Adaptive Wavelet Decompositions of Stationary (Gaussian) Processes

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General Remarks

References

• [DP06a] "Gaussian stationary processes: adaptive wavelet decompositions, discrete approximations and their convergence", G. Didier and V. Pipiras, 2006, Preprint.

• [DP06b] "Adaptive wavelet decompositions of stationary time series", G. Didier and V. Pipiras, 2006, Preprint.

About

Particular, called adaptive, wavelet decompositions (AWD), of general stationary (Gaussian) processes X in continuous time [DP06a] and discrete time [DP06b]. They are characterized by:

- independent (uncorrelated) detail coefficients $d_{j,n}$;
- possibly correlated approximations coefficients $X_{j,n}$;
- non(bi)orthogonal wavelet basis (adapted to the covariance of X);
- FWT-like algorithm (with filters possibly depending on scale j).

In addition,

• [DP06a]: $2^{j/2}X_{j,[2^jt]} \approx X(t)$ at small scales ("wavelet crime"). Practical conditions for this to take place are provided;

• [DP06b]: associated low and high pass filters decay to zero fast. Zero moments play here a key role.

Other issues addressed

- Riesz bases [DP06a].
- Convergence of discrete approximations $X_{j,n}$ [DP06a].
- Many examples [DP06a,b].

On significance of AWD

- New wavelet-based decompositions with independent coefficients.
- It works for general stationary processes; AWD in continuous time was previously obtained only for fractional Brownian motion.
- It allows for practical applications: simulations in [DP06a,b], MLE in [DP06b], etc.

Other references

• V. Ahn et al. "Fractional-order regularization and wavelet approximation to the inverse estimation problem for random fields", Journal of Multivariate Analysis, 2003.

• A. Benassi, and S. Jaffard, "Wavelet decomposition of one- and several-dimensional Gaussian processes", Recent Advances in Wavelet Analysis, 1994.

• D. Donoho, "Nonlinear solution of linear inverse problems by wavelet-vaguelette decomposition", Applied and Computational Harmonic Analysis, 1995.

• Y. Meyer, F. Sellan, and M. Taqqu, "Wavelets, generalized white noise and fractional integration: the synthesis of fractional Brownian motion", The Journal of Fourier Analysis and Applications, 1999.

• J. Zhang, and G. Walter, "A wavelet-based KL-like expansion for wide-sense stationary random processes", IEEE Transactions on Signal Processing, 1994, [ZW94].

[DP06a] Example: Ornstein-Uhlenbeck Process

Representation and earlier decomposition

It is known that

$$X(t) = \int_{-\infty}^{t} e^{-\lambda(t-u)}dB(u) = \int_{-\infty}^{\infty} e^{itx}\frac{1}{\lambda+ix}d\widehat{B}(x)$$
$$\overset{[ZW94]}{=} \sum_n a_{J,n}\theta^J(t-2^{-J}n) + \sum_{j\geq J}\sum_n d_{j,n}\Psi^j(t-2^{-j}n),$$

where $a_{J,n}$, $d_{j,n}$ are independent $N(0,1)$ and (with a scaling function ϕ and a wavelet ψ of an orthogonal MRA)

$$\widehat{\theta}^J(x) = (\lambda+ix)^{-1}2^{-J/2}\widehat{\phi}(2^{-J}x), \quad \widehat{\Psi}^j(x) = (\lambda+ix)^{-1}2^{-j/2}\widehat{\psi}(2^{-j}x).$$

AWD

$$X(t) \overset{[DP06a]}{=} \sum_n X_{J,n}\Phi^J(t-2^{-J}n) + \sum_{j\geq J}\sum_n d_{j,n}\Psi^j(t-2^{-j}n),$$

where X_J is AR(1) time series with

$$\widehat{X}_J(x) = 2^{-J}(1-e^{-(\lambda 2^{-J}+ix)})^{-1}\widehat{a}_J(x), \quad \widehat{\Phi}^J(x) = \frac{2^{J/2}(1-e^{-2^{-J}(\lambda+ix)})}{\lambda+ix}\widehat{\phi}(2^{-J}x).$$

FWT-like algorithm (at reconstruction)

$$X_{j+1} = u_j * \uparrow_2 X_j + v_j * \uparrow_2 d_j,$$

where (with Conjugate Mirror Filters u, v)

$$\widehat{u}_j(x) = \frac{1}{2}(1+e^{-(\lambda 2^{-(j+1)}+ix)})\widehat{u}(x), \quad \widehat{v}_j(x) = 2^{-(j+1)}(1-e^{-(\lambda 2^{-(j+1)}+ix)})^{-1}\widehat{v}(x).$$

Because of zero moments, v_j becomes essentially finite at small scales as $j \to \infty$ (if v is finite).

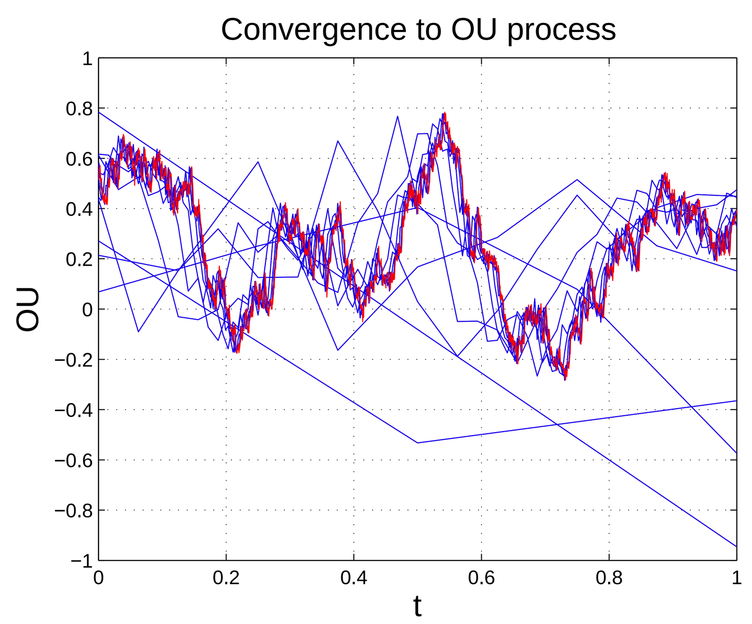
"Wavelet crime"

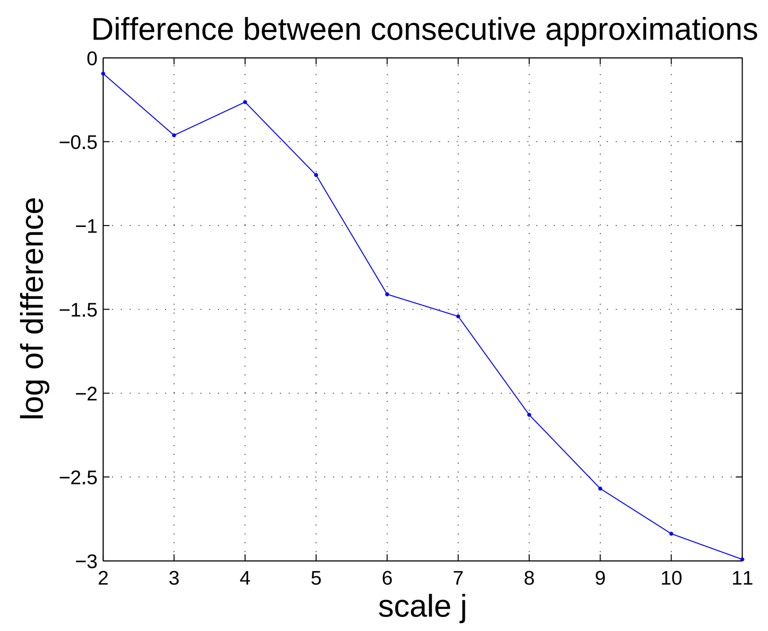
$$\sup_{t\in K}|2^{J/2}X_{J,[2^Jt]} - X(t)| \leq C2^{-J\gamma} \quad \text{a.s.},$$

where K is a compact interval and $\gamma > 0$ is a sample path smoothness parameter. The key here is that

$$2^J\frac{(1-e^{-2^{-J}(\lambda+ix)})}{\lambda+ix} \approx 1, \quad \text{as} \quad J\to\infty.$$

Simulation based on AWD





[DP06b] Example: FARIMA$(0, s, 0)$ time series

Representation

$$X_0 = (I-B)^{-s}\epsilon^0, \quad s\in(-1/2, 1/2)$$

AWD

$$X_j = \downarrow_2(\overline{U} * X_{j-1}), \quad d_j = \downarrow_2(\overline{V} * X_{j-1}), \quad j\geq 1,$$

with

$$\widehat{U}(\omega) = (1+e^{i\omega})^{-s}\widehat{u}(\omega), \quad \widehat{V}(\omega) = (1-e^{i\omega})^{s}\widehat{v}(\omega).$$

One can show that d_j are independent white noise sequences, and X_j are FARIMA$(0, s, 0)$ time series.

Filter decay vs zero moments

$(1+e^{i\omega})^{-s}$, $(1-e^{i\omega})^{s}$ decay extremely slowly, but not $\widehat{U}(\omega)$, $\widehat{V}(\omega)$, since, for instance,

$$\widehat{U}(\omega) = (1+e^{i\omega})^{-s+N}\widehat{u}_{0,N}(\omega),$$

where N is the number of zero moments of the orthogonal MRA.

In practice

Replace $*$ by $\circledast$. Fast decay of U, V implies smaller border effect.

Gaussian MLE based on AWD

For a data row vector X^0, the negative log-likelihood is

$$\log|\Sigma_\theta| + X^0\Sigma_\theta^{-1}X^{0\prime} \approx \frac{T}{\pi}\int_0^{2\pi}\log|\widehat{a}_\theta(\omega)|d\omega + Y_\theta Y_\theta',$$

where θ indicates unknown parameters, T is the size of X^0,

$$Y_\theta = X^0M_\theta$$

is the detail vector in AWD with $\circledast$, and M_θ is an invertible AWD matrix. The key to have above is the approximate covariance matrix factorization

$$\Sigma_\theta^{-1} \approx M_\theta M_\theta'.$$

Potential advantages of MLE based on AWD

Advantages over MLE based on orthogonal wavelet decompositions: the exact MLE based on orth. wavelet decompositions involves complex dependence structure of its detail coefficients. By construction, detail coefficients are independent, $N(0,1)$ variables in AWD.

Advantages over non-wavelet based MLE: MLE based on AWD is potentially **invariant to polynomial trends** in data.

MLE results for FARIMA$(0, s, 0)$

True parameter	Sample size T	Whittle bias	Whittle rMSE	N zero moments	AWD bias	AWD rMSE	Orth. WD bias	Orth. WD rMSE
$s = 0.3$	2^{10}	0.0003	0.0246	2	-0.0069	0.0270	-0.0079	0.0256
				6	-0.0055	0.0258	-0.0082	0.0253
	2^{14}	0.0002	0.0062	2	-0.0028	0.0068	-0.0026	0.0065
				6	-0.0027	0.0067	-0.0028	0.0068