, the driving force behind the speed-up argument, does not, however, imply a necessary increase in market power for the seller. We are still able to support efficient equilibria in which buyers receive any payoff  $u_1 \in [0, \delta S_1]$  and, as a result, the range of equilibrium payoffs for seller continues to include the flow dominance lower bound. The speed-up argument does change the qualitative nature of the equilibrium support. In particular, no equilibrium can be supported with a constant buyer continuation payoff (i.e.,  $u_1 = u_2 = \dots = \bar{u}$ ); the relatively low continuation value implies that buyers will not reject a deviation offer of a price above  $p_1$  (for the candidate  $u_1$ ). Instead, equilibrium requires a relatively high continuation payoff for buyers so that the seller cannot successfully increase price in period one.

The following proposition formalizes the above argument.

**Proposition 3** *Suppose  $\delta \leq 1/2$ . Then every  $u_1 \in [0, \delta S_1]$  can be supported in an efficient equilibrium with a continuation utility of  $u_\tau = \bar{u} \equiv u_1(1 - \delta)/\delta$ , for all  $\tau \geq 2$ .*

Consider the limiting case as innovation becomes less and less frequent,  $\Delta \rightarrow \infty$ . This provides a useful reference point for market power. As  $\Delta$  rises, the value of the future sequence of innovation upgrades declines (innovation occurs less frequently) and the current flow value of  $v$  comes to dominate the future surplus of  $\delta S_1$ . In the limit, as  $\Delta \rightarrow \infty$ , we find that  $v \rightarrow \lambda/r$  and we have a unique equilibrium outcome in which  $\pi_1 = p_1 = v$  and  $u_1 = 0$ . Formally, this is now equivalent to a static model and the seller is able to extract all surplus from the buyers. Thus, the flow dominance lower bound on the seller’s payoff can be viewed in terms of limiting the seller’s market power to that of a static monopolist who derives no added value from the upgrade market.

## 5 Delay and Generational Equilibria

We now examine generational equilibria, the case of a  $t$  – cycle where  $t \geq 2$ . In this case, multiple innovations are bundled together for sale, after a period of delay, in

a “new generation” of the product. By Proposition 1, every equilibrium is a  $t$  – *cycle* equilibrium and, by implication, the only form of inefficiency that can arise in equilibrium is that of periodic delay with  $t \geq 2$ . In addition to the inefficient outcome, we will show generational equilibria also have the important property of necessarily compressing payoffs on both sides of the market. On the equilibrium path, we have no sales in periods 1 through  $t - 1$  and  $(1, 0)$  through  $(t - 1, 0)$  are “delay states.” In period  $t$ , when the quality gap has reached  $t$ , there is a sale of  $t$  units at a price  $p_t$ . Thus, the continuation state after the sale is  $(t + 1, t)$  which, by Markovian behavior, reduces to state  $(1, 0)$ . This returns the quality gap to 1 and the cycle begins again.

Payoffs in a  $t$  – *cycle* equilibrium are then  $\pi_t = p_t/(1 - \delta^t)$  for the seller, as the revenue flow of  $p_t$  is received once every  $t$  periods, and  $u_t = \frac{1}{1 - \delta^t} [\frac{vt}{1 - \delta} - p_t]$  for the buyers, as a purchase of  $t$  units at a price  $p_t$  is made once every  $t$  periods. Due to delay, the realized joint surplus in a  $t$  – *cycle* equilibrium is less than the maximal surplus  $S_1$ . Delay also implies that the continuation path from any state in a  $t$  – *cycle* equilibrium always has a smaller realized surplus than that of the efficient path. These are short run efficiency losses since each cycle resolution ends with buyers holding all feasible units as of the sale date. The realized joint surplus in equilibrium at the time of a sale provides a measure of the resulting loss in efficiency. From the above seller and buyer payoffs, joint surplus is given by

$$\Psi_t \equiv \pi_t + u_t = \frac{vt}{(1 - \delta)(1 - \delta^t)}.$$

Thus, we will be able assess the relative efficiency loss by comparing the joint surplus  $\Psi_t$  to the maximal surplus  $S_1$ . Note that  $\Psi_\tau = \delta^{t-\tau} \Psi_t$  for  $\tau = 1, \dots, t - 1$  in the delay states. Thus,  $\Psi_1 < \Psi_2 < \dots < \Psi_t$  as we approach the sale date of  $t$ , and similarly for seller profits and buyer utilities.

We first show that innovation needs to be sufficiently frequent for delay to occur;  $\delta > 1/2$  is necessary for delay. We then consider seller and buyer incentives in the delay states and derive approach conditions that must be satisfied in an equilibrium. Next, as with efficient equilibria, we construct a “cash-in” continuation support for (off-equilibrium) states in which the quality gap exceeds  $t$ . We then derive conditions for the sale at date  $t$  and, finally, show existence.

## 5.1 Delay Equilibria and Upgrade Frequency

An equilibrium with delay has no sales until period  $t > 1$ . What prevents the seller from profitably deviating to make a ‘speed-up’ offer? Suppose the seller were to offer a bundle of  $t - 1$  units for a price of  $\hat{p}$  in state  $(t - 1, 0)$ . If all other buyers reject such an offer, then a deviating buyer would accept if

$$v(t - 1) - \hat{p} + \delta \max \left\{ \frac{v(t - 1)}{1 - \delta}, u_t \right\} > \delta u_t.$$

Intuitively, if  $\delta u_t$  is relatively small when  $\delta$  is small then this will hold and the offer of  $\hat{p}$  would necessarily be accepted by all buyers in any symmetric equilibrium. Would the offer then be profitable for the seller relative to delaying and selling in the next period? When  $\delta \leq 1/2$ , this is necessarily the case.

**Proposition 4** *Suppose  $\delta \leq 1/2$ . Then there does not exist an equilibrium with delay.*

When  $\delta \leq 1/2$  the seller and the buyers both value current flows more heavily than future ones. Intuitively, upgrade innovations are sufficiently infrequent that a mutually beneficial speed-up deviation to avoid delay is possible. An individual buyer with  $t - 1$  units on hand would not purchase the  $t$  bundle in state  $(t, 0)$  and this makes the seller’s speed-up offer for  $t - 1$  units in state  $(t - 1, 0)$  attractive to an individual buyer. When all buyers accept, the current revenue dominates the payoff from waiting to sell in the next period.

The longer the delay, the easier it is to find a speed-up deviation for a given  $\delta$ . That is, as the delay  $t$  rises, a speed-up deviation destroys a delay equilibrium for a range of  $\delta$  that exceeds  $1/2$ . This suggests that for a given delay, we will need a sufficiently high discount factor to support the equilibrium: as feasible upgrades become more frequent, it is possible that they are bundled on the equilibrium path.

## 5.2 Delay and Approach Conditions

Equilibrium delay requires that the seller can find no offer that is acceptable to buyers and also profitable relative to waiting to sell in state  $(t, 0)$ . Thus, we must derive approach conditions for the equilibrium. In analogy with the support conditions for efficient equilibria, buyer strategies follow cut-off rules of when to accept offers and in conjunction the seller does not find it profitable to make a sale before the quality gap reaches  $t$  units.

The problem of satisfying these buyer and seller delay incentives reduces to finding a buyer payoff  $u_t$  that satisfies

$$(1 - \delta^\tau)(\Psi_t - u_t) \geq \frac{v\tau(\delta^{\tau-t} - 1)}{(1 - \delta)} + \max \left[ \frac{v\tau}{(1 - \delta)}, u_t \right] - u_t \quad (7)$$

for  $\tau = 1, \dots, t-1$ . As developed in Appendix D, when the buyer payoff satisfies (7), there exist cut-off prices such that delay is supported. The analysis of (7) is involved because the buyer and seller delay incentives can change significantly as the quality gap rises. To begin, note that (7) implies a lower as well as an upper bound on  $u_t$ . As  $u_t$  approaches 0, (7) necessarily fails (at all  $\tau$ ). In this case, the incentive for an individual buyer to jump ahead is very strong when  $u_t$  is small and the seller can profitably attract buyers as early as  $\tau = 1$ . If the buyers get too much of the available surplus ( $u_t$  approaches  $\Psi_t$ ), then (7) necessarily fails (at all  $\tau$ ). Intuitively, recalling that  $\pi_t = \Psi_t - u_t$ , this is because the seller can exploit flow dominance to sell early and profitably attract buyers by offering  $\tau$  units at a price of  $v\tau$ .

An extreme payoff on either the buyer or the seller side thus allows the seller to profitably induce a speed up. As a consequence, both buyer and seller payoffs are compressed relative to the range of payoffs for efficient equilibria.

**Lemma 5** *The set of payoffs for all generational equilibria is a strict subset of the set of payoffs for efficient equilibria.*

In order to be willing to wait until period  $t$ , a buyer must decline seller deviation offers. This requires a strictly positive payoff for buyers. By contrast, there is an efficient equilibrium in which buyers receive no surplus. Similarly, the seller must be willing to delay and this includes foregoing the option to sell units prematurely at flow value. Thus, the seller necessarily earns more than the flow dominance lower bound. The upper bounds then follow from the lower bound on the other side of the market. Thus, while delay generates less surplus, both sides of the market necessarily receive a larger payoff relative to the minimum subgame perfect equilibrium payoff.

Satisfying the approach conditions for a  $t$ -cycle (delay) equilibrium reduces to finding a buyer utility for the sale date,  $u_t$ , that satisfies (7) for a given  $t$  and  $\delta$ . Note that buyer incentives can change significantly during the approach when  $u_t$  lies between  $\frac{v}{1-\delta}$  and  $\frac{v(t-1)}{1-\delta}$ . Initially, when  $\tau$  is small, a deviating buyer who acquired  $\tau$  units would be willing to purchase again at the sale date  $t$ . In this case, a sufficiently large quality increase will

accumulate by the sale date that an (early) deviating buyer would choose to keep up with the market. However, when  $\tau$  is closer to  $t$ , a deviating buyer would not purchase again at  $t$  since the incremental surplus from  $t - \tau$  units is insufficient and this buyer would choose to fall behind the market.

This change in buyer deviation incentives highlights the fact that the approach condition can have a binding constraint at an interior  $\tau$ . This is important for maintaining the seller's incentives during the approach. Intuitively, when  $u_t$  is too large relative to  $\delta^t$  the seller is unwilling to wait until  $t$  for a sale and (7) will fail at  $\tau = 1$  as a profitable deviation offer will exist: a deviating buyer would purchase again and a cut-off price has a significant component in the form of the interim surplus. On the other hand, when  $u_t$  is too small relative to  $\delta^t$  the seller will find a profitable deviation at  $\tau$  closer to  $t$ . While a complete characterization of (7) is quite involved, it turns out that many of the complications only arise at relatively low discount factors.<sup>14</sup> In Appendix D, we develop a sufficient condition on  $\delta^t$  such that if  $\delta^t$  is above a threshold,  $d^*$ , then the approach conditions (7) are satisfied for an interval,  $(\underline{u}^A, \bar{u}^A)$ , of utility levels.

We then have:

**Lemma 6** *If  $\delta^t > d^*$ , then there exist bounds  $\underline{u}^A$  and  $\bar{u}^A$  such that the approach conditions (7) are satisfied for any  $u_t \in (\underline{u}^A, \bar{u}^A)$  in a  $t - cycle$  equilibrium.*

Numerically, the threshold,  $d^*$ , is about .439. For example, if  $t = 2$ , then  $\delta$  must be at least  $\sqrt{.439} = 0.663$ . One can interpret the  $t - cycle$  as having two stages, the approach and the sale date, where the discount factor between stages is  $\delta^t$ . Hence, the longer delay in equilibrium, the higher must be  $\delta$  so that the seller will not find a profitable deviation.

Lemma 6 provides a lower and upper bound on the payoff to buyers. The bounds on utility,  $\underline{u}^A$  and  $\bar{u}^A$ , depend on  $\delta$  and  $t$  and they are derived in Appendix D. At  $d^*$ ,  $\underline{u}^A = \bar{u}^A$ , and for all  $t$  and  $\delta$  pairs where  $\delta^t > d^*$ , we have  $\underline{u}^A < \bar{u}^A$ . This pattern is shown for  $t = 2$  in Figure 2.

### 5.3 Existence and Payoffs for Delay Equilibria

We must also consider the cash-in (off equilibrium) support conditions for inefficient  $t - cycle$  equilibria in addition to the approach conditions. The buyer cut-off rules and seller profits must satisfy the analog of (3). The main difference relative to the efficient case

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<sup>14</sup>This is because, in general, when condition (7) holds at  $\tau = 1$  and at  $\tau = t - 1$  it does not necessarily follow that (7) holds at  $1 < \tau < t - 1$ . This can be seen by explicitly solving the cases of  $t = 2$  and  $t = 3$ .

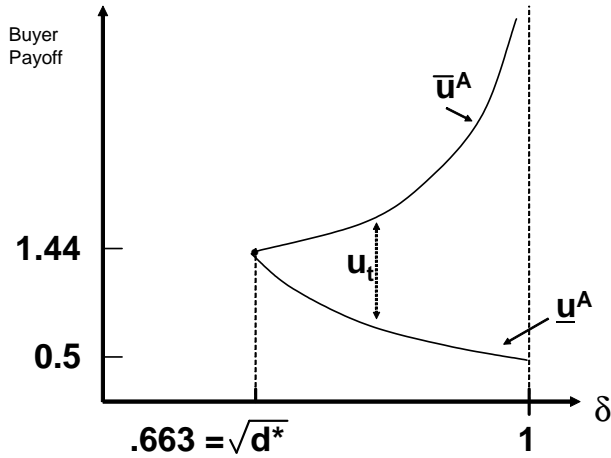


Figure 2: Approach Conditions ( $v = 1$  and  $t = 2$ )

is that we must replace the efficient surplus  $S_\tau$  with the realized surplus,  $\Psi_\tau$ . Furthermore, the set of support utilities for off the equilibrium path states where the quality gap exceeds  $t$  need to be modified for delay equilibria. We provide the details in Appendix D and now turn directly to the existence of delay equilibria. Sufficient conditions for approach states,  $(\tau, 0)$  where  $\tau < t$ , are provided in Lemma 6 and summarized by the bounds,  $\underline{u}^A < u_t < \bar{u}^A$ , on the equilibrium payoff. We can combine the approach conditions with the cash-in support conditions to establish the existence of delay equilibria.

**Proposition 5** *Let  $t \geq 2$  and suppose that  $\delta^t > d^*$ . Then every  $u_t \in (\underline{u}^A, \bar{u}^A)$  can be supported in a  $t$ -cycle inefficient equilibrium.*

Thus, the binding constraints on what buyer utilities can be supported are generated solely by the approach incentives. Intuitively, as  $\delta$  rises we can employ longer support lengths to support higher buyer payoffs.

The relationship between delay equilibria and the division of surplus is best understood in terms of the rate of innovation,  $\Delta$ . The length of time between sales in a  $t$ -cycle equilibrium is  $D \equiv \Delta t$ . As we let  $\Delta$  decline, so that innovation is more rapid, any given  $t$ -cycle will continue to exist, since  $\delta^t = e^{-r\Delta t} > d^*$  still holds, but the delay length  $D$  will go to zero. However, longer length cycles can be supported as  $\Delta$  declines. As with efficient equilibria, we assess seller market power as the ratio of seller profits relative to realized

equilibrium surplus; we also assess the efficiency loss from delay, namely  $\delta^{t-1}\Psi_t/S_1$ .

The utility bounds relative to realized equilibrium surplus,  $\bar{s}_b \equiv \bar{u}^A/\Psi_t$  and  $\underline{s}_b \equiv \underline{u}^A/\Psi_t$ , index the range of equilibrium payoffs. Simplifying, we find an invariance property in that the bounds depend only on the length of delay,  $D$ , and not on  $t$  and  $\Delta$  individually. The limiting cases are when delay vanishes,  $D \rightarrow 0$  (for example when  $\Delta \rightarrow 0$  for a given  $t$  – cycle) and when delay is maximized,  $D \rightarrow D_{\max} \equiv (-\ln d^*)/r$ . Straightforward limit calculations yield the following result.

**Corollary 2** *As equilibrium delay vanishes, the bounds on buyer utility converge to those for efficient equilibria,  $\bar{s}_b \rightarrow 1$  and  $\underline{s}_b \rightarrow 0$  as  $D \rightarrow 0$ . As equilibrium delay approaches  $D_{\max}$ , the bounds on buyer utility converge to each other at approximately .13 of the realized surplus. Further, the maximum efficiency loss from delay is approximately .35 and occurs as  $D \rightarrow D_{\max}$  and  $\lambda \rightarrow 0$ .*

In the limit where the buyer utility is 13% of the realized surplus and thus the seller receives 87% of this surplus, the 35% efficiency loss at the maximum delay implies that the seller would prefer 57% of the efficient surplus to 87% of the smaller delay surplus.

## 6 Discussion of Upgrade Structure

In this section, we consider how our endogenous bundling framework corresponds to observed practice in a variety of upgrade markets. We also discuss our results in relation to assumptions on bundling and the upgrade literature including price contingencies, bundling restrictions, network externalities, compatibility, and product complementarities.

### 6.1 Bundling in Practice

In practice, the upgrade process varies greatly with respect to how buyers move to higher quality levels. In our model, the seller is free to offer an unrestricted set of bundles. In equilibrium, offers take the form of upgrades, defined as a bundle of the form  $\{\sigma + 1, \dots, \sigma + k\}$ . Thus, a buyer with units  $\{1, \dots, \sigma\}$  purchases the upgrade offer and the new units are used in conjunction with the current ones. Effectively, the upgrade is an add-on. Alternatively, the seller in our model can offer bundles that constitute new versions, bundles of the form  $\{1, \dots, \sigma, \dots, \sigma + k\}$ . As we will show, there is a very close

relationship between upgrade offers and versions once downward complementarity and pricing contingent on a buyer's current holdings are taken into account.

A number of familiar examples help to illustrate the nature of upgrade (add-on) and version offers. For example, products such as media players from Windows and Real.com, or browsers such as Internet Explorer and Mozilla Firefox, are add-on programs in the sense that they can be installed on an existing computer. On the other hand, quality improvements for a given software program often involve replacing an older version with a newer version. However, as the anti-trust debate illustrates, the distinction between upgrades and versions is not necessarily driven by technological considerations. In this regard, recall that one of the United States's anti-trust cases against Microsoft centered on bundling of the internet browser and the operating system, while in Europe concern focused on the bundling of the media player with the operating system. Other examples demonstrate how the capabilities that were provided by a separate program, such as creating a PDF with Acrobat or presentation slides with Beamer have become part of word processing programs. Thus, the form of offered bundles, whether determined by firms or policy makers, can be viewed as an endogenous outcome.

Contract contingencies, especially with respect to a buyer's current holdings, are frequently observed in seller offers. MacKichan, for example, offers the technical word processor Scientific Word 5.5 in a number of versions differentiated by features and each version has an upgrade price for prior users (serial number required) as well as a (higher) price for new users. Airlines typically offer seat upgrades, club memberships, and other amenities at prices that vary with frequent flier status, a result of past purchases.

A new version with a price contingent on a buyer's current version is very close and often will be equivalent to an upgrade offer. Consider a buyer who holds units  $\{1, \dots, \sigma\}$  and two offers. One is an upgrade bundle  $\{\sigma + 1, \dots, \sigma + k\}$  for price  $p$ . The other is a new version  $\{1, \dots, \sigma, \dots, \sigma + k\}$  at price  $p$  that is only available to buyers who hold units  $\{1, \dots, \sigma\}$ . The direct value to the buyer is the same with either bundle. Now, consider the same offers, but suppose that the buyer does not hold any units. By downward complementarity, the upgrade bundle has no direct value (non-contiguous units) while the buyer does not qualify for the other offer. Further, observe that if the pricing contingency is stated as a minimum requirement then buyers who hold at least the minimum will place a common value on the two bundles. Such a minimum requirement is common in practice. Microsoft allows any 2000-2007 Office program or suite to qualify a buyer for Office Professional at the upgrade (discount) price.

Thus, our equilibria will be robust to allowing price contingencies if we can demonstrate that it is still optimal for the seller to make the same offers (either in the upgrade form or the appropriate version form with a holding contingency). This is easy to see in a finite horizon setting and the same unique outcome (efficient path and all surplus accrues to the seller) will continue to prevail. With our downward complementarity structure, the ability to offer partial upgrade bundles (lower quality units are not included) already allows the seller to induce buyers with different holdings to accept different bundles.<sup>15</sup> Thus, whether we regard the offers as upgrade bundles or versions with price contingencies, as in Fudenberg and Tirole (1998) or Ellison and Fudenberg (2000), does not matter.<sup>16</sup>

In our primary case with an infinite horizon, we need to consider the possibility that, by conditioning offers on current holdings, a seller may be able to curtail the credible threat of buyers to reject offers with high prices and thus eliminate equilibria with low seller payoffs. In this regard, our equilibria are robust. First, recall that buyers all have the same holdings on any equilibrium path and a contractual contingency in this regard has no force. With respect to the support for the equilibrium path, the same observation applies to the cash-in support for states of the form  $(\tau, 0)$ . Finally, for states in which buyers have asymmetric holdings, we constructed continuation equilibria with offers in which the seller captured the available surplus and, as a result, they are robust to the addition.

At a more intuitive level, recall that the credible threat to reject seller offers with high prices is based on the expectation of a sufficiently high future surplus (when all buyers reject the offer). Consider, for example, the support condition (3) for an efficient equilibrium and the impact of allowing contract contingencies on buyer holdings. An individual buyer who fails to purchase when others do will fall behind the equilibrium path, but such a buyer is already extracted in our analysis. On the other hand, a buyer who purchases when others do not will jump ahead of the market and, in our analysis, such a buyer does have a strict preference for purchasing the subsequent equilibrium support cash-in offer from the seller. The seller could then employ a contractual contingency to isolate such a buyer and eliminate the (valuable) option to accept an offer designed for

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<sup>15</sup>The proof is very similar to the finite case. The only potential difference is that the seller can use a contingency to prevent a buyer who is ahead of the market from purchasing bundles that are intended for buyers with lower holdings. However, since no buyer earns an incremental surplus in any continuation, the contingency has no impact.

<sup>16</sup>In both Fudenberg and Tirole and Ellison and Fudenberg the second period offers can distinguish between buyers who purchased in period 1 and those who did not. These models allow for buyer heterogeneity and, thus, the contract contingency is much more salient for influencing buyers with different valuations than in our model where buyers are homogenous.

buyers with fewer units and “rejoin” the equilibrium path. But if the contingency is used to make an offer that extracts the deviating buyer then this will only serve to reduce further the deviation payoff to purchasing when others do not (refer to the max condition for  $g(\tau, \tau)$  in (3)). Thus, the support condition (3) continues to be satisfied by our  $T$ -stage utility path and buyers retain a credible threat based on future surplus to reject offers with high prices.

In short, the ability to identify customers based on past purchases does not affect the equilibrium structure. In terms of information structures, it is worth noting that our original offer set corresponds to the semi-anonymous regime of Fudenberg and Tirole (1998), where harsher terms cannot be imposed on buyers who hold more units than others.

This establishes that our results do not depend on the precise form of the seller’s offers. Bundles can be presented as upgrades or as versions with a contingency on past purchases. In this regard, our formulation of the offer space is simpler as it avoids the added complications of contingencies. We note, however, that limiting the offer space to generation version bundles while adding price contingencies will also result in the same set of equilibria.

## 6.2 Network and Compatibility effects

Network effects and compatibility issues figure prominently in discussions of software and technology markets and are seen by many observers as critical factors in market success.<sup>17</sup> We now introduce network externalities and compatibility issues and argue that our basic conclusions on equilibrium outcomes are robust. Suppose that a buyer’s utility flow in a period in which he has  $\sigma$  units is  $v\sigma$  if the rest of the market has  $\sigma$  or fewer units but that the buyer’s flow utility falls to  $\lambda v\sigma$  if the rest of the market has strictly more than  $\sigma$  units, where  $\lambda < 1$ . These preferences reflect a compatibility and a network effect. First, as is standard practice in the software industry and also a standard assumption in the literature (see Ellison and Fudenberg (2000)), the good is backward compatible. The good is not, however, fully forward compatible and the buyer loses some utility by falling behind the market. For example, a buyer enjoys the full network benefits if he can use the files of others, but loses some network benefits by falling behind

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<sup>17</sup>See, for example, the thorough discussions in Shapiro and Varian (1999) and Liebowitz and Margolis (2001), and the related policy discussion on network effects and compatibility in Anton and Biglaiser (2010).

the market. In a recent episode, Word 03 (unable to read ".docx" files of Word 07) lacked full forward compatibility while Word 07 was backward compatible with Word 03 (a patch from Microsoft's website subsequently enabled Word 03 users to read .docx files).

As we have emphasized, market power hinges on the ability of the seller to tempt a buyer to move ahead of the market and not on a buyer's fear of falling behind the market. We can extend this argument to account for the presence of network effects and some lack of forward compatibility by considering how these effects alter buyer incentives. First, as is intuitive, the network and incompatibility effects only reinforce the incentive to keep up with the market; falling behind not only precludes any incremental surplus but also leaves the buyer with a reduced fraction ( $\lambda$ ) of the value for all currently held units. More importantly, with respect to jumping ahead of the market, consider how (3) changes. Initially, backward compatibility implies no immediate impact, as the value  $v\sigma$  in  $g(\sigma, u_{\tau+1})$  is not reduced by  $\lambda$ . In contrast, the option for subsequent purchases, as reflected by the max condition in  $g(\sigma, u_{\tau+1})$ , will be impacted. Recall that the option value,  $\max[v\sigma/(1-\delta), u_{\tau+1}]$ , represents the continuation utility of a buyer who is ahead of the market, holding  $\sigma$  units, entering period  $\tau+1$  when the rest of the market will move up to the state of the art via the cash-in support. The first term is the buyer's continuation utility if he does not make a purchase in  $\tau+1$ , thus falling behind the market and receiving no further utility increments, while the second term is the utility from buying along with others and returning to the equilibrium path. With network effects and the lack of perfect forward compatibility, the first term is reduced to  $\lambda v\sigma/(1-\delta)$  and this necessarily reduces the overall value of  $g(\sigma, u_{\tau+1})$ . Thus, support condition (3) is actually relaxed in that a wider range of cut-off prices can support any given utilities. Put differently, the addition of network effect and some lack of forward compatibility makes it *more* difficult to tempt a buyer to move ahead of the market (equilibrium coordination is facilitated by these effects). In fact, any sequence of buyer support utilities that originally satisfied (3) will necessarily satisfy the relaxed version, thus preserving the credible threat to reject high prices.

Adoption costs can also be incorporated and they lead to a similar conclusion. Suppose a buyer incurs an initial cost of  $c_\sigma$  for an upgrade of  $\sigma$  units. Examples include a fixed cost of  $c_\sigma = C$  for all  $\sigma > 0$  and a proportional cost of  $c_\sigma = c\sigma$ . Consider once again the impact on (3). In this case, the second term in  $g(\sigma, u_{\tau+1})$  involving the option value is unaffected but the initial term is reduced to  $v\sigma - c_\sigma$ . Since the overall value of  $g(\sigma, u_{\tau+1})$  declines, this aspect of the credible threat remains intact. Intuitively, by reducing the

immediate net benefit, the adoption cost reduces the incentive of an individual buyer to jump ahead of the market.<sup>18</sup>

### 6.3 Unbreakable version offers

Suppose that, as an exogenous condition, all bundles must be versions and no contingencies on purchase are allowed. That is, any offer for the current quality increment, unit  $\tau$ , is necessarily also an offer for units  $\{1, \dots, \tau\}$  and similarly for any lower quality level. This might reflect a necessary property of the production technology, where a quality increment cannot be ‘broken out’ for separate sale. This offer structure is examined in Waldman (1996) for a two-period model and in Fishman and Rob (2000) for an infinite horizon model, both of whom focus on innovation incentives.

In our setting, where buyers have identical preferences, this “unbreakable” upgrade structure necessarily limits the market power of the seller. The reason is that a buyer always has the option of passing on a current offer and waiting to purchase a later offer. As long as the seller eventually offers a higher quality, the cost of waiting (relative to purchasing when other buyers do) is the lost flow value. Any subsequent higher-quality offer that attracts prior buyers will necessarily provide the buyer who delays with a strictly larger surplus than that received by prior buyers. As a result, there is no equilibrium in which a monopoly seller is able to capture the full surplus from buyers in a market where upgrades are unbreakable.

In the simplest case of our model with two-periods and unbreakable goods, it is straightforward to derive the equilibrium outcomes. For large discount factors, the seller delays until period 2 and then offers units  $\{1, 2\}$  at the extraction price for the remaining surplus. For low discount factors, however, unit 1 is sold in period 1 at price  $v$  and then period 2 has extraction pricing for units  $\{1, 2\}$ . In both cases, the seller is unable to capture the full surplus, in contrast to the finite horizon equilibrium with upgrade offers or generation offers with price contingencies.

In the case of an infinite horizon with non-contingent pricing, ongoing quality growth necessarily imposes a more severe limit on the seller. Fishman and Rob (2000) point out

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<sup>18</sup>For the network and compatibility effect, the multiplicative specification is convenient because the expression for the efficient surplus does not change. Adoption costs, however, do reduce the efficient surplus. We can still construct a  $T$  – stage support by making the seller indifferent to delay, but the buyer payoff range will reflect the adoption costs. For example, a 1 – stage support covers buyer payoffs up to  $[v - c_1 + \delta(c_1 - c_2)] / (1 - \delta)$ .

that the option to wait implies that the seller can charge no more than the flow value.<sup>19</sup> In their rational expectations equilibrium, they find that the low price leads to a rate of innovation that is inefficiently low. In our analysis, where bundling is endogenous, the option to wait (fall behind the market) is inconsequential. We find that the basis of a credible threat for buyers resides with the extent to which the seller can tempt a buyer to jump ahead of other buyers.

## 6.4 Independent goods

The case of ‘independent’ goods involves a buyer who receives a flow utility of  $v$  from a good independently of whether the buyer holds any other units. With a complete absence of complementarity across quality levels, units 1, 2, ... are effectively independent goods. Then, as one might expect, the speed-up logic of Fudenberg, Levine, and Tirole (1985) implies that the seller regains the ability to extract buyers due to the lack a credible threat for refusing a seller offer.<sup>20</sup> The essential difference is that a buyer can accept a current offer when others do not, skip the subsequent cash-in offer from the seller, and then resume purchasing. Because the goods are independent, there is no payoff consequence due to complementarity from any missing units. Thus, with independent goods, we necessarily have  $u_1 = 0$ . Intuitively, complementarity is essential for building a credible threat to refuse price increases as a deviating individual buyer faces the extra cost of having to acquire the missing unit to avoid falling behind the market. This is why, in our upgrade structure, it matters to an individual buyer whether or not other buyers are expected to purchase the seller’s offer.

## 7 Directions for Future Work

We have assumed homogeneous buyers in our model. This was done primarily to focus on the structure of credible threats for buyers in a dynamic upgrade model in

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<sup>19</sup>Full extraction of total surplus by the seller requires implementing the efficient path and, in the unbreakable version of our model, the seller would be limited to prices that reflect only the flow value and not the present discounted value to buyers from quality increments. Formally, consider the efficient path and let  $p(\tau, \tau - 1)$  be an equilibrium price for version  $\tau$  when all buyers hold  $\{1, \dots, \tau - 1\}$ . By rejecting an offer and resuming purchases next period, an individual buyer obtains  $v(\tau - 1) + \delta u(\tau + 1, \tau)$ . Purchasing today yields  $v\tau - p(\tau, \tau - 1) + \delta u(\tau + 1, \tau)$ . Combining, we must have  $p(\tau, \tau - 1) \leq v$ .

<sup>20</sup>Consider, for example, payoffs in a Markov equilibrium for the efficient path and suppose the equilibrium has a cash-in support at price  $p_\tau$  in state  $(\tau, 0)$ . Much like our analysis of the single-buyer benchmark, in order to reject a price increase we must have  $(\delta + (1 - \delta)/\delta^\tau) u_1 \leq u_{\tau+1}$  and this implies  $u_1 = 0$ .

what one would expect to be the ideal situation for a seller to capture the full (efficient) joint surplus. Allowing for buyer heterogeneity is an important direction for subsequent work. In practice, it is common for sellers in upgrade markets to offer simultaneously different versions or quality levels of their products. This is typically taken to be a form of price discrimination. As noted before, several papers examine a finite horizon model, but there has been very little theoretical work on infinite horizon models in which buyers are always in the market and quality improvements are ongoing. We are currently exploring this problem in our model by allowing for buyer heterogeneity with high or low valuation buyers who are privately informed of their type. This allows for an endogenous determination of pricing and whether the buyer segments remain distinct over time or whether the seller chooses to price over a cycle that periodically brings high and low types together at a common quality level (a generational cycle). An interesting feature of equilibrium price discrimination in this dynamic context is that incentive constraints can bind in both directions (with low-value buyer types choosing to mimic high-value buyer purchases as well the standard downward incentive constraint).

We also assumed an exogenous rate for the increase in quality. Of course, a model that addresses the question of how rewards for a given quality innovation are determined is a necessary step toward an endogenous determination of quality change. We are currently studying innovation and pricing incentives in a model where innovations can be generated not only by an incumbent but also by potential entrants. In this setting, property rights for innovation in relation to imitation incentives are crucial for buyer decisions regarding adopting the products of an incumbent or an entrant and, in turn, for assessing public policy choices and welfare in upgrade markets.

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## 8 Appendix

### 8.1 Appendix A - Formal Structure

In this appendix, we present the formal structure of the model. First, we define bundles and offers. Next, we define strategies and histories. Finally, we present payoffs and formally define equilibrium.

(i) *The Bundle Offer Structure.*

Consider the feasible offer set for the seller in period  $\tau$ . Let  $\mathcal{P}_\tau \equiv \mathcal{P}(\{1, 2, \dots, \tau\})$  denote the power set for the first  $\tau$  integers. Any set  $z \in \mathcal{P}_\tau$  is called a *bundle*. An *offer* is a collection of bundles and associated (non-negative) prices,  $(z, p_z)_{z \in Z}$  for some  $Z \in \mathcal{P}(\mathcal{P}_\tau)$ . Define the *offer set*  $\Omega_\tau$  by

$$\Omega_\tau \equiv \{\omega \in \mathcal{P}(\mathcal{P}_\tau \times R_+) \mid (i) (\emptyset, 0) \in \omega, (ii) \text{ if } (z, p) \in \omega \text{ and } (z, p') \in \omega, \text{ then } p = p'\}.$$

By (i), we are including the null bundle in every offer by the seller. This is for two reasons: first, the seller can make no offer by choosing only the null bundle and, second, it streamlines the buyer choice formalism, as a buyer chooses to make no purchase by selecting the null bundle. By (ii), every offered bundle has a unique price. Clearly, if two prices were offered for the same bundle, no buyer would want to choose the higher price. This implies that buyers act as price takers and that the overall market does not respond

to the actions of an individual buyer. An acceptance choice by a buyer is an element of the set  $\mathcal{P}(\mathcal{P}_\tau)$ .

Consider the maximal contiguous quality. For any  $z \in \mathcal{P}_\tau$ , define  $M : \mathcal{P}_\tau \rightarrow \{0, 1, \dots, \tau\}$  by finding the unique  $m \in \{0, \dots, \tau\}$  such that  $m' \in z \forall m' \leq m$  and  $m + 1 \notin z$ , and set  $M(z) = m$ . Clearly,  $M(z)$  is the maximal contiguous quality held by a buyer and  $M(z)$  exists for any bundle  $z$ . For an arbitrary sequence of holdings  $z_\tau$  for  $\tau \geq 1$ , define  $q_\tau = M(z_\tau)$ .

(ii) *Strategies and Histories.*

A (pure) strategy for the seller is a *sequence* of offers,  $\mathcal{O} = (\mathcal{O}_\tau)$ . Each offer is a map from the *history* of play up through period  $\tau - 1$  into the offer set  $\Omega_\tau$ . A history is the sequence of previous offers by the seller and acceptances by the buyers. Letting  $\mathcal{H}_\tau$  denote the space of all histories up through period  $\tau - 1$ , we have

$$\mathcal{O}_\tau : \mathcal{H}_\tau \rightarrow \Omega_\tau.$$

That is, if  $h_\tau \in \mathcal{H}_\tau$  is the observed history of play, then the seller's strategy specifies the offer  $\omega_\tau = \mathcal{O}_\tau(h_\tau)$ .

A buyer (pure) strategy profile is a *sequence* of acceptance decisions,  $\mathcal{A} = (\mathcal{A}_\tau)$ . Given a history  $h_\tau$  and a seller offer  $\omega_\tau$ , each buyer  $x \in [0, 1]$  needs to choose which bundles in  $\omega_\tau$  to accept. Thus, we have acceptance strategies for each buyer

$$\mathcal{A}_\tau^x : \mathcal{H}_\tau \times \Omega_\tau \rightarrow \mathcal{P}(\mathcal{P}_\tau).$$

Hence, for observed history  $h_\tau \in \mathcal{H}_\tau$  and in response to a seller offer of  $\omega_\tau \in \Omega_\tau$ , buyer  $x$  chooses to accept the set of bundles  $\mathcal{A}_\tau^x(h_\tau, \omega_\tau) \subseteq \mathcal{P}(\mathcal{P}_\tau)$ . Of course, any accepted bundle,  $z \in \mathcal{A}_\tau^x(h_\tau, \omega_\tau)$ , must have been offered by the seller,  $(z, p) \in \mathcal{O}_\tau(h_\tau)$  for some  $p$ . This is a feasibility restriction. Note that a buyer is free to accept one or more of the bundles (i.e., any subset) included in an offer  $\omega_\tau$ . For example, by "accepting" only the null bundle, a buyer makes no purchase in period  $\tau$ . Finally, we use  $\mathcal{A}_\tau$  for the strategy profile across buyers.

We need to specify the history space  $\mathcal{H}_\tau$ . First, define  $\Omega^\tau \equiv \Omega_1 \times \Omega_2 \times \dots \times \Omega_\tau$ ; this product space contains each feasible sequence of previous offers. Second, we need to calculate acceptance sets from buyer bundle purchases and this entails a measurability assumption on buyer strategies.

Let  $\mathcal{F}_\tau$  denote the set of Borel measurable functions for  $[0, 1] \rightarrow \mathcal{P}(\mathcal{P}_\tau)$ . By definition,

$f_\tau : [0, 1] \rightarrow \mathcal{P}(\mathcal{P}_\tau)$  is Borel measurable (that is,  $f_\tau \in \mathcal{F}_\tau$ ) if for any  $z \in \mathcal{P}_\tau$  we have  $\mathcal{X}_\tau(z) \in \mathcal{B}$  (the Borel sets of  $[0, 1]$ ), where  $\mathcal{X}_\tau(z) = \{x \in [0, 1] \mid z \in f_\tau(x)\}$ . Thus, the set of buyers who chose bundle  $z$  is a Borel set and we can calculate market share and revenues by using standard Lebesgue measure. Define the product space  $\mathcal{F}^\tau \equiv \mathcal{F}_1 \times \mathcal{F}_2 \times \dots \times \mathcal{F}_\tau$ .

Then the history space is specified by  $H_1 = \emptyset$  and for  $\tau > 1$ ,

$$\mathcal{H}_\tau = \Omega^{\tau-1} \times \mathcal{F}^{\tau-1}.$$

Note that the bundles and prices offered by the seller are recorded in  $\Omega^{\tau-1}$  while the bundles accepted by each buyer are recorded in  $\mathcal{F}^{\tau-1}$ . Thus, we know the price a buyer paid for a bundle from the history. We assume that for each  $h_\tau \in \mathcal{H}_\tau$ , and  $\omega_\tau \in \Omega_\tau$ , we have  $\mathcal{A}_\tau \in \mathcal{F}_\tau$ , i.e.  $\mathcal{A}_\tau^x(h_\tau, \omega_\tau)$  is a Borel measurable function on  $x \in [0, 1]$ . An equivalent, but less convenient, formulation would be to assign an index number to each element in the finite set  $\mathcal{P}(\mathcal{P}_\tau)$  and then define measurability in the standard way for a real valued function.

(iii) *Payoffs and Equilibrium.*

Turning to the calculation of player payoffs, we begin with the buyers. First, for each  $h_{\tau+1}$ , calculate the units acquired by buyer  $x$  in each period  $k = 1, \dots, \tau$ . These units are given by  $Z_k(x) = \{i \in \{1, \dots, k\} \mid i \in z \text{ for some } z \in \mathcal{A}_k^x(h_k, \omega_k)\}$ , the bundles accepted by buyer  $x$ . Thus, the set of units that buyer  $x$  has accumulated through the end of period  $\tau$  is given by

$$Z^\tau(x) \equiv \bigcup_{k=1}^{\tau} Z_k(x) \subseteq \mathcal{P}_\tau.$$

Recalling that  $M(z)$  is the maximal contiguous quality for any subset  $z$  of  $\{1, \dots, \tau\}$ , we see that the maximal contiguous quality unit held by buyer  $x$  is given by  $m_\tau(x) \equiv M(Z^\tau(x))$ .

Next, the total expenditure of buyer  $x$  in period  $\tau$  is given by  $p_\tau(x) \equiv \sum_{z \in \mathcal{A}_\tau^x(h_\tau, \omega_\tau)} p_z$ , which is the sum of the payments for each bundle that the buyer accepted. Thus, the payoff to buyer  $x$  from strategy  $\mathcal{A}^x$  when other buyers follow  $\mathcal{A}^{\sim x}$  and the seller follows  $\mathcal{O}$  is the present discounted value of surplus from the maximal unit held less expenditures in each period:

$$U(\mathcal{O}, \mathcal{A}^x, \mathcal{A}^{\sim x}) = \sum_{\tau=1}^{\infty} \delta^{\tau-1} [vm_\tau(x) - p_\tau(x)].$$

The infinite sum is always well defined, since (i) the sequence of maximal holdings  $m_\tau$  is

non-decreasing in  $\tau$ , (ii)  $m_\tau \leq \tau$ , and (iii)  $\sum_{\tau=1}^{\infty} \delta^{\tau-1} \tau = 1/(1-\delta)^2$ .

We now compute the seller payoff. Given a history and an offer by the seller,  $\mathcal{X}_\tau(z)$  as defined above is the set of buyers for whom  $z \in \mathcal{A}_\tau^x(h_\tau, \omega_\tau)$ . The Lebesgue measure of such buyers is  $\alpha_\tau(z) \equiv \int_{\mathcal{X}_\tau(z)} dx$ . Thus, the revenue of the seller in period  $\tau$  is

$$r_\tau = \sum_{z \in \mathcal{P}_\tau} \alpha_\tau(z) p_z.$$

Note that  $\alpha_\tau(z) = 0$  must hold if the seller did not offer the bundle  $z$  or if no buyer purchased  $z$ .

The seller payoff under strategies  $(\mathcal{O}, \mathcal{A})$  is then

$$\Pi(\mathcal{O}, \mathcal{A}) = \sum_{\tau=1}^{\infty} \delta^{\tau-1} r_\tau$$

The definitions for Nash and subgame perfect equilibrium are standard. The strategies  $(\mathcal{O}, \mathcal{A})$  form a Nash equilibrium if

$$\begin{aligned} \Pi(\mathcal{O}, \mathcal{A}) &\geq \Pi(\widehat{\mathcal{O}}, \mathcal{A}) \quad \text{for all } \widehat{\mathcal{O}}, \\ U(\mathcal{O}, \mathcal{A}^x, \mathcal{A}^{\sim x}) &\geq U(\mathcal{O}, \widehat{\mathcal{A}}^x, \mathcal{A}^{\sim x}) \quad \text{for all } \widehat{\mathcal{A}}^x. \end{aligned}$$

A subgame perfect equilibrium requires that  $(\mathcal{O}, \mathcal{A})$  form a Nash equilibrium at any given  $h_\tau$ , where the seller makes an offer, and at any given  $h_\tau$  and  $\omega_\tau$ , where the buyers respond to the offer.

In general, Markovian strategies could depend on non-contiguous buyer holdings. For each bundle  $z$ , the set of buyers who hold  $z$ ,  $Q_\tau(z)$ , is found directly from  $Z^\tau(x)$ , the units accumulated by buyer  $x$  through  $\tau$ . Hence, the allocation of buyers across quality units, as given by  $(Q_\tau(z))_{z \in \mathcal{P}_\tau}$ , identifies all payoff relevant information as of the beginning of period  $\tau + 1$ . This form of Markovian behavior would allow non-contiguous holdings to affect strategies. While a non-contiguous unit does not affect current buyer flow payoffs, it would change a future flow payoff if missing intermediate units were acquired. As noted in the text, however, this does not generate any additional payoffs relative to those arise when Markovian strategies depend only on contiguous buyer holdings.

When buyers are distributed across maximal holdings, the state is given by  $(\tau, (Q_\tau^m)_{m=0, \dots, \tau-1})$ , where  $Q_\tau^m$  is the set of buyers with maximal contiguous quality  $m$ . More generally, when buyers are distributed as  $(Q_\tau^m)_{m=0, 1, \dots, \tau}$ , then the translated state is given by  $(\tau +$

$1 - \underline{m}_\tau, (Q_\tau^m)_{m=\underline{m}_\tau, \dots, \tau}$  where  $\underline{m}_\tau$  is the smallest index of  $Q_\tau^m$  with a non-zero measure.

## 8.2 Appendix B - Preliminary Results

In this appendix we prove Lemmas 1, 2, and 3 as well as Proposition 1.

*Proof of Lemma 1.* Depending on the history, buyers may also hold a subset, possibly null, of units  $\{Q + 2, \dots, \tau - 1\}$ . Without unit  $Q + 1$ , a buyer who rejects the upgrade offer will receive a flow payoff of  $vQ$  and have the same quality holdings in period  $\tau + 1$ . A buyer who accepts will receive a flow payoff of  $v\tau$  in period  $\tau$  and hold  $\{1, \dots, \tau\}$  next period. We will show that accepting yields a strictly higher payoff than rejecting, for any strategy choices of other buyers and the seller following the upgrade offer.

Obviously, accepting yields a higher flow payoff in period  $\tau$  since  $v\tau - p > vQ$ . Consider the continuation payoff. An accepting buyer (i) begins with more units than a non-accepting buyer and (ii) has the option of mimicking the strategy of any non-accepting buyer (any seller offer may be purchased by any buyer). By the upgrade payoff structure, the continuation payoff of an accepting buyer is therefore at least as large as that of a non-accepting buyer. This is because the mimicking option implies the same payments, but the accepting buyer will never hold fewer units.

It is now clear that every buyer will choose to accept the upgrade offer. Given any strategy choices of other buyers and the seller following the upgrade offer, a buyer who accepts always has a weakly larger payoff from  $\tau + 1$  onward, with respect to the continuation sequence of offers implied by the strategies, and a strictly larger flow payoff in period  $\tau$ . Note that, by the zero measure property, the continuation sequence of offers does not depend on the choice of a specific individual buyer to accept or reject the upgrade offer. ■

*Proof of Lemma 2.* By Lemma 1, at the start of the game the seller can offer unit 1 for a price of  $p_1 < v$  and every buyer will accept. Also, by Lemma 1, in period 2 when all buyers hold unit 1 the seller can offer unit 2 for a price of  $p_2 < v$  and every buyer will accept. By induction, in any period  $\tau$  and for any history in which all buyers hold units  $\{1, \dots, \tau - 1\}$  we can apply Lemma 1 to see that the seller can sell unit  $\tau$  for a price  $p_\tau < v$ . Each price can be arbitrarily close to  $v$ , so letting  $v - \epsilon = p_\tau$  for all  $\tau$ , the seller's payoff from the start of the game must be at least  $(v - \epsilon)(1 + \delta + \delta^2 + \dots) = (v - \epsilon)/(1 - \delta)$ . As this must hold for any  $\epsilon > 0$ , we are done.

For the continuation result, simply apply Lemma 1 with  $p_\tau = (v - \epsilon)(\tau - Q)$  in period  $\tau$  and then use the same argument as above in period  $\tau + 1$ . ■

*Proof of Lemma 3.* It is sufficient to prove the result for states of the form  $(\tau, 0)$ , since any state of the form  $(\sigma, Q)$  has the same quality increments and payments as  $(\sigma - Q, 0)$ . Consider state  $(\tau, 0)$  and a continuation path  $(\sigma, q_{\sigma-1})$  for  $\sigma \geq \tau + 1$ . By Lemma 2, we know the seller's payoff in state  $(\tau, 0)$  is positive. This implies that  $q_{\sigma-1} > 0$  for some  $\sigma$ . Otherwise, we have  $q_{\sigma-1} = 0$  for all  $\sigma$  and buyers never acquire unit 1. Hence, buyer payments to the seller must be zero. As a result, the seller's payoff would be zero, which is not possible. Thus,  $q_{\sigma-1} > 0$  for some  $\sigma$ . Relabel so that  $\sigma$  denotes the first such period, and we have  $q_{\sigma-1} > 0$  and  $q_{\sigma'-1} = 0$  for  $\sigma' < \sigma$ . Thus, the quality gap rises from  $\tau$  in state  $(\tau, 0)$  to  $\sigma - 1$  in state  $(\sigma - 1, 0)$  and then goes to a gap of  $\sigma - q_{\sigma-1}$ , which is less than or equal to the previous gap of  $\sigma - 1$ , in state  $(\sigma, q_{\sigma-1})$ . By equilibrium, the continuation path from  $(\sigma, q_{\sigma-1})$  has the same quality increments as state  $(\sigma - q_{\sigma-1}, 0)$ . Thus, the quality gap will thereafter cycle repeatedly from size  $\sigma - q_{\sigma-1}$  up to  $\sigma - 1$  and the continuation path from  $(\tau, 0)$  has a bounded quality gap. ■

*Proof of Proposition 1.* Starting from state  $(1, 0)$ , we know from Lemma 3 that the quality gap is bounded and, therefore, that there is a first date, say  $t$ , at which a sale involving unit 1 takes place. If  $t = 1$ , we are done as equilibrium (MPE) implies we have a 1-cycle equilibrium. So, consider  $t > 1$ . By construction, the maximal quality held by buyers before period  $t$  is zero, so the state is  $(t, 0)$ , and  $q_t > 0$  results from sales in period  $t$ . A potential complication with state  $(t, 0)$  that does not occur with  $(1, 0)$  is that  $(t, 0)$  corresponds to histories in which buyers acquired no quality units as well as histories in which they acquired some subset of  $\{2, \dots, t - 1\}$ . By definition, however, Markovian behavior requires that the seller offer in  $(t, 0)$  and buyer acceptance choice(s) are the same across these histories since strategies only depend on the state  $(t, 0)$ .

Suppose that the sale at date  $t$  does not result in  $q_t = t$  or, in other words, buyers do not acquire the full feasible set of units  $\{1, 2, \dots, t\}$ . This implies that, for some  $\tau$  where  $1 \leq \tau < t$ , buyers hold units  $\{1, \dots, \tau\}$  but not unit  $\tau + 1$  as they enter period  $t + 1$ . Also, let  $p$  denote the total payment made by a buyer to the seller for all bundles purchased in state  $(t, 0)$ . Finally, note that whether or not any of the units in  $\{\tau + 2, \dots, t\}$  are held by buyers before period  $t$  or acquired in  $t$ , the state in period  $t + 1$  will be  $(t + 1, \tau)$ .

By construction, the equilibrium buyer continuation payoff from state  $(t, 0)$  is given by

$$u(t, 0) = v\tau - p + \delta u(t + 1, \tau)$$

as the quality flow utility is  $v\tau$  and the payment is  $p$  in  $(t, 0)$ , and next period's state is  $(t + 1, \tau)$ . We will show that the seller has a profitable deviation in period  $t - 1$ , namely,

offering the feasible (as  $\tau < t$ ) upgrade bundle to quality  $\tau$  (units  $1, \dots, \tau$ ) for some price  $\hat{p}$ .

Before proceeding with the main argument, we need to develop two properties of buyer payoffs. First, the equilibrium path will follow a cycle, since state  $(t + 1, \tau)$  has the same quality gap as state  $(t + 1 - \tau, 0)$ . Thus, the maximal buyer quality remains at  $\tau$  until period  $t + \tau$ , when the state reaches  $(t + \tau, \tau)$ , at which time the maximal buyer quality rises to  $2\tau$  and the cycle begins again. Equilibrium also implies that a buyer only needs to make purchases in the states where maximal quality rises in order to achieve the equilibrium buyer payoff. As noted above, the history of play only matters to the extent that it impacts maximal buyer quality. Thus, the bundle(s) offered by the seller in any state of the form  $(t + k\tau, k\tau)$ , where  $k = 1, 2, \dots$ , must, at a minimum, always include the next  $\tau$  units of quality. In particular, this is true for the history where buyers hold exactly the first  $k\tau$  quality units (and no other units), since the maximal quality for this history is  $k\tau$ . Thus, an individual buyer never needs to hold more than these units in order to be able to reach the next equilibrium path level of maximal quality via purchases in state  $(t + k\tau, k\tau)$ . Furthermore, such a buyer can always choose from the same offered bundle(s) and price as any other buyer. It follows directly that the continuation payoff of a buyer only depends on holding the current maximal quality and it is independent of whether the buyer holds higher but non-contiguous units. This is the first property of buyer payoffs that we will need.

The second property is that, in equilibrium, the seller only receives revenues in states of the form  $(t + k\tau, k\tau)$ , where  $k = 1, 2, \dots$ . In  $(t + k\tau, k\tau)$ , in equilibrium, the seller offer must include units  $\{k\tau + 1, \dots, k\tau + \tau\}$  and all buyers must acquire these units. Thus, no buyer ever pays a positive price for any bundle in states  $(1, 0)$  through  $(\tau, 0)$ , since only units in  $\{2, \dots, \tau\}$  can be offered by the seller in equilibrium and these units will necessarily be acquired via the upgrade bundle in state  $(t, 0)$  when buyers also acquire unit 1. The same logic then applies for the next  $\tau$  units, and so on.

We now proceed with the main deviation argument. To keep things simple, let us first consider the case where the history for state  $(t, 0)$  has buyers holding no quality units. For the seller deviation in period  $t - 1$ , choose the price  $\hat{p}$  for the bundle of units  $\{1, \dots, \tau\}$  so that

$$\hat{u} \equiv v\tau - \hat{p} + \delta v\tau + \delta^2 u(t + 1, \tau) = \delta u(t, 0) + \epsilon,$$

for a small  $\epsilon > 0$ . Combining this with the earlier expression for  $u(t, 0)$ , we find that  $\hat{p} = v\tau + \delta p - \epsilon$ . We claim that in any equilibrium continuation after this offer, all buyers will

accept. By symmetry of strategies, in response to this offer in state  $(t - 1, 0)$ , all buyers must either accept or reject. Suppose the buyer strategy calls for a rejection and consider the decision of an individual buyer. Because no other buyer accepts, the continuation state will be  $(t, 0)$ . By accepting, the payoff for an individual buyer is  $\delta u(t, 0) + \varepsilon$ . To see this, note first that the individual buyer receives a flow of  $v\tau - \hat{p} = -\delta p + \varepsilon$  in period  $t - 1$ . In period  $t$  by (optimally) making no purchase in period  $t$ , the buyer receives a flow of  $v\tau$ .

Finally, consider the continuation state  $(t + 1, \tau)$ . A complication is that, in addition to the first  $\tau$  units, the outcome in state  $(t, 0)$  may involve buyers acquiring units in  $\{\tau + 2, \dots, t\}$ . By making no purchases in period  $t$ , the deviating buyer will lack these units in the future while other buyers possess them. But, this is of no consequence in equilibrium: a buyer holding exactly  $\tau$  units obtains the same continuation payoff of  $u(t + 1, \tau)$ , by our first property of buyer payoffs.

Now, adding the terms in periods  $t - 1$ ,  $t$ , and  $t + 1$  for a deviating buyer, we arrive at  $\hat{u}$  as in the above equation. Thus, accepting the seller's deviation offer in period  $t - 1$  for  $\tau$  units results in a higher payoff than rejecting and waiting whenever  $\varepsilon > 0$ . Thus, all buyers rejecting the offer is never an equilibrium continuation. In a symmetric equilibrium, it must be that all buyers accept the offer in  $t - 1$ .

Now, to see that the deviation is profitable for the seller, note that the payoff to the deviation offer in period  $t - 1$  (where all buyers accept) is

$$\hat{\pi} = \hat{p} + \delta\pi(t, \tau) = v\tau + \delta p - \varepsilon + \delta^2\pi(t + 1, \tau) = v\tau - \varepsilon + \delta\pi(t, 0) > \delta\pi(t, 0),$$

where we have used the definition of  $\hat{p}$  and the equilibrium hypothesis for  $(t, 0)$ , which implies  $\pi(t - 1, 0) = \delta\pi(t, 0)$  and  $\pi(t, 0) = p + \delta\pi(t + 1, \tau)$ . Thus, we cannot have  $\tau < t$  in equilibrium.

Finally, we must verify that the same deviation will work for the seller when the history for state  $(t, 0)$  has buyers holding quality units (but not unit 1). By equilibrium, seller's payoffs  $\pi(t, \tau)$  and  $\pi(t + 1, \tau)$  are independent of these holdings. The only remaining possible complication is that the deviation offer in  $t - 1$  sacrifices revenues that would otherwise have been received by the seller from an offer of units in  $\{2, \dots, t - 1\}$ . Equilibrium, however, rules out any such revenues for the seller as we showed above. ■

### 8.3 Appendix C - Efficient Equilibria

In this appendix we first prove Lemma 4, which provides sufficient conditions for equilibrium existence. We then prove several lemmas dealing with properties of the  $T$  – stage support. This is followed by proofs of Propositions 2 and 3 for existence of equilibrium with frequent and infrequent innovation, respectively.

*(i) Sufficient Conditions for Equilibrium*

*Proof of Lemma 4.* To establish existence of an equilibrium we need to show that (i) our candidate upgrade offer is optimal for the seller with respect to the offer set  $\Omega_\tau$ , and (ii) construct a continuation equilibrium for states where buyers are distributed asymmetrically providing, for any such state, both an optimal offer for the seller and optimal buyer responses for any given seller offer.

Now, define an upgrade offer  $\mathcal{B}$  as any offer in  $\Omega_\tau$ , with the property that if  $(z, p_z) \in \mathcal{B}$ , then  $M(z) = \sup\{i \mid i \in z\}$ . By construction, every bundle in  $\mathcal{B}$  is an upgrade bundle, since the maximal contiguous quality in  $z$  coincides with the largest quality unit in  $z$ . Thus, we can denote any  $(z, p_z) \in \mathcal{B}$  by  $(b, p_b)$ , where  $b \equiv \sup\{i \mid i \in z\}$  is the upgrade level for  $z$  and  $p_b \equiv p_z$  is the price. Note that an upgrade offer need not include all of the feasible upgrade bundle levels  $1, \dots, \tau$ . A buyer will optimally choose at most one bundle in  $\mathcal{B}$ .

The restriction to upgrade offers can be shown to be without loss of generality. This is because we construct continuation equilibria in which only upgrade offers are made by the seller. Thus, even if a period  $\tau$  offer includes non-upgrade bundles, all players expect that every possible period  $\tau + 1$  continuation state will involve only upgrade offers. Furthermore, in every possible period  $\tau + 1$  continuation state, every buyer will move to a quality holding of at least  $\tau$  in the continuation equilibrium outcome. Consequently, a buyer will value bundles in an offer only to the extent that the bundle allows the buyer to move to a higher quality level in period  $\tau$ . If a buyer's purchases in  $\tau$  result in the acquisition of non-contiguous quality levels, these non-contiguous units have no current or future payoff effect due to the structure of the continuation equilibria.

More generally, if buyers are asymmetrically distributed across maximal quality levels and if buyers hold non-contiguous quality units (above a buyer's maximal level), we can still work with upgrade offers without loss of generality. This is because any offer in conjunction with a buyer's current maximal quality and non-contiguous holdings can always be reduced to an implied set of payments for achievable (higher) maximal quality levels. With the continuation equilibria noted above, any resulting non-contiguous quality

units will have no payoff impact.

To demonstrate (i), it is sufficient to show that the seller cannot profitably deviate in state  $(\tau, 0)$  to some other upgrade offer (with multiple upgrade bundles). First, define a buyer preference relation,  $\succeq_B$ , for any two upgrade bundles by

$$(b, p_b) \succeq_B (i, p_i) \iff vb - p_b + \delta u(\tau + 1, b) \geq vi - p_i + \delta u(\tau + 1, b \mid i),$$

where  $u(\tau + 1, b \mid i)$  is equal to  $\frac{vi}{1-\delta}$  if  $i < b$  and equal to  $\max\{\frac{vi}{1-\delta}, u(\tau + 1, b)\}$  if  $i \geq b$ . Note that the  $\succeq_B$  relation reflects implicit coordination in that an individual buyer has no incentive to choose  $(i, p_i)$  if all other buyers choose  $(b, p_b)$ . Now, define an upgrade  $(b, p_b) \in \mathcal{B}$  to be a buyer continuation equilibrium (*BCE*) in state  $(\tau, 0)$  for offer  $\mathcal{B}$  if  $(b, p_b) \succeq_B (i, p_i) \forall (i, p_i) \in \mathcal{B}$ . We must show that for any offer  $\mathcal{B}$  there exists a *BCE* such that the seller cannot gain in state  $(\tau, 0)$  by deviating to offer  $\mathcal{B}$  instead of making the cash-in offer of  $p_\tau$ . The proof is lengthy so we only provide a sketch. First, one shows that any two upgrade offers are comparable under  $\succeq_B$ . Next, one can show that

$$\arg \max_{(b, p_b) \in \mathcal{B}} \left\{ \frac{vb}{1-\delta} - p_b + \delta u_{\tau+1-b} \right\}$$

is a *BCE* (existence is trivial as  $\mathcal{B}$  has a finite number of bundles; if it is not unique then select the arg max with largest upgrade level). Essentially, this follows because the argmax is the highest possible coordinated payoff for buyers and because utility differences across a  $T$ -stage support satisfy the bound  $u_\sigma - u_{\sigma'} \leq \frac{v(\sigma - \sigma')}{1-\delta}$ .

We then have two cases for the offer  $\mathcal{B}$ . If every  $(b, p_b) \in \mathcal{B}$  satisfies  $p_b \geq G_b \equiv g(b, u_{\tau+1})$ , then it is easy to show that  $(0, 0)$ , where all buyers refuse to purchase, is a *BCE* for  $\mathcal{B}$ . This is equivalent to a delay outcome and we know from (3) that the seller prefers to make the cash-in offer  $p_\tau$ . If  $p_b < G_b$  for some  $(b, p_b) \in \mathcal{B}$  then we first find the arg max specified above for the subset of all such bundles in  $\mathcal{B}$ , call it  $(b^*, p_{b^*})$ . We then show that  $(b^*, p_{b^*}) \succeq_B (i, p_i) \forall (i, p_i)$  where  $p_i \geq G_i$ . Then,  $(b^*, p_{b^*})$  is a *BCE* for  $\mathcal{B}$  and the resulting seller payoff of  $p_{b^*} + \delta u_{\tau+1-b^*}$  is, by (3), not profitable relative to  $p_\tau$ . This completes (i).

Consider (ii) and any state in which buyers are asymmetrically distributed across quality levels. Since all buyers with the same maximal contiguous quality level are treated identically in the continuation it is sufficient to keep track only of market shares. Thus, let us denote such a state by  $(\tau, \alpha)$  where  $\alpha = (\alpha_0, \dots, \alpha_{\tau-1})$  specifies for each  $\sigma = 0, \dots, \tau - 1$  the fraction  $\alpha_\sigma \in [0, 1]$  of buyers entering period  $\tau$  with maximal quality level of  $\sigma$ . By

hypothesis,  $1 > \alpha_0 > 0$  and  $\sum_{\sigma=0}^{\tau-1} \alpha_\sigma = 1$ . We specify a continuation equilibrium for  $(\tau, \alpha)$  as follows. The seller makes an upgrade offer  $\{(b_\sigma, p_\sigma)\}_{\sigma=0, \dots, \tau-1}$  where each  $(b_\sigma, p_\sigma)$  is an upgrade from  $\sigma$  to  $\tau$ , that is the bundle  $\{\sigma + 1, \dots, \tau\}$ , for price

$$p_\sigma = \frac{v(\tau - \sigma)}{1 - \delta} + \delta u_1.$$

For buyer strategies, we specify that a buyer with  $\sigma$  units chooses to accept  $(b_\sigma, p_\sigma)$ . It is straightforward to verify that, when all other buyers follow this strategy, it is optimal for an individual buyer with  $\sigma$  to do so as well. Since these upgrade offers leave each buyer with a payoff of  $v\sigma / (1 - \delta)$ , the payoff to the seller is equal to the continuation surplus of  $S_\tau$  less these individual-rationality payoffs aggregated across buyers according to the distribution  $\alpha$ . By feasibility, this is an upper bound on the seller's payoff in any continuation.

Finally, to complete the argument that the above upgrade offer is an optimal choice for the seller, we need to specify a *BCE* if the seller makes some other upgrade offer. Allowing for partial upgrades, denote such an offer by  $\mathcal{B} = \{(b, \sigma; p_{b,\sigma}) \mid \sigma \leq b \leq \tau, 0 \leq p_{b,\sigma}\}$  in  $(\tau, \alpha)$ , where each  $(b, \sigma; p_{b,\sigma})$  denotes an upgrade bundle for units  $\{\sigma + 1, \dots, b\}$  at price  $p_{b,\sigma}$ . Since the offer  $\mathcal{B}$  has upgrades that begin at different levels and a buyer is free to purchase multiple bundles, we construct from  $\mathcal{B}$  for each possible buyer quality level  $\sigma = 0, \dots, \tau - 1$ , the set  $\mathcal{B}_\sigma$  of all upgrade bundles that move  $\sigma$  to a higher quality level; note  $\mathcal{B}_\sigma$  might contain only the refusal option for some  $\sigma$ . We then have each buyer with quality level  $\sigma$  choose to accept the (largest index)

$$\arg \max_{\mathcal{B}_\sigma} \left\{ \frac{vb}{1 - \delta} - p_{b,\sigma'} \right\}.$$

Then, these choices can be shown to form a *BCE* for  $\mathcal{B}$ . The proof is trivial if the buyer choices result in a nondegenerate distribution across quality levels since the buyers are then held to their individual-rationality payoffs in the continuation state. If not, then all buyers move to some common quality level, say  $\hat{b}$ , and we must use the continuation payoff  $u_{\tau+1-\hat{b}}$  from the  $T$ -stage support. ■

(ii) *Properties of the  $T$  - stage Support*

**Lemma A1** *Consider a  $T$ -stage support, where  $u_\tau = u_T \equiv \bar{u}$  for  $\tau \geq T$ . If the support condition (3) holds at  $\tau = T$  for  $\sigma = 0, \dots, T$ , then (3) holds at  $\tau > T$  for  $\sigma = 0, \dots, \tau$ .*

*Proof.* We begin by establishing two claims. These are sufficient to prove the lemma

for  $T = 1$ . We then establish a third claim and prove the lemma for  $T \geq 2$ .

First, we show that if (3) holds at  $(0, \tau)$ , then it holds at  $(0, \tau + 1)$ . By (3) at  $(0, \tau)$  we have

$$v\tau \geq u_\tau - \delta u_{\tau+1} + g(0, u_{\tau+1}) = (1 - \delta)\bar{u}$$

and this directly implies that (3) holds at  $(0, \tau + 1)$ , since

$$v(\tau + 1) \geq u_{\tau+1} - \delta u_{\tau+2} + g(0, u_{\tau+2}) = (1 - \delta)\bar{u}.$$

The second claim is that if (3) holds at  $(0, \tau)$ , then it holds at  $(\sigma, \tau)$  for  $\sigma = 1, \dots, \tau + 1 - T$ . Since  $\sigma \leq \tau + 1 - T$  if and only if  $T \leq \tau + 1 - \sigma$ , the quality gap in the continuation state,  $(\tau + 1, \sigma)$ , is at least  $T$  and therefore  $u_{\tau+1-\sigma} = \bar{u}$ . The support condition at  $(\sigma, \tau)$  is then

$$\begin{aligned} S_\tau - \delta S_{\tau+1-\sigma} &\geq u_\tau - \delta u_{\tau+1-\sigma} + g(\sigma, u_{\tau+1}) \Leftrightarrow \\ v\tau + \frac{\delta v\sigma}{1-\delta} &\geq \bar{u}(1-\delta) + g(\sigma, \bar{u}). \end{aligned}$$

At  $\sigma = 0$ , we have  $g(0, \bar{u}) = 0$ . Thus, the  $(0, \tau)$  condition is  $v\tau \geq \bar{u}(1 - \delta)$ . Now consider  $\sigma = 1, \dots, \tau + 1 - T$ . We have two cases. First, if  $\bar{u} \leq \frac{v\sigma}{1-\delta}$ , then support at  $\sigma$  becomes

$$\begin{aligned} v\tau + \frac{\delta v\sigma}{1-\delta} &\geq \bar{u}(1-\delta) + \frac{v\sigma}{1-\delta} - \delta\bar{u} \Leftrightarrow \\ v(\tau - \sigma) &\geq (1 - 2\delta)\bar{u}, \end{aligned}$$

which always holds for  $\tau \geq \sigma$  and  $\delta \geq 1/2$ . Second, if  $\bar{u} > \frac{v\sigma}{1-\delta}$ , then  $\max\{\frac{v\sigma}{1-\delta}, \bar{u}\} = \bar{u}$  and support at  $\sigma$  becomes

$$\begin{aligned} v\tau + \frac{\delta v\sigma}{1-\delta} &\geq \bar{u}(1-\delta) + v\sigma \Leftrightarrow \\ v\tau + v\sigma \left( \frac{\delta}{1-\delta} - 1 \right) &\geq \bar{u}(1-\delta). \end{aligned}$$

Since  $v\tau \geq \bar{u}(1 - \delta)$ , by support at  $\sigma = 0$ , and  $\frac{\delta}{1-\delta} \geq 1$  for  $\delta \geq 1/2$ , we have established the second claim.

We can now prove Lemma A1 for  $T = 1$ . By hypothesis, (3) holds for  $\sigma = 0$  and 1 at  $T = 1$ . Consider  $\tau > 1$ . By the first claim, (3) holds for  $\sigma = 0$  at  $\tau$ . By the second claim, (3) then holds for  $\sigma = 1, \dots, \tau + 1 - T$ . Since  $\tau + 1 - T = \tau$  in this case, we are done.

From now on take  $T \geq 2$ . The third claim is that if (3) holds at  $\tau$  for  $\sigma = (\tau + 2 - T)$

to  $\sigma = \tau$ , then it holds at  $\tau + 1$  for  $\sigma = (\tau + 2 - T) + 1$  to  $\sigma = \tau + 1$ . Note that the quality gap in the continuation state  $(\tau + 1, \sigma)$  is less than  $T$  exactly when  $\sigma \geq \tau + 1 - T$ . Hence, we have  $u_{\tau+1-\sigma} \in \{u_1, \dots, u_{T-1}\}$ , and we are on the rising utility part of the  $T$ -stage support. Also, note that  $u_{\tau+1} = \bar{u}$ .

Condition (3) holds at  $(\sigma, \tau)$  if and only if

$$\begin{aligned} S_\tau - \delta S_{\tau+1-\sigma} &\geq u_\tau - \delta u_{\tau+1-\sigma} + g(\sigma, u_{\tau+1}) \Leftrightarrow \\ v\tau + \frac{\delta v\sigma}{1-\delta} &\geq \bar{u} - \delta u_{\tau+1-\sigma} + g(\sigma, \bar{u}), \end{aligned}$$

and at  $(\sigma + 1, \tau + 1)$  if and only if

$$v(\tau + 1) + \frac{\delta v(\sigma + 1)}{1-\delta} \geq \bar{u} - \delta u_{\tau+1-\sigma} + g(\sigma + 1, \bar{u}).$$

For the  $(\sigma, \tau)$  condition to imply the  $(\sigma + 1, \tau + 1)$  condition, it is sufficient to show that

$$\frac{v}{1-\delta} \geq g(\sigma + 1, \bar{u}) - g(\sigma, \bar{u})$$

There are three cases that we need to check.

1) If  $\bar{u} \leq \frac{v\sigma}{1-\delta}$ , then  $g(\sigma + 1, \bar{u}) = \frac{v(\sigma+1)}{1-\delta} - \delta\bar{u}$  and  $g(\sigma, \bar{u}) = \frac{v\sigma}{1-\delta} - \delta\bar{u}$ , hence the condition holds with equality.

2) If  $\frac{v\sigma}{1-\delta} < \bar{u} \leq \frac{v(\sigma+1)}{1-\delta}$ , then  $g(\sigma + 1, \bar{u}) - g(\sigma, \bar{u}) = \frac{v(\sigma+1)}{1-\delta} - \delta\bar{u} - v\sigma$ , which is less than  $\frac{v}{1-\delta}$  if and only if  $\frac{v\sigma}{1-\delta} < \bar{u}$ .

3) If  $\frac{v(\sigma+1)}{1-\delta} < \bar{u}$ , then  $g(\sigma + 1, \bar{u}) - g(\sigma, \bar{u}) = v(\sigma + 1) - v\sigma < \frac{v}{1-\delta}$ .

Thus, we have established the third claim.

To complete the proof, consider  $\tau = T + 1$  and suppose (3) holds at  $T$  for  $\sigma = 0, \dots, T$ . We know from the third claim that since, (3) holds at  $T$  for  $\sigma = 2, \dots, T$ , it must hold at  $T + 1$  for  $\sigma = 3, \dots, T + 1$ . From the first claim, the support holds at  $(0, T + 1)$ , since it holds at  $(0, T)$ . From the second claim, we know the support at  $(0, T + 1)$  is sufficient for support at  $(\sigma, T + 1)$  for  $\sigma = 1, \dots, \tau + 1 - T$ . But,  $\tau + 1 - T = 2$  in this case and thus (3) holds for all  $(\sigma, T + 1)$ . The same logic applies between any  $\tau$  and  $\tau + 1$ . ■

**Lemma A2** Consider a  $T$ -stage support. Then for any  $\tau \leq T$ , we have

$$u_\tau = \frac{1}{\delta^{\tau-1}} \left[ u_1 - \frac{v}{1-\delta} \left( \frac{1-\delta^{\tau-1}}{1-\delta} - (\tau-1)\delta^{\tau-1} \right) \right].$$

*Proof.* It is clearly valid at  $\tau = 1$ . To verify for  $\tau + 1$ , assume it holds at  $\tau$  and we

then have

$$\begin{aligned}
u_{\tau+1} &= \frac{1}{\delta} [u_\tau - v\tau] = \frac{1}{\delta^\tau} \left[ u_1 - \frac{v}{1-\delta} \left( \frac{1-\delta^{\tau-1}}{1-\delta} - (\tau-1)\delta^{\tau-1} \right) \right] - \frac{v\tau}{\delta} \\
&= \frac{1}{\delta^\tau} \left[ u_1 - \frac{v}{1-\delta} \left( \frac{1-\delta^{\tau-1}}{1-\delta} - (\tau-1)\delta^{\tau-1} + \tau\delta^{\tau-1}(1-\delta) \right) \right] \\
&= \frac{1}{\delta^\tau} \left[ u_1 - \frac{v}{1-\delta} \left( \frac{1-\delta^\tau}{1-\delta} - \tau\delta^\tau \right) \right]
\end{aligned} \tag{8}$$

and the formula is valid for  $\tau + 1$ . ■

**Lemma A3** *The  $T$ -stage support sequence  $(\underline{u}_1, \dots, \underline{u}_T)$  defined by  $\underline{u}_1 = (1 - \delta^{T-1})S_1$  satisfies (i)  $\underline{u}_\tau = (1 - \delta^{T-\tau})S_1 + \frac{v(\tau-1)}{1-\delta}$ , (ii)  $\underline{u}_\tau \geq \frac{v\tau}{1-\delta}$  if and only if  $\delta \geq \delta^{T-\tau}$  and (iii)  $\underline{u}_{T-1} \equiv \frac{v(T-1)}{1-\delta}$  and  $\underline{u}_{T-2} \equiv \frac{v(T-1)}{1-\delta} - v$ .*

*Proof.* We have  $\underline{u}_1 \equiv \frac{v}{1-\delta} \left[ \frac{1-\delta^{T-1}}{1-\delta} \right]$  and  $\underline{u}_\tau = v\tau + \delta\underline{u}_{\tau+1}$  for  $\tau = 1, \dots, T$ . We first show (i). This holds for  $\underline{u}_1$  by construction. Assume that it holds for  $\underline{u}_\tau$  and consider  $\underline{u}_{\tau+1}$ . Then

$$\begin{aligned}
\underline{u}_{\tau+1} &= \frac{1}{\delta} [\underline{u}_\tau - v\tau] = \frac{1}{\delta} \left[ \frac{v}{1-\delta} \left( \frac{1-\delta^{T-\tau}}{1-\delta} + (\tau-1) \right) - v\tau \right] \\
&= \frac{v}{\delta(1-\delta)^2} [\delta + \delta\tau - \delta^{T-\tau} - \delta^2\tau] = \frac{v}{1-\delta} \left[ \frac{1-\delta^{T-(\tau+1)}}{1-\delta} + \tau \right].
\end{aligned}$$

Next, we show (ii):  $\underline{u}_\tau \geq \frac{v\tau}{1-\delta}$  if and only if  $\delta \geq \delta^{T-\tau}$ . Using (i) for  $\underline{u}_\tau$ , we have

$$\begin{aligned}
\frac{v}{1-\delta} \left[ \frac{1-\delta^{T-\tau}}{1-\delta} + (\tau-1) \right] &\geq \frac{v\tau}{1-\delta} \Leftrightarrow \\
\frac{1-\delta^{T-\tau}}{1-\delta} + (\tau-1) &\geq \tau \Leftrightarrow \delta \geq \delta^{T-\tau}.
\end{aligned}$$

For (iii), apply the previous formula for  $u_\tau$  in terms of  $u_1$  at  $\tau = T - 1$  and simplify, and then use (5) to find  $\underline{u}_{T-2}$  in terms of  $\underline{u}_{T-1}$  and simplify. ■

**Lemma A4** *Consider a  $T$ -stage support with (i)  $u_\tau \geq \frac{v\tau}{1-\delta}$  for  $\tau = 1, \dots, T - 1$  and (ii)  $\frac{v(T-1)}{1-\delta} < u_T < \frac{vT}{1-\delta}$ . If the support condition (3) holds at  $(\tau, \tau)$  for  $\tau = 1, \dots, T$ , then the  $T$ -stage support satisfies (3) for all  $(\sigma, \tau)$ , where  $0 \leq \sigma \leq \tau$  and  $\tau \geq 1$ .*

*Proof.* Note that we choose  $T$  so that (i) and (ii) of Lemma A4 are satisfied. By Lemma A1, it is sufficient to show (3) holds for  $\tau \leq T$ . Consider  $\sigma = 0$  and  $\tau = 1, \dots, T$ . Since  $g(0, u_\tau) = 0$ , we see that (3) reduces to  $v\tau \geq u_\tau - \delta u_{\tau+1}$  for  $\tau < T$ . This holds with equality by construction of a  $T$ -stage support. At  $\tau = T$ , (3) reduces to  $vT \geq u_T(1 - \delta)$

and this holds by (i) of the lemma.

Now consider  $1 \leq \sigma \leq \tau$  with  $\tau < T$ . We claim (3) at  $(\sigma, \tau)$  implies (3) at  $(\sigma, \tau + 1)$ . The condition for  $(\sigma, \tau)$  is

$$\begin{aligned} S_\tau - \delta S_{\tau+1-\sigma} &\geq u_\tau - \delta u_{\tau+1-\sigma} + g(\sigma, u_{\tau+1}) \Leftrightarrow \\ v(\tau - \sigma) + \frac{\delta v \sigma}{1 - \delta} &\geq u_\tau - \delta u_{\tau+1-\sigma}, \end{aligned}$$

since, by (i) and (ii),  $u_{\tau+1} > \frac{v\sigma}{1-\delta}$  for any  $\sigma \leq \tau < T$ . At  $(\sigma, \tau + 1)$ , we need

$$\begin{aligned} S_{\tau+1} - \delta S_{\tau+2-\sigma} &\geq u_{\tau+1} - \delta u_{\tau+2-\sigma} + g(\sigma, u_{\tau+2}) \Leftrightarrow \\ v(\tau - \sigma + 1) + \frac{\delta v \sigma}{1 - \delta} &\geq u_{\tau+1} - \delta u_{\tau+2-\sigma}, \end{aligned}$$

since  $u_{\tau+2} > \frac{v\sigma}{1-\delta}$  for any  $\sigma \leq \tau < T$ . By the  $T$ -stage support, the  $(\sigma, \tau + 1)$  condition becomes

$$\begin{aligned} v(\tau - \sigma + 1) + \frac{\delta v \sigma}{1 - \delta} &\geq \frac{1}{\delta}(u_\tau - v\tau) - u_{\tau+1-\sigma} + v(\tau + 1 - \sigma) \Leftrightarrow \\ v\tau + \frac{\delta^2 v \sigma}{1 - \delta} &\geq u_\tau - \delta u_{\tau+1-\sigma}, \end{aligned}$$

Thus, we need only show that  $v\tau + \frac{\delta^2 v \sigma}{1 - \delta} \geq v(\tau - \sigma) + \frac{\delta v \sigma}{1 - \delta}$ . This clearly holds for any  $\delta \in [0, 1]$  and any non-negative  $\sigma$ .

Consequently, (3) at  $(1, 1)$  implies (3) at  $(1, \tau)$  for  $\tau = 2, \dots, T$ , (3) at  $(2, 2)$  implies (3) at  $(2, \tau)$  for  $\tau = 3, \dots, T$ , and so on up through  $(T - 1, T - 1)$ . Thus, (3) at  $(\tau, \tau)$  for  $\tau = 1, \dots, T$  is sufficient and the lemma is established. ■

(iii) *Existence of Equilibrium*

*Proof of Proposition 2.* First, note that conditions (i) and (ii) of Lemma A4 are valid when  $(1 - \delta^{T-1})S_1 \leq u_1 \leq (1 - \delta^T)S_1$ . This follows by applying Lemma A3 to the reference sequences,  $\underline{u}_\tau$ , for  $T$  and for  $T + 1$ . Then, by Lemma A4, it is sufficient to verify condition (3) at  $(\tau, \tau)$  for  $\tau = 1, \dots, T$ .

It is immediate that (3) at  $\tau = 1$  requires  $\delta S_1 \geq u_1$ . Hence, we are done if  $T = 1$ . Now, consider  $T \geq 2$  and note that the same observation implies that (3) holds at  $(1, 1)$ .

Now, consider (3) at  $(\tau, \tau)$  for  $\tau \leq T - 1$ . Then  $u_{\tau+1} > \frac{v\tau}{(1-\delta)}$  and we have  $g(\tau, u_{\tau+1}) = v\tau$ . Thus, the equilibrium support condition (3) becomes

$$\frac{\delta v \tau}{1 - \delta} + \delta u_1 \geq u_\tau.$$

We claim that condition  $(\tau, \tau)$  implies condition  $(\tau+1, \tau+1)$  for  $\tau \leq T-2$ . In other words, we claim that  $\frac{\delta v \tau}{1-\delta} + \delta u_1 \geq u_\tau$  implies  $\frac{\delta v(\tau+1)}{1-\delta} + \delta u_1 \geq u_{\tau+1}$ . Recall that  $u_{\tau+1} = \frac{1}{\delta}(u_\tau - v\tau)$ . So, condition  $(\tau+1, \tau+1)$  can be written as

$$\frac{\delta^2 v(\tau+1)}{1-\delta} + \delta^2 u_1 + v\tau \geq u_\tau.$$

Thus, it is sufficient to show that  $\frac{\delta^2 v(\tau+1)}{1-\delta} + \delta^2 u_1 + v\tau > \frac{\delta v \tau}{1-\delta} + \delta u_1$ . But, this holds if and only if  $\delta S_1 + \frac{v\tau}{\delta} > u_1$ , which is always the case for  $\tau \geq 1$ . Thus, (3) holds at  $(1, 1)$  and this implies (3) holds at  $(\tau, \tau)$  for  $\tau = 2, \dots, T-1$ .

We are then left with the  $(T, T)$  condition, which reduces to

$$\delta u_1 \geq (1-\delta)u_T,$$

since  $g(T, u_{T+1}) = \frac{vT}{1-\delta} - \delta u_T$  by Lemma A3. We know that the condition holds at  $(T-1, T-1)$  and we have

$$\begin{aligned} \frac{\delta v(T-1)}{1-\delta} + \delta u_1 &\geq u_{T-1} = v(T-1) + \delta u_T \Leftrightarrow \\ \frac{1}{\delta} \left[ \frac{\delta v(T-1)}{1-\delta} - (T-1)v + \delta u_1 \right] &\geq u_T. \end{aligned}$$

Thus, it is sufficient for  $(T, T)$  to show that

$$\frac{\delta}{1-\delta} u_1 \geq \frac{1}{\delta} \left[ \frac{\delta v(T-1)}{1-\delta} - (T-1)v + \delta u_1 \right]. \quad (9)$$

Simplifying and noting that  $\delta \geq 1/2$ , condition (9) holds if and only if  $\delta u_1 \geq v(T-1)$ .

From  $u_1 \geq (1 - \delta^{T-1})S_1$ , it is sufficient to show that  $\delta(1 - \delta^{T-1})S_1 \geq v(T-1)$ . At  $T=2$  this reduces to  $\delta \geq 1/2$ . Now, we carry out an induction: assume it holds for  $T$  if  $\delta + \delta^{T-1} > 1$  and show that it holds for  $T+1$  if  $\delta + \delta^T > 1$ . So, we must show that  $\delta(1 - \delta^T)S_1 \geq vT$  or, equivalently, that

$$\delta(1 + \dots + \delta^{T-1}) > T(1 - \delta).$$

The condition at  $T$  is  $\delta(1 - \delta^{T-1})S_1 \geq v(T-1)$ , which holds if and only if

$$(1 - \delta) + \delta(1 + \dots + \delta^{T-2}) > T(1 - \delta).$$

But,

$$\delta(1 + \dots + \delta^{T-1}) > (1 - \delta) + \delta(1 + \dots + \delta^{T-2}) \Leftrightarrow \delta + \delta^T > 1,$$

which establishes the induction. Thus, the  $(T, T)$  condition holds, and we have therefore shown that the  $T$ -stage support satisfies (3) for all  $(\sigma, \tau)$ , where  $0 \leq \sigma \leq \tau$  and  $\tau \geq 1$ .

To see that every buyer payoff,  $u_1 \in [0, \delta S_1]$  can be supported in this way, simply note that each  $\delta \in [1/2, 1]$  lies in exactly one of the  $\delta_\tau$  cutoff sequence intervals. Note that the cut-off sequence  $\delta_\tau$  is strictly increasing in  $\tau$ , from  $\delta_1 = 1/2$  to  $\lim_{\tau \rightarrow \infty} \delta_\tau = 1$ , and satisfies  $\delta^{\tau-1} < 1 - \delta < \delta^\tau$  for  $\delta \in (\delta_{\tau-1}, \delta_\tau)$ . With  $\delta \in [\delta_\tau, \delta_{\tau+1})$ , we then see that every  $u_1 \in [0, \delta S_1]$  lies in exactly one of the  $[(1 - \delta^{\tau-1}) S_1, (1 - \delta^\tau) S_1]$  intervals, where  $T$  ranges from 1 up to the index on the  $\delta_\tau$  root. ■

*Proof of Proposition 3.* By construction of an efficient equilibrium, for  $u_1 \in [0, \delta S_1]$  we have a payoff of  $\pi_1 = \frac{p_1}{1-\delta}$  for the seller and  $u_1 = \frac{1}{1-\delta} \left[ \frac{v}{1-\delta} - p_1 \right]$  for the buyers. It is sufficient for an equilibrium to show that the support condition

$$S_\tau - \delta S_{\tau+1-\sigma} \geq u_\tau - \delta u_{\tau+1-\sigma} + g(\sigma, u_{\tau+1})$$

holds for  $\sigma = 0, 1, \dots, \tau$  at each  $\tau \geq 1$ . To verify that the support condition holds, we note three useful facts: (i)  $\bar{u} \in [0, \frac{v}{1-\delta}]$ , as follows from the range for  $u_1$ , (ii)  $g(0, \bar{u}) = 0$  and  $g(\sigma, \bar{u}) = \frac{v\sigma}{1-\delta} - \delta \bar{u}$  for  $\sigma \geq 1$ , also by the range for  $u_1$  (iii)  $S_\tau - \delta S_{\tau+1-\sigma} = v\tau + \frac{\delta v \sigma}{1-\delta}$ , by definition of surplus.

Begin with  $\sigma = 0$ . For  $\tau = 1$ , the support condition reduces to  $v \geq \delta u_1$ . But  $\delta < 1/2$  implies  $v > \delta^2 S_1$ , so we are done as  $u_1 \leq \delta S_1$ . For  $\tau > 1$ , the support condition reduces to  $v\tau \geq (1 - \delta)\bar{u}$ . It is sufficient to show this holds at  $\tau = 2$ , which reduces to  $\frac{2v}{1-\delta} \geq \bar{u}$ , and we are done by the upper bound on  $\bar{u}$ .

Now consider  $\sigma = 1, \dots, \tau-1$  and  $\tau > 1$ . The condition reduces to  $v(\tau - \sigma) \geq (1-2\delta)\bar{u}$ . As  $\bar{u} \leq \frac{v}{1-\delta}$ , we need only show that at  $\tau = 2$  we have  $1 \geq \frac{1-2\delta}{1-\delta}$ . This reduces to  $\delta > 0$ .

Finally, the case of  $\sigma = \tau$  corresponds to the equilibrium path and continuation outcomes. The condition reduces, by construction, to

$$\pi_\tau = S_\tau - u_\tau \geq \delta (S_1 - u_1) + g(\tau, \bar{u}) = \delta \pi_1 + \frac{v\tau}{1-\delta} - \delta \bar{u}.$$

Since  $\pi_1 = \frac{p_1}{1-\delta}$ ,  $\pi_\tau = p_\tau + \delta \pi_1$ ,  $\bar{u} = \frac{v\tau}{1-\delta} - p_\tau + \delta u_1$  by construction of the candidate equilibrium, this reduces to  $\delta u_1 \geq (1 - \delta)\bar{u}$  and this holds with equality. ■

## 8.4 Appendix D - Inefficient Equilibria

This appendix has four parts. We begin with infrequent innovation and prove Proposition 4. Next, we prove results for the approach conditions. We then provide the analysis of the off equilibrium support. Finally, we prove Proposition 5 to show existence of equilibria for frequent innovation.

(i) *Infrequent Innovation* ( $\delta \leq 1/2$ ).

*Proof of Proposition 4.* We first show that  $\frac{v(t-1)}{1-\delta} > u_t$  is necessary in any delay equilibrium when  $\delta \leq 1/2$ . By equilibrium construction, a first sale in state  $(t, 0)$  of  $t$  units implies  $\pi_1 = \delta^{t-1}\pi_t$  and

$$\pi_t + u_t = \frac{vt}{(1-\delta)(1-\delta^t)}.$$

By flow dominance, we have  $\pi_1 \geq \frac{v}{1-\delta}$  and, hence,  $\pi_t \geq \frac{v}{\delta^{t-1}(1-\delta)}$ . Combining with the joint payoff expression from above, we have

$$u_t \leq \frac{vt}{(1-\delta)(1-\delta^t)} - \frac{v}{\delta^{t-1}(1-\delta)}.$$

It is then sufficient to show that  $\frac{v(t-1)}{1-\delta}$  exceeds the right-hand-side in the above expression. Simplifying the resulting inequality, this holds if

$$1 > \delta^{t-1} [1 + \delta + t\delta^t - \delta^t].$$

At  $t = 2$  this reduces to  $1 > \delta [1 + \delta + \delta^2]$ . It holds at  $\delta = 1/2$  since  $1 > 7/8$  and, hence, for all smaller  $\delta$ . By induction, it holds for all  $t > 2$  if

$$\begin{aligned} \delta^{t-1} [1 + \delta + t\delta^t - \delta^t] &> \delta^t [1 + \delta + (t+1)\delta^{t+1} - \delta^{t+1}] \Leftrightarrow \\ 1 &> \delta^2 + \delta^t + t\delta^t(\delta^2 - 1). \end{aligned}$$

But  $\delta^2 + \delta^t < 1$  for  $\delta \leq 1/2$  and the second term is negative. Therefore, we have  $\frac{v(t-1)}{1-\delta} > u_t$ .

Now suppose the seller offers  $t - 1$  units for a price of  $\hat{p} = \frac{v(t-1)}{1-\delta} - \delta u_t - \epsilon$  in state

$(t-1, 0)$ . It cannot be that all buyers reject this offer since

$$\begin{aligned} v(t-1) - \hat{p} + \delta \max \left\{ \frac{v(t-1)}{1-\delta}, u_t \right\} &> \delta u_t \Leftrightarrow \\ \frac{v(t-1)}{1-\delta} - \delta u_t &> \hat{p}. \end{aligned}$$

Thus, in a symmetric equilibrium, it must be that all buyers accept the offer and, hence, the continuation state is  $(t, t-1)$ . The payoff to the seller is then  $\hat{\pi} = \hat{p} + \delta\pi_1$ . This exceeds  $\delta\pi_t$  if

$$\frac{v(t-1)}{1-\delta} + \delta\pi_1 > \delta(\pi_t + u_t).$$

Substituting for the equilibrium joint surplus and using  $\pi_1 \geq \frac{v}{1-\delta}$ , by flow dominance, it is sufficient to show

$$\begin{aligned} \frac{v(t-1)}{1-\delta} + \delta \frac{v}{1-\delta} &> \frac{\delta vt}{(1-\delta)(1-\delta^t)} \Leftrightarrow \\ t(1-\delta) &> (1-\delta)(1-\delta^t) + t\delta^t. \end{aligned} \quad (10)$$

At  $t=2$ , (10) reduces to  $1 > \delta(1+\delta+\delta^2)$ , which is valid for all  $\delta \leq 1/2$ . To show that an induction argument holds, note that (10) at  $t+1$  is

$$\begin{aligned} (t+1)(1-\delta) &> (1-\delta)(1-\delta^{t+1}) + (t+1)\delta^{t+1} \Leftrightarrow \\ t(1-\delta) &> \delta^{t+2} + t\delta^{t+1}. \end{aligned}$$

So, to show that the induction argument is valid, and we need only show

$$\begin{aligned} (1-\delta)(1-\delta^t) + t\delta^t &> \delta^{t+2} + t\delta^{t+1} \Leftrightarrow \\ (1-\delta-\delta^t) + \delta^{t+1}(1-\delta) + t\delta^t(1-\delta) &> 0. \end{aligned}$$

The first term is positive for  $\delta \leq 1/2$  and the other two terms are clearly positive. Thus, the deviation is profitable for the seller. ■

*(ii) Approach Conditions.*

The cut-off rules for buyers in periods  $\tau < t$  are to reject any (upgrade) offer for  $\sigma \leq \tau$  units at a price greater than  $p(\sigma, \tau)$ , where  $p(\sigma, \tau)$  satisfies

$$\frac{v\sigma(1-\delta^{t-\tau})}{(1-\delta)} + \delta^{t-\tau} \max \left[ \frac{v\sigma}{(1-\delta)}, u_t \right] - \delta^{t-\tau} u_t \leq p(\sigma, \tau) \leq \frac{v\sigma}{(1-\delta)} + \delta^{t-(\tau-\sigma)} u_t. \quad (11)$$

Note that we have used the continuation properties  $\delta^{t-\tau}u_t = u(\tau + 1, 0)$  and  $\delta^{t-(\tau-\sigma)}u_t = u(\tau + 1 - \sigma, 0)$  as the first sale on the equilibrium path occurs in state  $(t, 0)$ . The left-hand-side of (11) provides the lower bound on the cut-off price; otherwise, an individual buyer would be better off accepting when other buyers reject. This bound reflects the difference in gross surplus for an individual buyer between buying and rejecting, since other buyers are expected to reject and, hence, the continuation state would be  $(\tau + 1, 0)$ . The first term is the buyer's interim flow payoff, from  $\tau$  until  $t$ , generated by  $\sigma$  units while the second terms correspond to the option of buying (or not) with the other buyers once the state reaches  $(t, 0)$ . The right-hand-side provides an upper bound on the cut-off price; if it failed, an individual buyer would be better off rejecting when others accept. Given that all other buyers are expected to buy the package and the state will be  $(t + 1, \sigma)$ , the bound reflects the payoff difference for an individual buyer between buying and not buying the offer for  $\sigma$  units. Clearly, a set of prices exists that satisfies (11) for  $\sigma = 1, \dots, \tau$  and  $\tau = 1, \dots, t - 1$ . As with efficient equilibria, this is due to the implicit coordination on cut-off prices among buyers.

Given the cut-off prices, delay must be optimal for the seller. Thus, in state  $(\tau, 0)$  for  $\tau < t$ , the seller prefers the equilibrium path payoff of  $\pi_\tau = \delta^{t-\tau}\pi_t$  to selling  $\sigma$  units in period  $\tau$  at a price of  $p = p(\sigma, \tau)$  and receiving a payoff of  $p(\sigma, \tau) + \delta^{t-(\tau-\sigma)}\pi_t$ . We then have

$$\delta^{t-\tau}(1 - \delta^\sigma)\pi_t \geq p(\sigma, \tau) \quad (12)$$

for  $\sigma = 1, \dots, \tau$  and  $\tau = 1, \dots, t - 1$ . The next lemma greatly simplifies the analysis of these buyer and seller approach conditions.

**Lemma A5.** If the buyer and seller approach conditions, (11) and (12), hold for  $\sigma = \tau$ , at each  $\tau = 1, \dots, t - 1$ , then the conditions hold for all feasible pairs  $(\sigma, \tau)$ .

*Proof of Lemma A5.* Beginning with the seller approach conditions, (12), note that  $\delta^{t-\tau}$  is strictly increasing in  $\tau$ . Thus, we can set  $p(\sigma, \tau) = p(\sigma, \sigma)$  and the condition holds for  $\sigma < \tau$ . Now, for the buyer approach conditions, (11), note that the right hand side is increasing in  $\tau$ . We claim that the left hand side is decreasing in  $\tau$ . When  $\frac{v\sigma}{(1-\delta)} \geq u_t$ , then the left hand side is  $\frac{v\sigma}{(1-\delta)} - \delta^{t-\tau}u_t$ , which is falling in  $\tau$ . If  $\frac{v\sigma}{(1-\delta)} < u_t$ , then the left hand side is  $\frac{v\sigma(1-\delta^{t-\tau})}{(1-\delta)}$ , which is also strictly decreasing in  $\tau$ . Again setting  $p(\sigma, \tau) = p(\sigma, \sigma)$  for  $\sigma < \tau$  is sufficient. ■

As a result, we need only find  $t - 1$  distinct prices,  $p(1, 1), \dots, p(t - 1, t - 1)$  and it is sufficient to deter the seller from selling the maximum feasible number of units, “cashing-in,” in each delay period. Intuitively, if it is not profitable to sell  $\tau$  units in period  $\tau$ , the

first time it is possible to do so, it will not be profitable to sell  $\tau$  units in a later delay state. For example, if the seller does not offer one unit in state  $(1, 0)$ , then there will be no temptation to sell one unit at a later date when the additional unsold units will create a longer delay in the continuation state.

In view of the need to provide a positive payoff for buyers, how can we ensure that the delay incentives are satisfied? Combining the buyer and seller approach conditions, (11) and (12), we see that supporting prices exist if and only if

$$\frac{v\tau(1 - \delta^{t-\tau})}{(1 - \delta)} + \delta^{t-\tau} \max \left[ \frac{v\tau}{(1 - \delta)}, u_t \right] - \delta^{t-\tau} u_t \leq p(\tau, \tau) \leq \delta^{t-\tau} (1 - \delta^\tau) \pi_t \quad (13)$$

for  $\tau = 1, \dots, t - 1$ . Condition (7) in the text follows from (13) upon simplifying and substituting with  $\pi_t + u_t = \Psi_t$ .

*Proof of Lemma 5.* We will show (i)  $0 < u_1 < \delta S_1$  and (ii)  $\max_{k=1, \dots, t-1} \left( \frac{\delta^{k-1} vk}{1 - \delta} \right) < \pi_1 < S_1$ . When  $u_t = 0$ , for any deviation offer by the seller in period  $\tau$  for  $\sigma$  units, condition (11) immediately reduces to  $p(\sigma, \tau) = \frac{v\sigma}{(1 - \delta)}$ . Thus, take  $\sigma = \tau = t - 1$  and consider condition (12). Since  $\pi_t = \Psi_t$ , when  $u_t = 0$ , we have

$$\delta(1 - \delta^{t-1})\Psi_t \geq \frac{v(t-1)}{(1 - \delta)} \Leftrightarrow \frac{1 - \delta^t}{1 - \delta} \geq t$$

after simplification. Clearly, this is always false for  $t \geq 2$  and any  $\delta$ .

To establish the lower bound for the seller's payoff we apply the bounds in (13) at  $\sigma = \tau = 1$  and obtain

$$(1 - \delta)\pi_t \geq \frac{v}{1 - \delta}(\delta^{1-t} - 1).$$

Since  $\pi_1 = \delta^{t-1}\pi_t$  on the equilibrium path, for  $\pi_1 \geq \delta^{k-t}vk/(1 - \delta)$  for  $k = 1, \dots, \tau - 1$ , it is sufficient to show

$$\frac{v}{(1 - \delta)^2}(\delta^{1-t} - 1) > \frac{\delta^{k-t}vk}{1 - \delta},$$

is satisfied. This holds if and only if

$$\frac{\delta - \delta^t}{1 - \delta} > k\delta^k,$$

which always holds, since  $k \leq t - 1$ .

Now, we establish the upper bounds on payoffs. First, for the buyer, by definition we have  $u_1 = \delta^{t-1}(\Psi_t - \pi_t) = \delta^{t-1}\Psi_t - \pi_1$ . Since  $\delta^{t-1}\Psi_t < S_1$  and  $\pi_1 > v/(1 - \delta)$ , from above,

we have  $u_1 < \delta S_1$ . For the seller, by definition  $\pi_1 = \delta^{t-1}(\Psi_t - u_t)$ . Since  $u_1 > 0$ , from above,  $\delta^{t-1}\Psi_t < S_1$  then implies  $\pi_1 < S_1$ . ■

*Proof of Lemma 6.* In (7),  $\tau$  assumes integer values  $1, \dots, t-1$ . Let us replace  $\tau$  with a continuous variable,  $x$ , that assumes values in the interval  $[0, t]$ . This greatly simplifies the derivation of the sufficiency condition. It is useful to define three functions:

$$A(x, u, \delta, t) \equiv (\delta^{t-x} - \delta^t) \left[ \frac{vt}{(1-\delta)(1-\delta^t)} - u \right]$$

$$B(x, \delta, t) \equiv \frac{vx}{1-\delta}(1-\delta^{t-x})$$

$$C(x, u, \delta, t) \equiv \frac{vx}{1-\delta} - \delta^{t-x}u,$$

where  $u \equiv u_t$ . For  $(\delta, t)$ , consider  $u \in [0, \frac{vt}{1-\delta}]$ ; we treat the case of  $u > \frac{vt}{1-\delta}$  later in the proof. In terms of  $x$ , condition (7) becomes

$$\begin{aligned} A(x, u, \delta, t) &\geq B(x, \delta, t) \quad \text{for } 0 \leq x \leq \frac{(1-\delta)u}{v}, \\ A(x, u, \delta, t) &\geq C(x, u, \delta, t) \quad \text{for } \frac{(1-\delta)u}{v} < x \leq t. \end{aligned}$$

First, consider when  $A(x, u, \delta, t) \geq B(x, \delta, t)$  for all  $x$  in the interval  $[0, \frac{(1-\delta)u}{v}]$ . Suppressing arguments, note that  $A$  is increasing and convex in  $x$ , and equals 0 when  $x = 0$ , while  $B$  is strictly concave in  $x$  and equals 0 when  $x = 0$ . Thus, if we have  $\frac{\partial A}{\partial x} \geq \frac{\partial B}{\partial x}$  at  $x = 0$ , then  $A \geq B$  must hold for all positive  $x$ . Calculating the partial derivatives, this yields the condition

$$\bar{u}^A \equiv \frac{v}{(1-\delta)} \left[ \frac{t}{1-\delta^t} - \frac{1-\delta^t}{(-\ln \delta)\delta^t} \right] \geq u. \quad (14)$$

Next, consider when  $A(x, u, \delta, t) \geq C(x, u, \delta, t)$  for all  $x$  in the interval  $[\frac{(1-\delta)u}{v}, t]$ . This condition simplifies to

$$\frac{\delta^t(1-\delta)u}{v} \geq x - \frac{t(\delta^{t-x} - \delta^t)}{1-\delta^t} \equiv h(x, t, \delta). \quad (15)$$

It is easy to show  $h(x, t, \delta)$  is strictly concave and equals 0 at  $x = 0$  and  $x = t$ . Thus,  $h(x, t, \delta)$  has a unique interior maximum at some  $x^*(\delta, t)$  which is implicitly defined by  $\delta^{x^*}(1-\delta^t) = -\delta^t \ln \delta^t$ . Note that when condition (14) holds, the term in brackets must

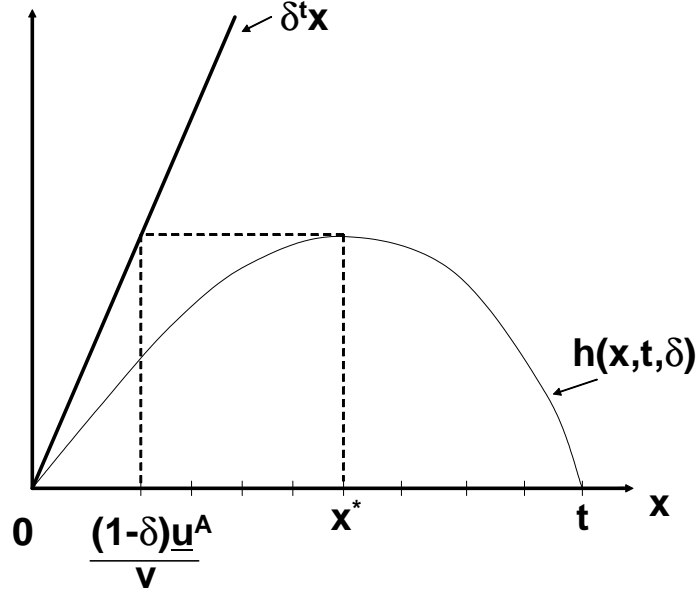


Figure 3: Payoff Lower Bound for Approach Conditions

be positive; this implies that  $\delta^t > \frac{\partial h}{\partial x}$  at  $x = 0$ . Refer to Figure 3.

Define  $\underline{u}^A$  by

$$\begin{aligned}
 \underline{u}^A &\equiv \frac{v}{\delta^t(1-\delta)} h(x^*, t, \delta) \\
 &= \frac{v}{\delta^t(1-\delta)} \left[ x^* - \frac{t\delta^{t-x^*} - t\delta^t}{1-\delta^t} \right] \\
 &= \Psi_t + \frac{v(1-a(\delta^t))}{\delta^t(1-\delta)\ln\delta},
 \end{aligned}$$

where the function  $a(\delta^t)$  is defined below. Then any  $u \geq \underline{u}^A$  will necessarily satisfy (15). See Figure 3.

In order to find when (14) and (15) hold at a candidate  $u$ , we compare the bounds to obtain,

$$\bar{u}^A \geq \underline{u}^A \quad \Leftrightarrow \quad \delta^t \geq a(\delta^t),$$

where we define  $a(d) \equiv -\ln \left[ -\frac{d \ln(d)}{1-d} \right]$  for  $d \in (0, 1)$ . It is easy to show that  $a(d)$  is strictly decreasing for  $d \in (0, 1)$  and, by L'Hospital's Rule, that  $\lim_{d \rightarrow 1} a(d) = 0$ . Then, the equation  $d = a(d)$  has a unique root,  $d^*$  in  $(0, 1)$ . Thus, we have established that

$\delta^t \geq d^*$  implies the interval  $(\underline{u}^A, \bar{u}^A)$  is non-empty.

The last step of the proof is to consider the range of values for  $u$  that can be supported. A straightforward argument establishes that  $\underline{u}^A < \frac{vt}{1-\delta}$  holds for  $\delta^t \geq d^*$ . The comparison of  $\bar{u}^A$  with  $\frac{vt}{1-\delta}$  reveals that  $\bar{u}^A$  crosses  $\frac{vt}{1-\delta}$  exactly one time, from below, at the root of the equation  $-\ln \delta^t = (\delta^{-t} - 1)^2$ ; numerically, the root is .572, which exceeds  $d^*$ . Thus, for  $\delta^t$  below this root, we have  $\bar{u}^A < \frac{vt}{1-\delta}$  and any  $u \in (\underline{u}^A, \bar{u}^A)$  satisfies the approach condition (7). For  $\delta^t$  above this root, we have  $\bar{u}^A > \frac{vt}{1-\delta} > \underline{u}^A$  and there are two cases. First, by the above analysis, any  $u \in (\underline{u}^A, \frac{vt}{1-\delta})$  satisfies the approach condition (7). Second, for the case of  $u \in [\frac{vt}{1-\delta}, \bar{u}^A)$ , the approach condition (7) requires that  $A(x, u, \delta, t) \geq B(x, \delta, t)$  for all  $x$  in the interval  $[0, t]$ ; note that since  $u$  is large, the case of  $\frac{(1-\delta)u}{v} < x \leq t$  never arises. Clearly, we have  $A(x, u, \delta, t) \geq B(x, \delta, t)$  and utilities in this payoff range satisfy (7). ■

(iii) *Off Equilibrium Support (Cash-in offers)*

Now, we deal with off equilibrium states  $(\tau, 0)$  where  $\tau > t$ . Since we are using cash-in supports, the payoffs are  $\pi_\tau = p_\tau + \delta\pi_1$  for the seller and  $u_\tau = v\tau - p_\tau + \delta u(\tau + 1, \tau) = \frac{v\tau}{1-\delta} - p_\tau + \delta u_1$  for buyers. Note that from  $(\tau, 0)$  the surplus on the continuation path is

$$\Psi_\tau \equiv \frac{v\tau}{1-\delta} + \delta\Psi_1 = \pi_\tau + u_\tau \text{ for } \tau > t$$

and that cash-in states contrast with delay states,  $\tau < t$ , where we have  $\Psi_\tau \equiv \delta^{t-\tau}\Psi_t$ .

Recall from the text that we must satisfy (1) and (2). Thus, we seek a utility sequence that satisfies the analog of (3), as given by

$$\Psi_\tau - \delta\Psi_{\tau+1-\sigma} \geq u_\tau - \delta u_{\tau+1-\sigma} + g(\sigma, u_{\tau+1}) \quad (16)$$

for  $\tau \geq t$  and  $\sigma = 0, \dots, \tau$ .

Analogous to our support utility sequence for efficient equilibria, we define a  $T$ -stage support sequence, where  $T \geq t + 1$ , for inefficient equilibria by

$$u_\tau = \delta u_{\tau+1} + \Psi_\tau - \delta\Psi_{\tau+1} \text{ for } \tau = t, \dots, T - 1, \quad (17)$$

and

$$u_\tau = u_T \text{ for } \tau \geq T.$$

For the special case of  $T = t$ , we specify a constant sequence  $u_\tau = u_t$  for all  $\tau$ . As before, the seller is indifferent with respect to delay and the cash-in up to period  $T$ , and the seller

prefers to cash-in in period  $T$  provided that the buyers' utilities,  $u_T$ , are not too large.

We can satisfy (16) by an appropriate choice of the support length  $T$  for a given  $\delta$  and buyer payoff. Recalling Figure 1 for efficient equilibria, we find a similar structure for the relationship between  $u_t$ ,  $\delta$ , and the length of a  $T$ . For any given  $\delta$  we can support successively higher buyer payoffs by increasing the support horizon,  $T$ . In particular as  $u_t$  rises, then  $T$  must also rise until we achieve  $u_t \in [\Psi_t - \delta^{T-t}S_1, \Psi_t - \delta^{T-t+1}S_1]$ . Thus, we demonstrate

**Lemma A6** *Let  $T \geq t$  and suppose  $\delta + \delta^T \geq 1$ . Let buyer utilities follow a  $T$ -stage support sequence of horizon  $T$ , where  $u_t$  satisfies (i)  $0 \leq u_t \leq \Psi_t - \delta S_1$  for  $T = t$  and (ii)  $\Psi_t - \delta^{T-t}S_1 \leq u_t \leq \Psi_t - \delta^{T-t+1}S_1$  for  $T > t$ . Then (16) holds for every  $\tau \geq t$  and  $\sigma = 0, \dots, \tau$ .*

*Proof.* We only sketch the arguments, since they are similar to those of the efficient case. First, we note that if expression (16) holds when buyer utility first becomes constant at  $\tau = T$  and for all  $\sigma \leq T$ , then it holds for all  $\tau > T$  for all  $\sigma \leq \tau$ . This follows from arguments very similar to those used in Lemma A1. We must, however, replace  $S_\tau$  with  $\Psi_\tau$ . Also, we need to take into account a new range of large  $\sigma$ ,  $\sigma \geq T + 1 - t$ , where the continuation state involves delay. Thus, it only remains to verify the support conditions for  $\tau = t, \dots, T$ .

Second, it is straightforward to show that every partial cash-in constraint, where  $0 < \sigma < \tau$ , is implied by the cash-in constraint  $\sigma = \tau$  and the delay constraint  $\sigma = 0$  for each  $\tau = t, \dots, T - 1$ . Recall that delay constraints for  $\tau = t, \dots, T - 1$  hold by construction of the  $T$ -stage support. The delay constraint, (16) for  $\tau = T$  and  $\sigma = 0$ , reduces to

$$\frac{\delta v}{1 - \delta} \leq (1 - \delta)(\Psi_T - u_T).$$

This holds if  $u_T \leq \Psi_T - \delta S_1$ , which is implied by the assumed range for  $u_t$ . Thus, only the cash-in constraints  $\tau = t, \dots, T$  remain to be checked. Also, (16) is easy to verify for the special case of  $t = T$ . Henceforth, we take  $T > t$ .

We begin with the cash-in constraint at  $\tau = T$ . The condition for  $\tau = T$  and  $\sigma = T$  is

$$g(T, u_T) \leq (1 - \delta^T)(\Psi_T - u_T). \tag{18}$$

By the range for  $u_t$  and the  $T$ -stage construction, we find that  $u_T$  must be less than

$\frac{vT}{1-\delta}$ . Thus, (18) becomes

$$(1 - \delta - \delta^T)u_T \leq (1 - \delta^T)\Psi_t - \frac{vT}{1 - \delta}.$$

This holds since the l.h.s. is negative, by the lemma hypothesis, and, via calculations, the r.h.s. is positive. Thus, we have verified that condition (16) holds for  $\tau = T$  and  $\sigma = T$ .

Verifying (16) is more subtle when  $\tau < T$ . We need the following two claims for the relationship between  $u_\tau$  and  $\frac{v(\tau-1)}{1-\delta}$  for  $\tau = t, \dots, T$  (the proofs involve only routine algebra with the  $T$ -stage utility path and are hence omitted). The first claim is

*Claim 1. (i) If  $u_\tau < \frac{v(\tau-1)}{1-\delta}$ , then  $u_{\tau+1} < \frac{v\tau}{1-\delta}$  for all  $\tau = t+1, \dots, T$ ; (ii) if  $u_{\tau+1} \geq \frac{v\tau}{1-\delta}$ , then  $u_\tau \geq \frac{v(\tau-1)}{1-\delta}$  for all  $\tau = t, \dots, T$ .*

This claim provides a simple algorithm for verifying the interior support constraints. Recall that  $u_T < \frac{vT}{1-\delta}$ . If  $u_T > \frac{v(T-1)}{1-\delta}$ , then Claim 1 implies  $u_\tau > \frac{v(\tau-1)}{1-\delta}$  for all  $\tau \leq T-1$ . If  $u_T < \frac{v(T-1)}{1-\delta}$ , however, then we move to  $\tau = T-2$  and compare  $u_{T-1}$  with  $\frac{v(T-2)}{1-\delta}$ . If  $u_{T-1}$  is greater, then  $u_\tau > \frac{v(\tau-1)}{1-\delta}$  for all  $\tau \leq T-2$ . If not, continue down until either  $u_\tau < \frac{v(\tau-1)}{1-\delta}$  or  $\tau = t$ . If at every  $\tau$ , it is always the case that  $u_\tau < \frac{v(\tau-1)}{1-\delta}$ , then we are in the "small" utility case. Otherwise, there always exists a  $\hat{\tau}$  such that if  $\tau \geq \hat{\tau}$ , then  $u_{\tau+1} < \frac{v\tau}{1-\delta}$  and if  $\tau < \hat{\tau}$ , then  $u_{\tau+1} > \frac{v\tau}{1-\delta}$ . Since  $u_T < \frac{vT}{1-\delta}$  there is always a  $\hat{\tau} \geq t$  such that  $u_\tau < \frac{v(\tau-1)}{1-\delta}$ .

*Claim 2. (i) Suppose  $u_\tau < \frac{v(\tau-1)}{1-\delta}$ , then if (16) holds at  $\tau$  it holds at  $\tau-1$  for any  $\tau \leq T$ . (ii) Suppose  $u_\tau > \frac{v(\tau-1)}{1-\delta}$ , then if (16) holds at  $\tau-1$  it holds at  $\tau$  for any  $\tau \leq T$ .*

We now employ these results to complete the proof of Lemma A5. We need to show (16) holds at  $\tau = t, \dots, T-1$ . There are two cases. First, suppose  $u_{t+1} > \frac{vt}{1-\delta}$ . Then it is straightforward to show that (16) holds at  $\tau = t$ . Now apply (ii) in Claim 1 at  $\tau = t$  and (ii) in Claim 2 to see that (16) holds at  $\tau = t+1, \dots, T-1$ . The second case is  $u_{t+1} \leq \frac{vt}{1-\delta}$ . From before, we know (16) holds at  $\tau = T$ . We can now apply the (i) in Claim 1 at  $\tau = T$  and (i) in Claim 2 to see that (16) holds at  $\tau = T-1, \dots, t$ . This completes the proof. ■

#### (iv) Equilibrium Existence

We have the following lemma that provides sufficient conditions for equilibrium existence.

**Lemma A7** *Suppose a sequence of buyer utilities  $u_\tau$  satisfies (7) and (16) for some  $t \geq 2$  and for all  $\sigma$  and  $\tau$ , such that  $0 \leq \sigma \leq \tau$  for any  $\tau \geq 1$ . Then there exists an inefficient  $t$ -cycle equilibrium with supporting prices  $p(\sigma, \tau)$ .*

*Proof.* The approach conditions are satisfied by Lemma 6 for  $u_t \in (\underline{u}^A, \bar{u}^A)$ . With (16)

satisfied as well, we know that no single-bundle deviation is profitable for the seller. As with the efficient case, we must now show that the seller cannot profitably deviate to an upgrade offer set in state  $(\tau, 0)$ . While the logic is similar to that for efficient equilibria, the proof is more complicated and we only provide a brief sketch. We must distinguish between approach states, where  $\tau < t$ , and support states where  $\tau \geq t$ . We define a *BCE* based on the same buyer preference relation  $\succeq_B$  but now  $u(\tau + 1, b)$  and  $u(\tau + 1, b | i)$  must account for whether there is a delay or a cash-in the continuation state. We can then show the same arg max construction yields a *BCE*. Next, note that any upgrade offer can be written in the form  $\mathcal{B} = \mathcal{I} \cup \mathcal{J} \cup \mathcal{K}$  where an upgrade to  $b$  is in  $\mathcal{I}$  if  $p_b \leq G_b$ , in  $\mathcal{J}$  if  $p_b \geq H_b$ , and in  $\mathcal{K}$  if  $G_b < p_b < H_b$ . The threshold prices, following the buyer and seller decisions in the text at (16), are specified by

$$G_b = \begin{cases} \frac{vb}{1-\delta} (1 - \delta^{t-\tau}) + \delta^{t-\tau} \max \left\{ \frac{vb}{1-\delta}, u_t \right\} - \delta^{t-\tau} u_t & \text{if } \tau < t \\ vb + \delta \max \left\{ \frac{vb}{1-\delta}, u_{\tau+1} \right\} - \delta u_{\tau+1} & \text{if } \tau \geq t \end{cases}$$

and

$$H_b = \begin{cases} \delta^{t-\tau} (1 - \delta^b) (\psi_t - u_t) & \text{if } \tau < t \\ (\psi_\tau - u_\tau) - \delta (\psi_{\tau+1-b} - u_{\tau+1-b}) & \text{if } \tau \geq t \end{cases} .$$

The proof then proceeds by considering the various cases for the decomposition  $\mathcal{B} = \mathcal{I} \cup \mathcal{J} \cup \mathcal{K}$ . For instance, if  $\mathcal{J}$  is empty then we are done as the argmax property can be applied to  $\mathcal{I} \cup \mathcal{K}$  and we can specify that buyers choose this *BCE*, which is priced below  $H_b$ . The most difficult case arises when none of the component sets in  $\mathcal{I} \cup \mathcal{J} \cup \mathcal{K}$  is empty. We can show that there always exists a *BCE* priced below  $H_b$  in this case.

The next step is to specify continuation equilibrium strategies at asymmetric buyer states,  $(\tau, \alpha)$ ; see part (i) of Appendix C for the definition. These states are complicated by the fact that joint surplus is not maximized in an inefficient  $t$ -cycle equilibrium. This contrasts with the efficient case, where the  $(\tau, \alpha)$  offers fully extracted buyers relative to equilibrium payoffs for the continuation state of  $(\tau + 1, \tau)$ , since feasibility will no longer imply this is an optimal offer for the seller. Instead, we specify a continuation equilibrium that leaves buyers asymmetric. Thus, this part of the support neither returns to the equilibrium path nor to a cash-in support state.

Let  $\alpha_m \equiv \inf \{ \alpha_\sigma \mid \alpha_\sigma > 0, \}$  denote the smallest non-zero mass of buyers in  $\alpha$  and if the inf is not unique let  $m$  be the largest index (e.g., if  $\alpha_\sigma = 1/\tau$  for all  $\sigma$  then  $\alpha_m = 1/\tau$  and  $m = \tau - 1$ ). We specify that the seller offers a collection of upgrades

$\{(b_\sigma, p_\sigma)\}_{\sigma=0, \dots, \tau-1}$  with prices of

$$p_m = \frac{v(\tau - 1 - m)}{1 - \delta}, \quad \text{and} \quad p_\sigma = \frac{v(\tau - \sigma)}{1 - \delta} \quad \text{for } \sigma \neq m.$$

It is straightforward to verify that all buyers optimally accept and that the continuation state is of the form  $(\tau + 1, \alpha')$  where  $\alpha' = (0, \dots, 0, \alpha_m, 1 - \alpha_m)$ . Thus, in period  $\tau + 1$  the smallest mass group is at quality level  $\tau - 1$  units, while all other buyers are at  $\tau$ . We then apply the same logic to specify offers for  $(\tau + 1, \alpha')$  and all resulting states so that the distribution of buyers is always concentrated in the same proportions  $(\alpha_m, 1 - \alpha_m)$  at the two highest quality levels.

The argument that this is an optimal strategy for the seller is involved, so we only offer a brief sketch. For any arbitrary seller offer in  $(\tau, \alpha)$  and pattern of buyer choices, one can bound the seller payoff via individual rationality of the buyer choices, using the hypothesis that the seller makes the offer in the continuation state in period  $\tau + 1$  that induces a concentration of buyers at the two highest quality levels (if the buyer choices lead to a degenerate distribution, then payoffs must instead follow the  $t - cycle$  equilibrium path and cash-in support). Then, since the repeated concentration of buyers in proportions  $(\alpha_m, 1 - \alpha_m)$  at the two highest quality levels allows the seller to extract all surplus generated on this path, the associated seller payoff can be shown to be at least as large as the bound derived from individual buyer rationality for any other seller offer. In this regard, note that every equilibrium with delay ( $t \geq 2$ ) has a buyer quality level that is strictly below the mean for the  $(\alpha_m, 1 - \alpha_m)$  concentration.

Finally, finding a *BCE* for an arbitrary upgrade offer  $\mathcal{B}$  in state  $(\tau, \alpha)$  follows the same basic approach as that for an efficient equilibrium. The only difference is that we must take care to account for any possible delay if the continuation state is degenerate. ■

*Proof of Proposition 5.* Define  $B(t, T, \delta) \equiv \Psi_t - \delta^{T-t} S_1$ , the upper bound of the payoffs that can be supported with a  $T - stage$  support. We will show that  $B(t, T, \delta) \geq \bar{u}^A(t, \delta)$ , whenever  $\delta$  satisfies the approach conditions, thus proving the proposition. Comparing,  $B(t, T, \delta) \geq \bar{u}^A(t, \delta) \Leftrightarrow$

$$\frac{(1 - \delta^t)(1 - \delta)}{\delta^T} \geq -\ln \delta \tag{19}$$

It is clear that both sides of (19) are positive, falling in  $\delta$  and equal to 0 at  $\delta = 1$ . The l.h.s. of (19) is rising in  $T$  and  $t$ . Differentiating both sides of (19) with respect to  $\delta$  we

find that the slope of the l.h.s. is greater/less than the slope of the r.h.s. if and only if

$$\delta^T + \delta^{t+1} \gtrless T(1 - \delta) + \delta - (T - t)\delta^t(1 - \delta) \quad (20)$$

In (20), the l.h.s. is positive and convex in  $\delta$  and the r.h.s. is decreasing in  $\delta$ . Thus, the l.h.s. and the r.h.s. of (19) cross a single time in  $\delta$ . It easy to see that (19) is negative in a neighborhood of 1 and positive at  $\delta = \sqrt[t]{d^*}$  for any  $T \geq t$ . So, there is a  $b(t, T) \in (\sqrt[t]{d^*}, 1)$  such that (19) holds whenever  $\delta \leq b(t, T)$ . Clearly,  $b(t, T)$  is increasing in  $T$ .

Now, we will show that  $b(t, T) > \delta_T$  for all  $t \geq 2$ . At  $b = b(t, T)$  we have

$$\frac{1 - b}{b^T} = \frac{-\ln b}{1 - b^t}$$

The r.h.s. is falling in  $b$ . Since for any  $b$  this is less than 1,  $b(t, T) > \delta_T$ , the result is proved. ■