Ex 7.19/ Suppose $\delta_x \sim \delta_y$ for all $x, y \in Z$.

Show that for all $p, q \in L_0(Z)$, $p \sim q$.

Proof: Fix any $p, q \in L_0(Z)$.

~~Note~~ Let $Z_p = supp(p) = \{z_1, z_2, \ldots z_{N_p}\}$

$Z_q = supp(q) = \{z_1', z_2', \ldots z_{N_q}'\}$

Fix some $z \in Z$.

Note $p = (p_1, p_2, \ldots p_{N_p})$

$q = (q_1, q_2, \ldots q_{N_q})$

We can think of p and q as compound lotteries such that for example with probability p_j you get the degenerate lottery δ_{z_j} under p.

Therefore we can write $p = \sum_{n=1}^{N_p} p_n \delta_{z_n}$ and $q = \sum_{n=1}^{N_q} q_n \delta_{z_n'}$

also note that $\sum_{n=1}^{N_p} p_n = 1$ and $\sum_{n=1}^{N_q} q_n = 1$.

And we know that for all $z_j \in Z_p$ $\delta_{z_j} \sim \delta_z$

and for all $z_j' \in Z_q$ $\delta_{z_j'} \sim \delta_z$ (given in the question)

Therefore by substitution lemma

$$p = \sum_{n=1}^{N_p} p_n \delta_{z_n} \sim \sum_{n=1}^{N_p} p_n \delta_z = \delta_z \sum_{n=1}^{N_p} p_n = \delta_z$$

i.e $p \sim \delta_z$

similarly

$$q = \sum_{n=1}^{N_q} q_n \delta_{z_n'} \sim \sum_{n=1}^{N_q} q_n \delta_z = \delta_z \sum_{n=1}^{N_q} q_n = \delta_z$$

i.e $q \sim \delta_z$

Then by transitivity of "$\sim$" we have the result;

$p \sim q$ //

Ex 7.21/ Show that if "there exists a function $u: Z \to \mathbb{R}$ such that for any lotteries $p, q \in L_0(Z)$, $p \succsim q$ iff and only

$$V(p) = \sum_z p(z) u(z) \geq \sum_z q(z) U(z) = V(q)$$ "

then $\succsim$ satisfy IND and CONT.

Proof Assume that the condition inside " " above holds.

Let $p = (p_1, \dots p_N)$ $q = (q_1, \dots q_M)$ and $r = (r_1, \dots r_M)$

Then $p \succsim q$ iff $\sum_z p(z) u(z) \geq \sum_z q(z) u(z)$

or we can write

$$\sum_{n=1}^{N} p_n u(z_n) \geq \sum q_n u(z_n)$$

This inequality is equivalent [for any $\alpha \in [0,1]$]

$$\alpha \left(\sum_{n=1}^{N} p_n u(z_n) \right) + (1-\alpha) \left(\sum_n r_n U(z_n) \right) \geq \alpha \left(\sum_{n=1}^{N} q_n u(z_n) \right) + (1-\alpha) \left(\sum_{n=1}^{M} r_n u(z_n) \right)$$

this holds if and only if $\alpha p + (1-\alpha) r \succsim \alpha q + (1-\alpha) r$.

Hence $p \succsim q$ if and only if $\alpha p + (1-\alpha) r \succsim \alpha q + (1-\alpha) r$

i.e. the Independence axiom holds.

ii) Suppose $p \succ q \succ r$.

This holds if and only if

$$\sum_n p_n v(z_n) > \sum_n q_n v(z_n) > \sum_n r_n v(z_n)$$

Since these are real numbers $\exists\, \alpha \in (0,1)$ s.t.

$$\sum_n q_n v(z_n) = \alpha \left(\sum_n p_n v(z_n) \right) + (1-\alpha) \left(\sum_n r_n v(z_n) \right)$$

$$= \sum_n \left(\alpha p_n + (1-\alpha) r_n \right) v(z_n)$$

By cond. "..." this holds if and only if

$q \sim \alpha p + (1-\alpha) r$.

Hence if $p \succ q \succ r$, then there exists $\alpha \in (0,1)$ such that $q \sim \alpha p + (1-\alpha) r$,

Thus, the Contunity axiom holds.

(See also MGW Ex. 6.B.2)

8.13/ Prove Prop 8.12 for lotteries with infinite support.

where <u>Prop. 8.12</u> Let $\succsim$ be a preference on $L(Z)$, represented by a VN-M utility function. Then $\succsim$ is risk-averse if and only if u is concave.

<u>Proof</u> (i) ~~Assume Assume~~ $\succsim$ is risk averse $\Rightarrow$ u is concave.

Same ~~as~~ the proof for lotteries for finite support given in the class notes.

(ii) u is concave $\Rightarrow$ $\succsim$ is risk averse.

When u is concave by Jensen's inequality we have

$$u\left(\int x \, dF(x)\right) \geq \int u(x) \, dF(x)$$

$$\| \qquad\qquad \|$$

$$u\left(E_p[x]\right) \geq E_p[u]$$ hence $\succsim$ is risk averse.

8.14

ii) Show that if $\succsim$ is (strictly) risk averse, then CE is unique.

$\succsim$ risk averse means $u(\cdot)$ is strictly concave.

For contradiction suppose for some lottery p there are two certainty equivalents: CE_1 and CE_2. s.t

$$E_p[u] = u(CE_1) = u(CE_2)$$

then since $u(\cdot)$ is strictly increasing this directly implies that $CE_1 = CE_2$.

8.15/ Risk Premium (RP) is defined as:

$$RP(p) = E_p[x] - CE(p)$$

Show that $RP \geq 0$ iff and only if $\succsim$ is risk-averse.

Proof

(i) Suppose $RP \geq 0 \Rightarrow$ (by def). $E_p[X] \geq CE(p)$ (greater than or equal)

since $u(\cdot)$ ~~peppe~~ is monotone this implies that

$$E_p[x] \succsim CE(p) \sim p$$

(preference)

then by transitivity of $\succsim$ $E_p[x] \succsim p$ i.e. $\succsim$ is risk averse.

(ii) Conversely suppose that $\succsim$ is risk averse. Then $E_p[x] \succsim p$ and since by definition $p \sim CE(p)$ by transitivity of $\succsim$ we have $E_p[x] \succsim CE(p)$ and since $\succsim$ is monotone this implies $E_p[X] \geq CE(p)$ $\Rightarrow R(p) \geq 0$ //

Ex 8.20 Show that CRA1 is equivalent to CRA2, if preferences are monotone.

The class notes show CRA1 $\Rightarrow$ CRA2. To show the equivalence of the two definitions hence we just need to show CRA2 $\Rightarrow$ CRA1.

Suppose CRA2 holds. For contradiction suppose that $CE_1(p) > CE_2(p)$. Note by definition $p \sim_1 \delta_{CE_1(p)}$ and by monotonicity $\delta_{CE_1(p)} \succ_1 \delta_{CE_2(p)}$ then by transitivity of $\succ$ we have $p \succ_1 \delta_{CE_2(p)}$ (strictly preffered)

But then CRA2 implies that $p \succ_2 \delta_{CE_2(p)}$

which is a contradiction since $p \sim_2 \delta_{CE_2(p)}$ by definition.

8.32/ Let G be a mean preserving spread of F. Show that every risk averter will prefer F to G.

Proof w.l.o.g. $G(\cdot)$ being a MPS of $F(\cdot)$ implies that $G(\cdot)$ is obtained adding a random return z to any outcome x under $F(\cdot)$ s.t. z is distributed with $H_x(z)$ with a mean zero. i.e. $\int z \, dH_x(z) = 0$

[note we allow distr. of $H_x(z)$ to depend on outcome x]

Therefore, if $u(\cdot)$ is concave (person is risk averse)

$$\int u(x) \, dG(x) = \int \left(\int u(x+z) \, dH_x(z) \right) dF(x) \quad \text{(by def. of } G(\cdot))$$

$$\leq \int u\left(\int (x+z) \, dH_x(z) \right) dF(x) \quad \text{by Jensen's inequality}$$

$$= \int u(x) \, dF(x) \quad \text{since } \int z \, dH_x(z) = 0$$

//

(please see MGW p.197 example 6.D.2)

8.33/ Let $p_0 = \delta_0$ and

$$p_n = \frac{1}{2^n}\delta_{-2^{n-1}} + \frac{2^{n-1}-1}{2^{n-1}}\delta_0 + \frac{1}{2^n}\delta_{2^{n-1}} \text{ for all } n.$$

1. Show that p_1 is a mean preserving spread of p_0.

$$p_1 = \frac{1}{2}\delta_{-2^0} + \frac{2^0-1}{2^0}\delta_0 + \frac{1}{2}\delta_{2^0}$$

$$= \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$$

Therefore p_1 is lottery which gives -1 ~~with~~ and 1 with equal probabilities. Hence mean return is 0.

On the other hand $p_0 = \delta_0$

~~For~~ Hence p_0 is the lottery that gives 0 for sure.

Since p_1 has the same mean as p_0 but more risk it is a mean preserving spread of p_0.

33. ii) Show that p_{n+1} is a mean preserving spread of p_n.

$$p_n = \frac{1}{2^n}\delta_{-2^{n-1}} + \frac{2^{n-1}-1}{2^{n-1}}\delta_0 + \frac{1}{2^n}\delta_{2^{n-1}}$$

Hence p_n is the lottery which gives $\$(-2^{n-1})$ w/p $\frac{1}{2^n}$

$\$\,2^{n-1}$ w/p $\frac{1}{2^n}$ and $\$0$ w/p $\frac{2^{n-1}-1}{2^{n-1}}$

$$E_{p_n}[x] = \frac{1}{2^n}\cdot(-2^{n-1}) + \frac{1}{2^n}\cdot 2^{n-1} + \frac{2^{n-1}-1}{2^{n-1}}\cdot 0$$

$$= -\frac{1}{2} + \frac{1}{2} = 0 \quad : \text{(mean return of } p_n\text{)}$$

Now note that

$$p_{n+1} = \frac{1}{2^{n+1}}\delta_{-2^n} + \frac{2^n-1}{2^n}\cdot\delta_0 + \frac{1}{2^{n+1}}\delta_{-2^n}$$

$$= \frac{1}{2^{n+1}}\delta_{-2^n} + \overbrace{\left[\underbrace{\frac{1}{2^n}}_{\frac{1}{2^{n+1}}+\frac{1}{2^{n+1}}} + \frac{2^{n-1}-1}{2^{n-1}}\right]}^{\frac{2^n-1}{2^n}}\delta_0 + \frac{1}{2^{n+1}}\delta_{-2^n}$$

$$p_{n+1} = \frac{1}{2^{n+1}}\delta_{-2^n} + \left[\frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} + \frac{2^{n-1}-1}{2^{n-1}}\right]\delta_0 + \frac{1}{2^{n+1}}\delta_{-2^n}$$

$$= \underbrace{\frac{1}{2^n}\left(\frac{1}{2}\delta_{-2^n} + \frac{1}{2}\delta_0\right)}_{I} + \underbrace{\frac{2^{n-1}-1}{2^{n-1}}\delta_0}_{II} + \underbrace{\frac{1}{2^n}\left(\frac{1}{2}\delta_{2^n} + \frac{1}{2}\delta_0\right)}_{III}$$

Now, notice that the term in the middle is the same as the term in the middle in p_n.

To see that p_{n+1} is a mean preserving spread of p_n note that p_{n+1} is a lottery that gives $\$0$ w/p $\frac{2^{n-1}-1}{2^{n-1}}$ as in p_n, additionally with probabobility $\frac{1}{2^n}$ the result is another lottery which gives $\$(-2^n)$ w/p $\frac{1}{2}$, and $\$0$ w/p $\frac{1}{2}$.

The mean return of this second stage lottery is

$$\frac{1}{2}\cdot(-2^n) + \frac{1}{2}\cdot 0 = -2^{n-1}$$

Recall that p_n is giving -2^{n-1} w/p $\frac{1}{2^n}$ without a second stage lottery.

~~Atra~~ Hence p_{n+1} adds more risk on this outcome ~~of~~ p_n but preserves the mean return.

Lastly, the third outcome of p_{n+1} is another second stage lottery w/p $\frac{1}{2^n}$, which is $\$2^n$ w/p $\frac{1}{2}$ and $\$0$ w/p $\frac{1}{2}$.

Mean return of this second stage lottery is

$$\frac{1}{2}\cdot 2^n + \frac{1}{2}\cdot 0 = 2^{n-1}$$

Again p_{n+1} made the third outcome of p_n more risk at the same time preserved the mean return.

Therefore, p_{n+1} gives the same return as p_n, but it is more risky.

Hence p_{n+1} is a mean preserving spread of p_n.

8.33. iii. Show that $p_n \xrightarrow{w} p_0$.

$$p_n = \underbrace{\frac{1}{2^n}\delta_{-2^{n-1}}}_{\mathrm{I}} + \underbrace{\frac{2^{n-1}-1}{2^{n-1}}\delta_0}_{\mathrm{II}} + \underbrace{\frac{1}{2^n}\delta_{2^{n-1}}}_{\mathrm{III}}.$$

Note that probabilities of 1^{st} and 3^{rd} outcomes $\frac{1}{2^n} \to 0$ as $n \to \infty$. Hence these outcomes becomes ignorable as $n \to \infty$.

And the probability of middle outcome which is 0

$$\frac{2^{n-1}-1}{2^{n-1}} = 1 - \frac{1}{2^{n-1}} \longrightarrow 1 \text{ as } n \to \infty$$

(since $\frac{1}{2^{n-1}} \to 0$ as $n \to \infty$)

Therefore as $n \to \infty$, p_n ~~become~~ converges to a lottery which gives \$0 for sure. This is exactly the lottery p_0.

Ex. 827 (See MGW Ex. 6.C.19 as well) $u(x) = -e^{-\alpha x}$ $\alpha > 0$

ω fixed

β_n : the wealth invested in risky asset n

$\omega - \sum_{n=1}^{2} \beta_n$: wealth invested in riskless assets with return 1.

Portfolio $\beta = (\beta_1, \beta_2) \in \mathbb{R}^2$

This result depends on the utility function and the joint distribution of return of risky assets. You can't say that it holds in general!

$$FW = (\omega - \beta_1 - \beta_2) + \beta_1 z_1 + \beta_2 z_2 \equiv X$$

$$\begin{pmatrix} z_{1} \\ z_2 \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \right) \equiv N(\mu, V)$$

$$X \sim N(FW, \beta V \beta')$$

$$\max_X E\left[-e^{-\alpha X}\right]$$

Aside

$$y \sim N(\mu, \sigma^2)$$

$$M_y(t) = E(e^{ty}) \to \text{Moment generating function}$$

$$= e^{t\mu + \frac{1}{2}\sigma^2 t^2}$$

$$= -\exp\left[(-\alpha)(FW) - \beta V \beta' (-\alpha)^2/2\right]$$

$$\Rightarrow \beta^* = \frac{1}{\alpha} V^{-1}(\mu - e) = \frac{1}{\alpha}\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}^{-1} \begin{bmatrix} \mu_1 - 1 \\ \mu_2 - 1 \end{bmatrix}$$

If $\alpha_1 < \alpha_2 \Rightarrow \beta_1^* > \beta_2^*$ where $\beta_1^* = \begin{bmatrix} \beta_{11}^* \\ \beta_{12}^* \end{bmatrix}$ etc.

$\Downarrow$ 1st guy is less risk averse than 2nd guy $\Rightarrow$ wealth invested in risk assets by 1st guy is more than invested by 2nd guy.