Ex. 2.16. Prove the equivalence of definitions C1 – C4

We'll show $C1 \Leftrightarrow C2 \Leftrightarrow C4 \Leftrightarrow C3$

- i) $C1 \Leftrightarrow C2$ see. the class notes

ii) $C4 \Leftrightarrow C3$

By definition, a set A is closed if A^c is open. Then since

$\{y \in X : y \succ x\}$ open $\Leftrightarrow \{y \in X : x \succsim y\}$ is closed

$\{y \in X : x \succ y\}$ open $\Leftrightarrow \{y \in X : y \succsim x\}$ is closed

because these sets are complements to each other.

Therefore we have $C4 \Leftrightarrow C3$. //

iii) $C2 \Leftrightarrow C4$

First a definition: A is closed if and only if the limit of every converging sequence $\langle x_n \rangle$ with $x_n \in A$ for all n belongs to A.

Let $U(x) \equiv \{y \in X : y \succsim x\}$

$L(x) = \{y \in X : x \succsim y\}$.

Suppose C2 holds. Fix some $x \in X$ and let

$\langle x_n \rangle = \{x, x, \dots x, \dots\} \to x$

$\langle y_n \rangle = \{y_1, y_2, \dots y_n, \dots\} \to y$

be two sequences such that $y_n \succsim x$ for all n,

then also $y_n \succsim x_n$ $\forall n$.

Then $y_n \in U(x)$ for all n.

Since $(x_n, y_n) \to (x, y)$ and $y_n \succsim x_n$ $\forall n$, then

by C2 we have $y \succsim x$ and hence $y \in U(x)$

which implies $U(x)$ is closed.

Similarly we can show $L(x)$ is closed as well.

Thus $\succsim$ is cts per C4.

<u>$C4 \Rightarrow C2$</u> Assume $\succsim$ is CTS per C4. Thus

$U(x)$ and $L(x)$ are closed for all $x \in X$.

Then for all $\langle y_n \rangle \in U(x)$ s.t $y_n \to y$ we have

$y \in U(x)$ ie. $y \succsim x$ and similarly for all

$\langle z_n \rangle \in L(x)$ s.t $z_n \to z$ we have $z \in L(x)$

ie. $x \succsim z$.

Ex 2.3 Let X be nonempty and $u, v: X \longrightarrow \mathbb{R}$ represents $\succsim$ on X. Show that $\exists$ a strictly increasing function $f \in \mathbb{R}^{u(X)}$ s.t. $v = f \circ u$.

Proof Since both u and v represent $\succsim$, we have

$x \succsim y$ iff $u(x) \geq u(y)$ iff $v(x) \geq v(y)$

We know that $\exists$ a function $f \in \mathbb{R}^{u(X)}$ s.t

$v = f \circ u$ i.e. $v(x) = f(u(x))$. For contradiction suppose f is not strictly increasing. Then $\exists$ x, y s.t $u(x) \geq u(y)$ and

hence $x \succsim y$ (since $u(\cdot)$ repr. $\succsim$) but $f(u(x)) \leq f(u(y))$

$\Longleftrightarrow$ $v(x) \leq v(y)$ (where the last relation is by def. of v)

But since we assumed that $v(\cdot)$ represents $\succsim$, this is a contradiction. Hence, $\exists$ a str. increasing function $f \in \mathbb{R}^{u(X)}$ s.t. $v := f \circ u$. //

3. C. 6 (Mas-Colell)

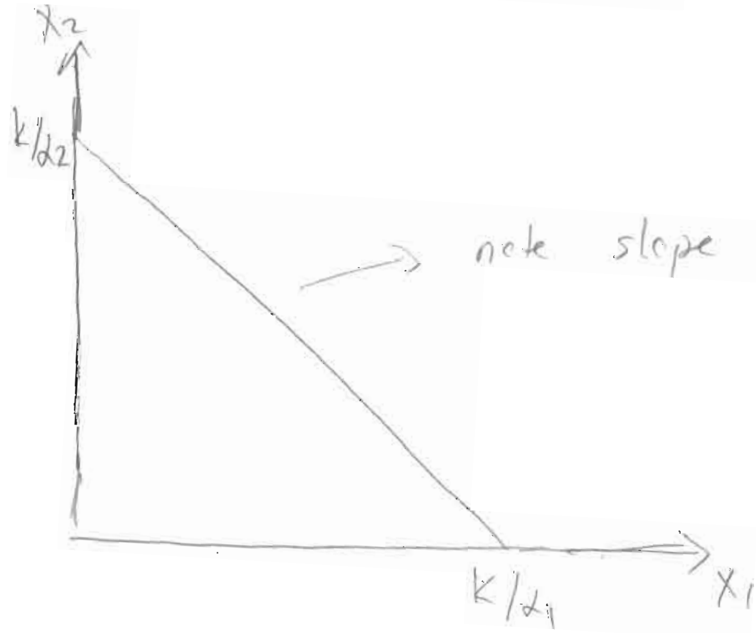
Let $X = \mathbb{R}^2_+$ and $u(x) = [\alpha_1 x_1^\rho + \alpha_2 x_2^\rho]^{1/\rho}$ (CES).

i) Show that when $\rho = 1$, indifference curves becomes linear

For $\rho = 1$, we have $u(x) = \alpha_1 x_1 + \alpha_2 x_2$, Thus the indifference curves becoms linear

i.e Let $u(x) = \alpha_1 x_1 + \alpha_2 x_2 = k$ an indifference set

then $x_2 = -\dfrac{\alpha_1}{\alpha_2} x_1 + \dfrac{k}{\alpha_2}$



note slope is $-\dfrac{k/\alpha_2}{k/\alpha_1} = -\dfrac{\alpha_1}{\alpha_2}$

3.C.6.b Show that as $\rho \to 0$, $u(x) \to x_1^{\alpha_1} x_2^{\alpha_2}$

Note: Every monotonic transformation of a utility function represents the same preferences.

It is easy to work with log's. So let

$$\tilde{u}(x) = \ln u(x) = (1/\rho) \ln(\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho})$$

By L'Hopital's rule

$$\lim_{\rho \to 0} \tilde{u}(x) = \lim_{\rho \to 0} \frac{\partial [\ln(\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho})] / \partial \rho}{d\rho / d\rho}$$

$$= \lim_{\rho \to 0} \frac{\alpha_1 x_1^{\rho} \ln x_1 + \alpha_2 x_2^{\rho} \ln x_2}{\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho}} \qquad (2)$$

$$= \frac{\alpha_1 \ln x_1 + \alpha_2 \ln x_2}{\alpha_1 + \alpha_2}$$

note in the second step (2) we've used that

$$\frac{\partial (b.a^x)}{\partial x} = b.a^x . \ln a \quad \text{and} \quad \frac{d\rho}{d\rho} = 1$$

then since $\exp((\alpha_1 + \alpha_2)\, \tilde{u}(x)) = x_1^{\alpha_1} x_2^{\alpha_2}$ we've obtained a Cobb-Douglas utility function.

3.C.6.c Show that as $p \to -\infty$

$u(x_1, x_2) \to \min\{x_1, x_2\}$

Wlog let $x_1 \le x_2$. We want to show that (WTS)

$$x_1 = \lim_{p \to -\infty} (\alpha_1 x_1^p + \alpha_2 x_2^p)^{1/p}$$

Let $p < 0$. Since $x_1 \geq 0$ and $x_2 \geq 0$ we have

$\alpha_1 x_1^p \le \alpha_1 x_1^p + \alpha_2 x_2^p$. Thus

$\alpha_1^{1/p} x_1 \geq (\alpha_1 x_1^p + \alpha_2 x_2^p)^{1/p}$

Also, since $x_1 \le x_2$ we have

$\alpha_1 x_1^p + \alpha_2 x_2^p \le \alpha_1 x_1^p + \alpha_2 x_1^p = (\alpha_1 + \alpha_2) x_1^p$

Hence $(\alpha_1 x_1^p + \alpha_2 x_2^p)^{1/p} \geq (\alpha_1 + \alpha_2)^{1/p} x_1$. Therefore

$\alpha_1^{1/p} x_1 \geq (\alpha_1 x_1^p + \alpha_2 x_2^p)^{1/p} \geq (\alpha_1 + \alpha_2)^{1/p} x_1$

Letting $p \to -\infty$ we obtain $\lim_{p \to -\infty} (\alpha_1 x_1^p + \alpha_2 x_2^p) = x_1$

b/c $\lim_{p \to -\infty} \alpha_1^{1/p} x_1 = \lim_{p \to -\infty} (\alpha_1 + \alpha_2)^{1/p} x_1 = x_1$. //