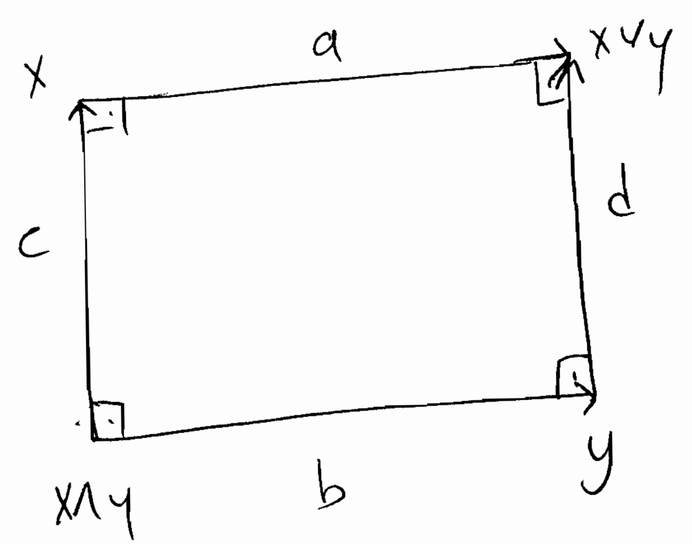
M. II/ Let $x, y \in \mathbb{R}^n$. Show that $x, y, x \vee y, x \wedge y$ form a two dimensional rectangle in $\mathbb{R}^n$.

Let $v := x \vee y - x$ and suppose $f : X \to \mathbb{R}$ is super-modular. Show that

$$f(x \vee y) - f(y) \geq f(x + \lambda v) - f(x \wedge y + \lambda v) \text{ for all } \lambda \in [0,1].$$

$$x \vee y = (\max\{x_1, y_1\}, \ldots, \max\{x_n, y_n\})$$

$$x \wedge y = (\min\{x_1, y_1\}, \ldots, \min\{x_n, y_n\})$$



let $a = x \vee y - x$

$b = y - x \wedge y$

$c = x - x \wedge y$

$d = x \vee y - y$

Show $a^T d = 0$ $\quad a^T c = 0$

$b^T c = 0$ $\quad b^T d = 0$

i.e. a is orthogonal to ~~b~~, d and c and

b is orthogonal to d and c

$$a_i = (x \vee y - x)_i = \begin{cases} 0 & \text{if } x_i \geq y_i \\ y_i - x_i & \text{if } x_i < y_i \end{cases}$$

$$b = (y - x \wedge y)_i = \begin{cases} 0 & \text{if } x_i \geq y_i \\ y_i - x_i & \text{if } x_i < y_i \end{cases}$$

$$c_i = (x - x \wedge y)_i = \begin{cases} x_i - y_i & \text{if } x_i \geq y_i \\ 0 & \text{if } x_i < y_i \end{cases}$$

$$d_i = (x \vee y - y)_i = \begin{cases} x_i - y_i & \text{if } x_i \geq y_i \\ 0 & \text{if } x_i < y_i \end{cases}$$

Now look at $a^T d = a_1 . d_1 + a_2 d_2 + \cdots + a_n . d_n$

if $x_i \geq y_i$ $\quad a_i = 0, \quad d_i = x_i - y_i \Rightarrow a_i . d_i = 0$

if $x_i < y_i$ $\quad a_i = y_i - x_i, \quad d_i = 0 \Rightarrow a_i . d_i = 0$

Therefore $a^T d = 0$.

Similarly it can be shown that $a^T c = 0$

$b^T c = 0$ and $b^T d = 0$. (But if you show above that $a = b$ and $c = d$, then showing $a^T d = 0$ will be enough)

ii) Show that $f(x \vee y) - f(y) \geq f(x+\lambda v) - f(x \wedge y + \lambda v)$

To show this we need to show:

$(x+\lambda v) \vee y = x \vee y$ and $(x+\lambda v) \wedge y = x \wedge y + \lambda v$

Note that $v_i = y_i - (x \wedge y)_i = \begin{cases} y_i - x_i & \text{if } x_i < y_i \\ 0 & \text{if } x_i \geq y_i \end{cases}$

Note that $(x+\lambda v) \wedge y = ((1-\lambda)x + \lambda(x \vee y)) \wedge y$

$= \min\{(1-\lambda)x_i + \lambda . \max\{x_i, y_i\}, y_i\}$ for i^{th} coordinate (1)

Suppose $x_i \geq y_i$, then we can write (1) as

$= \min\{(1-\lambda)x_i + \lambda x_i, y_i\} = \min\{x_i, y_i\} = y_i$

$= (x \wedge y)_i + \lambda v_i$

(since $v_i = 0$ in this case)

Suppose $x_i < y_i$, then (1) becomes

$= \min\{(1-\lambda)x_i + \lambda y_i, y_i\} = (1-\lambda)x_i + \lambda y_i$

$= x_i + \lambda(y_i - x_i)$

$= x_i + \lambda(y_i - (x \wedge y)_i)$ [since $x = x \wedge y$]

$= (x \wedge y)_i + \lambda v_i$

For the second equality

$$(x+\lambda v) \vee y = ((1-\lambda)x + \lambda(x \vee y)) \vee y$$

$$\equiv \max\{(1-\lambda)x_i + \lambda \max\{x_i, y_i\}, y_i\} \text{ for } i^{th} \text{ coordinate} \quad (2)$$

$$= \max\{(1-\lambda)x_i + \lambda x_i, y_i\} \quad \text{if } x_i \geq y_i$$

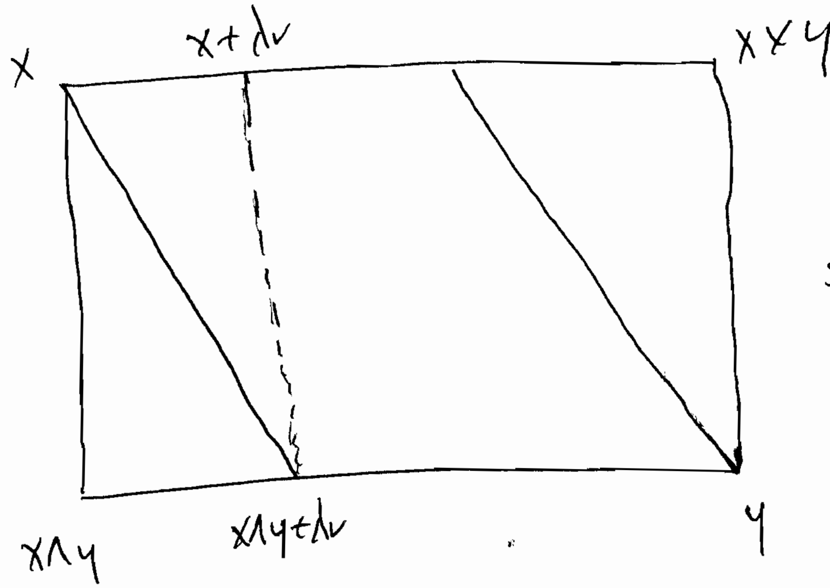
$$= \max\{x_i, y_i\} = x_i = (x \vee y)_i$$

Suppose $x_i < y_i$, then (2) becomes

$$\max\{(1-\lambda)x_i + \lambda y_i, y_i\} = y_i = (x \vee y)_i$$

note that since $x_i < y_i$, we have $(1-\lambda)x_i + \lambda y_i < y_i$

Therefore $(x+\lambda v) \vee y = x \vee y$. //



= (Basicly we show that spm holds in the rectangle formed by $x+\lambda v$, $x \vee y$, y, $x \wedge y + \lambda v$)

(wherever f is spm)

(I) Given (p, w) and (p', w')

if $\langle x'(p', w'), p \rangle \leq w$ then we must have

$\langle x(p, w), p' \rangle > w'$! whenever $x(p, w) \neq x(p', w')$

a) $w^0 = p^0 \cdot x^0 = 1.4 + 3.2 = 10$

$p^0 = (1, 3) \quad x^0 = (4, 2)$

$w' = 3.3 + 5.1 = 14$

Is x^0 affordable under $(p', w') = ((3,5), 14)$

$4.3 + 2.5 \overset{?}{<} 14 \Rightarrow 12 + 10 \not< 14$ not affor.

Is x' affordable under $(p, w) = ((1,3), 10)$

$3.1 + 1.3 = 6 < 10$ affordable

$\Rightarrow$ a satisfies WARP!

b) $p^0 = (1,2)$ $x^0 = (3,1)$; $p' = (2,2)$ $x' = (1,2)$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$w^0 = 1.3 + 2.1 = 5$

$w' = 2.1 + 2.2 = 6$

i) Is x^0 affordable under (p', w')

$x^0 . p' = 3.2 + 1.2 = 8 > 6 = w'$ not affordable;

ii) Is x' afferdable under $(p^0 w^0)$?

$x' . p_0 = 1.1 + 2.2 = 5$ afferdable.

$\Rightarrow$ Both are not afforedable at the same time.

satisfies WARP!

Def A function $x(p, w)$ satisfies WARP

if $\langle p, x(p', w') \rangle \leq w$ and $x(p, w) \neq x(p', w')$

implies $\langle p', x(p, w) \rangle > w'$.

IV. A firm is completing the development of a new product and is valuating how long it should wait before launching it. A longer development time allows the firm to improve its production technology, which results in cost savings and better product quality. On the other hand, the firm knows that a direct competitor is working on a similar product, and it realizes that whoever introduces its product first will capture a significant share of the market.

Specifically, suppose that if the competitor enters the market first, the firm will be left with profits equal to π, while if it introduces its product at time t and the competitor hasn't entered yet, it enjoys a profit of $\pi(t) > \pi$ (where $\pi'(t) > 0$). Finally, the firm believes that its competitor's time of entry is exponentially distributed with parameter θ (where $\theta > 1$):

$$\Pr\left(\text{competitor enters at time } t\right) = \theta e^{-\theta t}$$

1 a) Write down the probability, $F(t, \theta)$, that the competitor has not entered by time t.

1.5 b) Write down the profit $\Pi(\theta, t)$ from entering at time t.

3 c) Assume that $F(t, \theta)$ has increasing differences in (t, θ). Show that $\Pi(t, \theta)$ has increasing differences in (t, θ).

1.5 d) How will the firm react to an increase in (its estimate of) θ?

a) $F(t,\theta) = 1 - \int_0^t \theta e^{-\theta s}\, ds = e^{-\theta t}$

b) $\Pi(\theta, t) = \pi \cdot \Pr(\text{Comp. has entered before } t) + \pi(t) \cdot \Pr(\text{comp. has not entered by } t)$

$= \pi \cdot (1 - F(t,\theta)) + \pi(t) \cdot F(t,\theta)$

c) $\pi(\theta, t) = \underbrace{F(t,\theta)}_{\text{has increasing diff.}} \cdot \underbrace{[\pi(t) - \pi]}_{\substack{\text{increasing in } t \\ \text{(does not depend on } \theta)}} + \underbrace{\pi}_{\text{constant}}$

$\Downarrow$

$\pi(\theta, t)$ has increasing diff

d) Since $\pi(\theta, t)$ has increasing differences a rise in the parameter θ, will lead to rise in the choice variable (t)

Therefore, the firm waits more to enter if θ increases