# Adaptive Wavelet Decompositions of (Gaussian) Stationary Processes


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# Adaptive Wavelet Decompositions

Adaptive Wavelet Decompositions (AWD) are particular wavelet decompositions of general stationary (Gaussian) processes $X$ in continuous and discrete time.

Our work on AWD is based on two papers:

• [DP06a] "Gaussian stationary processes: adaptive wavelet decompositions, discrete approximations and their convergence", G. Didier and V. Pipiras, 2006, Preprint. (Continuous time).

• [DP06b] "Adaptive wavelet decompositions of stationary time series", G. Didier and V. Pipiras, 2006, Preprint. (Discrete time).

# Outline

- Adaptive Wavelet Decompositions (AWD) in general terms
- AWD for **FARIMA(0,s,0)**
- Applications in discrete time

# General aspects of AWD

AWD are characterized by:

- independent (uncorrelated) detail coefficients $d_{j,n}$;
- possibly correlated approximation coefficients $X_{j,n}$;
- non(bi)orthogonal wavelet basis (adapted to the covariance of $X$);
- FWT-like algorithm (with filters possibly depending on scale $j$).
- in [DP06a]: $2^{j/2}X_{j,[2^jt]} \approx X(t)$ at small scales ("wavelet crime"). Practical conditions for this to take place are provided;
- in [DP06b]: associated low and high pass filters decay to zero fast. Zero moments play a key role here.

## General aspects of AWD

Other issues addressed:

- Riesz bases [DP06a].
- Convergence of discrete approximations $X_{j,n}$ [DP06a].
- Many examples [DP06a,b]: Ornstein-Uhlenbeck process, processes with rational spectral densities, FARIMA time series, etc.

On the significance of AWD:

- New wavelet-based decompositions with independent coefficients.
- It works for general stationary processes;
- It allows for practical applications: simulations in [DP06a,b], MLE in [DP06b], etc.

# Definition of AWD in discrete time: FARIMA(0,s,0)

- Representation $X_0 = (I-B)^{-s}\epsilon^0, \quad s \in (-1/2, 1/2)$
- AWD at decomposition

$$X_j = \downarrow_2 (\overline{U}_d * X_{j-1}), \quad d_j = \downarrow_2 (\overline{V}_d * X_{j-1}), \quad j \geq 1,$$

with

$$\widehat{U}_d(\omega) = (1+e^{i\omega})^{-s}\widehat{u}(\omega), \quad \widehat{V}_d(\omega) = (1-e^{i\omega})^{s}\widehat{v}(\omega).$$

- AWD at reconstruction

$$X_j = (\overline{U}_r * \uparrow_2 X_{j+1}) + (\overline{V}_r * \uparrow_2 d_{j+1}), \quad j \geq 0,$$

with

$$\widehat{U}_r(\omega) = (1+e^{i\omega})^{s}\widehat{u}(\omega), \quad \widehat{V}_r(\omega) = (1-e^{i\omega})^{-s}\widehat{v}(\omega).$$

Here, $d_j$ are independent white noise sequences and $X_j$ are a **FARIMA(0,s,0)** time series, and $u$ and $v$ are the Conjugate Mirror Filters of the underlying orthogonal MRA.

# Definition of AWD in discrete time: FARIMA(0,s,0)

Two issues here:

- Filter decay vs zero moments (for example, at decomposition)

$(1+e^{i\omega})^{-s}, (1-e^{i\omega})^{s}$ decay extremely slowly, but not $\widehat{U}_d(\omega)$, $\widehat{V}_d(\omega)$, since, for instance,

$$\widehat{U}_d(\omega) = (1+e^{i\omega})^{-s+N}\widehat{u}_{0,N}(\omega),$$

where $N$ is the number of zero moments of the orthogonal MRA.

- In practice

Replace $*$ by $\circledast$. Fast decay of $U$, $V$ implies smaller border effect.

# First Application: Gaussian MLE based on AWD

For a data row vector $X^0$, the negative log-likelihood involves

$$X^0 \Sigma_\theta^{-1} X^{0\prime} \approx Y_\theta Y_\theta',$$

where $\theta$ indicates unknown parameters, $T$ is the size of $X^0$,

$$Y_\theta = X^0 M_\theta$$

is the detail vector in AWD with $\circledast$, and $M_\theta$ is an invertible AWD matrix. The key to have the above is the approximate covariance matrix factorization

$$\Sigma_\theta^{-1} \approx M_\theta M_\theta'.$$

# First Application: Gaussian MLE based on AWD

• Advantage over MLE based on orthogonal wavelet decompositions (OWD): the exact MLE based on OWD involves the complex dependence structure of its detail coefficients. By construction, detail coefficients are independent, $N(0,1)$ variables in AWD.

• Advantage over non-wavelet based MLE: MLE based on AWD is potentially invariant to polynomial trends in data.

• MLE results for FARIMA$(\mathbf{0}, \mathbf{0.3}, \mathbf{0})$

| Sample size $T$ | Whittle bias | Whittle rMSE | $N$ zero moments | AWD bias | AWD rMSE | OWD bias | OWD rMSE |
|---|---|---|---|---|---|---|---|
| $2^{10}$ | 0.0003 | 0.0246 | 2 | -0.0069 | 0.0270 | -0.0079 | 0.0256 |
| | | | 6 | -0.0055 | 0.0258 | -0.0082 | 0.0253 |
| $2^{14}$ | 0.0002 | 0.0062 | 2 | -0.0028 | 0.0068 | -0.0026 | 0.0065 |
| | | | 6 | -0.0027 | 0.0067 | -0.0028 | 0.0068 |

# Second Application: Simulation based on AWD

Two methods:

1. For a given $J > 0$, use some simulation method to generate an initial time series vector, apply FWT based on AWD at reconstruction and drop terms affected by border effect to obtain a time series of size $2^J$

2. Replace $*$ with $\circledast$ to obtain a circular time series whose covariance structure is close enough to the one of the desired stationary time series, and take only the first $T$ observations, where $T$ is small relative to the total size of the series.

• The advantage of both methods: computational order is $O(2^J)$, as compared to $O(2^J \log(2^J))$ of Circular Matrix Embedding

# References

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