Chapter 3 - Consumer Preferences -

Def: A preference relation $\succsim$ is monotone if $x \geq y$ implies $x \succsim y$ and if $x \gg y$ implies $x \succ y$.

Def (S-MON): A preference $\succsim$ is strongly monotone if $x > y$ implies $x \succ y$.

$$\text{S-MON} \Rightarrow \text{MON}$$

If $\succsim$ is S-MON then $x \succ y$ whenever $x > y$, then $x \succ y$ when $x \gg y$ of course. $\rightarrow$ $\succsim$ is MON.

Example $u(x) = \min\{x_1, x_2\}$ is MON but not S-MON

$u(x) = x_1 + x_2$ is S-MON.

Prop 4 Let $\succsim$ be CONT and satisfy MON. Then it has a continuous utility representation.

– Marginal Rate of Substitution –

Marginal Utility of good i $\frac{\partial u}{\partial x_i}(\mathbf{x}) \equiv MU_i(\mathbf{x})$

depends on the bundle X.

$$MRS_{\ell k}(x^*) = \frac{\partial u(x^*)/\partial \ell}{\partial u(x^*)/\partial k}$$

Marginal rate of substitution of good ℓ for good k at x

Idea: ~~How many~~ The amount of good k the consumer must be given to ~~keep~~ compensate her for a one unit marginal reduction in her consumption of good ℓ.

Alternatively: The amount of good k the consumer must give up to get an extra unit of good ℓ, ~~that~~ ~~and~~ such that her utility level won't be changed.

Shortly: How much of good k does the consumer have to give up so that he remains in the same utility level.
(when given a small amount of good ℓ)

MRS - continued

Derivation If utility is unchanged with differential changes in x_ℓ and x_k, dx_ℓ and dx_k, then

$$\frac{\partial u(x)}{\partial x_\ell} \cdot dx_\ell + \frac{\partial u(x)}{\partial x_k} \cdot dx_\ell = 0$$

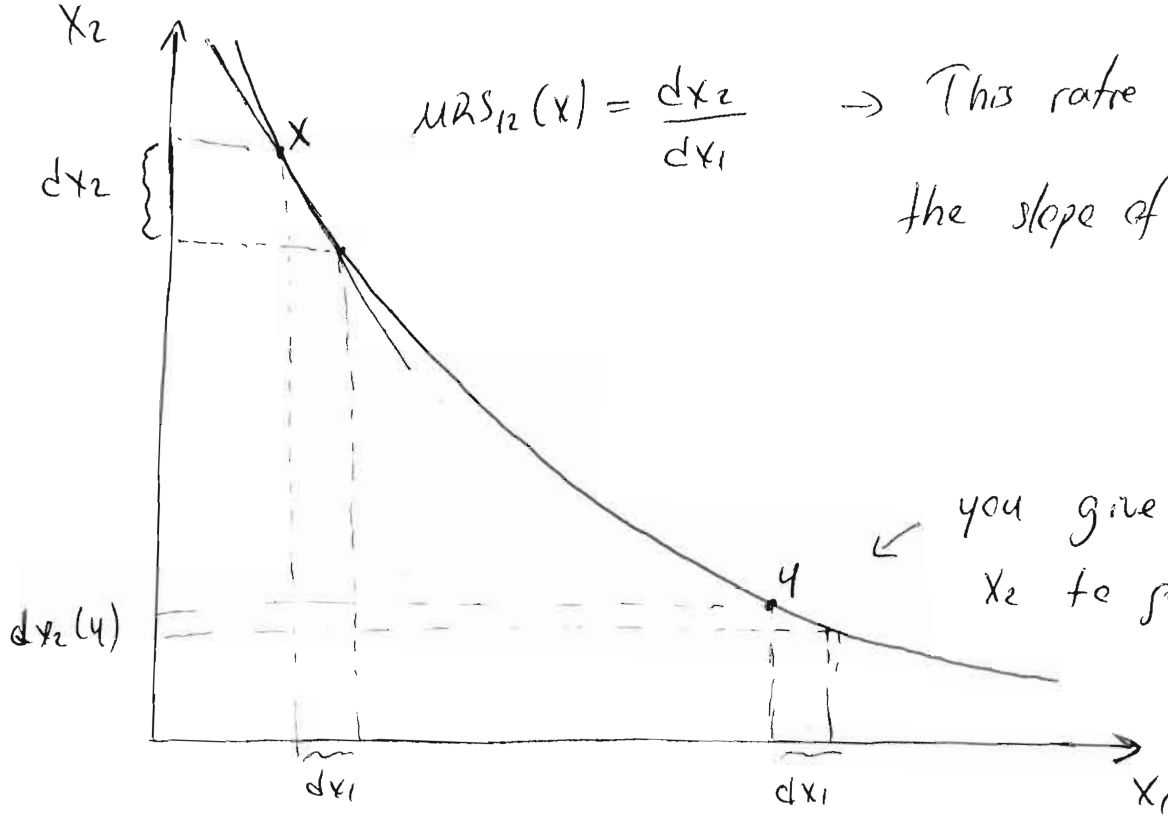
Rearranging gives

$$\frac{dx_\ell}{dx_k} = -\frac{\dfrac{\partial u(x)}{\partial x_\ell}}{\dfrac{\partial u(x)}{\partial x_k}} \equiv MRS_{\ell k}(x)$$

Thus when x_ℓ falls by an amount $dx_\ell < 0$, the increase required in x_k to keep utility unchanged is precisely

$$dx_k = MRS_{\ell k}(x^*)\,(+dx_\ell)$$

Geometric Interpretation of MRS



$MRS_{12}(x) = \frac{dx_2}{dx_1}$ → This ratio is given by the slope of tangent at X.

you give up less of X_2 to get an extra X_1.

$|MRS|$ is decreasing as X increases.

You are willing to give up less of good 2, for an extra unit of good 1, if the amount of X_1 you have increases.

This can be seen by the decrease of the absolute value of the slope of tangent to ind. curve as X increases.

6 Assumption Preferences are convex so that indifference (5)
curves are either linear or bowed toward the origin.

7 Exercise Find the MRS of the linear, Leontief, Cobb-Douglass and quasi-linear utility functions given above at bundle (1,2).

Linear $u(x) = a_1 x_1 + a_2 x_2$ $\qquad z^* = (1,2)$

$$MRS_{12} = -\frac{MU_1(z^*)}{MU_2(z^*)} = -\frac{a_1}{a_2} \qquad \frac{\partial |MRS_{12}|}{\partial x_1} = 0 \rightarrow \text{constant}$$

MRS is constant at an ind. curve

Cobb-Douglass $u(x) = x_1^{\alpha} x_2^{\beta}$

$$MRS_{12} = -\frac{\alpha \cdot x_1^{\alpha-1} x_2^{\beta}}{\beta \cdot x_1^{\alpha} x_2^{\beta-1}} = -\frac{\alpha \cdot x_2}{\beta \cdot x_1}$$

$$\frac{\partial |MRS|}{\partial x_1} = -\frac{\alpha \cdot x_2}{\beta \cdot x_1^2} < 0$$

Q-linear $u(x) = x_1 + v(x_2)$ $\qquad$ ex. $u(x) = x_1 + \ln x_2$

$$MRS_{12} = -\frac{1}{v'(x_2)} = -\frac{1}{\frac{1}{x_2}} = -x_2 \qquad \frac{\partial |MRS|}{\partial x_1} = 0$$

↓ MRS is constant.

8 Exercise A reasonable conjecture might be that $|MRS(x_1, x_2)|$ is decreasing in x_1. Consider the function $u(x) = x_1^2 x_2^4$. (6)

$$|MRS_{12}| = \frac{2 \cdot x_1 \cdot x_2^4}{4 \cdot x_1^2 x_2^3} = \frac{x_2}{2x_1}$$

$$\frac{\partial |MRS|}{\partial x_1} = -\frac{x_2}{2x_1^2} < 0$$ so it fits the conjecture!

$\partial |MRS| / \partial x_1 < 0$ implied by convexity of $\succsim$.

9 Def CONV-I A preference rel. $\succsim$ is convex if $x \succsim y$ and $\alpha \in (0,1)$ implies $\alpha x + (1-\alpha) y \succsim y$.

Def CONV-II A pref. rel. $\succsim$ is convex if the set $U(y) := \{x : x \succsim y\}$ is convex for each $y \in X$.

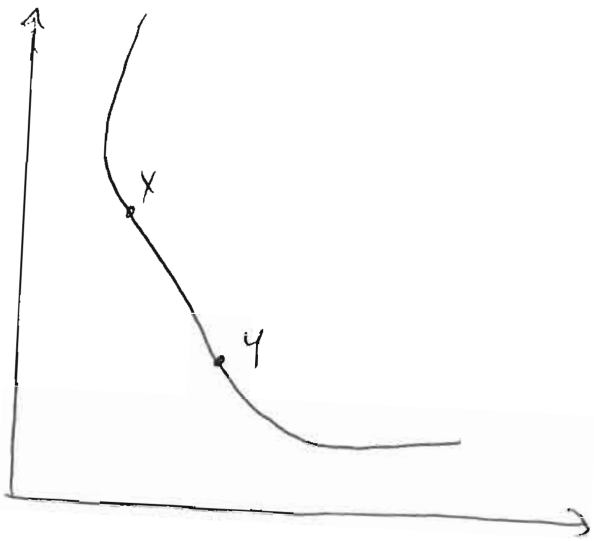
11 Prop CONV I iff CONV 2

i) CONV I $\Rightarrow$ CONV II : Fix $y \in X$, and suppose $a \succsim b \succsim y$. Then $\alpha a + (1-\alpha) b \succsim b \succsim y$ so that $\alpha a + (1-\alpha) b \in U(y)$.

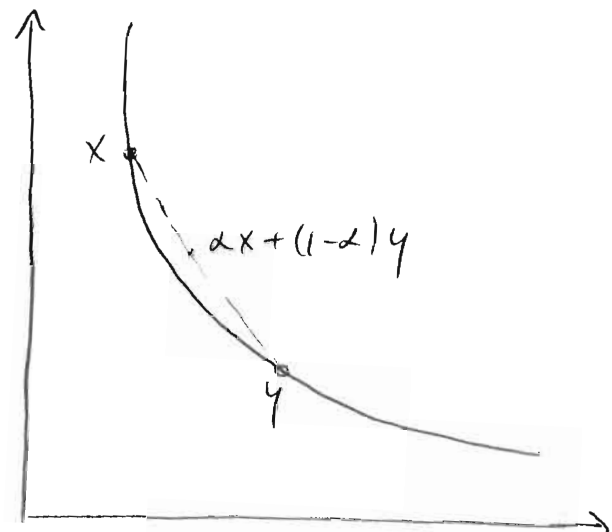
ii) CONV 2 $\Rightarrow$ CONV I : Suppose $x \succsim y$ so that $x, y \in U(y)$. Since $U(y)$ is CONV, $\alpha x + (1-\alpha) y \in U(y)$ i.e. $\alpha x + (1-\alpha) y \succsim y$.

Convexity - Continued

Def (S-CONV) Pref. $\succsim$ is strictly convex if $a, b \in U(y)$, $a \neq b$ and $\alpha \in (0,1)$ then $\alpha a + (1-\alpha) b \succ y$.



CONVEX Pref



S-CONV pref.

Examples i) $u(x) = \sqrt{x_1} + \sqrt{x_2}$ is S-CONV.

Take $x = (4, 9)$ and $y = (9, 4)$ so $u(x) = \sqrt{4} + \sqrt{9} = u(y) = 5$

let $\alpha \in (0,1)$. $\alpha x + (1-\alpha) y = (\alpha \cdot 4 + (1-\alpha) \cdot 9, \ \alpha \cdot 9 + (1-\alpha) \cdot 4)$

$$u(\alpha x + (1-\alpha) y) = \sqrt{\alpha \cdot 4 + (1-\alpha) 9} + \sqrt{\alpha \cdot 9 + (1-\alpha) \cdot 4}$$

$$> \alpha \cdot u(x) + (1-\alpha) u(y) = \alpha \cdot [\sqrt{4} + \sqrt{9}] + (1-\alpha)(\sqrt{4} + \sqrt{9})$$

$$= \sqrt{4} + \sqrt{9}$$

$\Rightarrow \alpha x + (1-\alpha) y \succ x$

ii) $u(x) = \min\{x_1, x_2\}$ is CONV but not S-CONV.

<u>14 exercise</u> Show that Lexicographic pref. rel. is S-CONV.

(8)

Fix some $x \in \mathbb{R}^n$ and let $y \succ x$ wlog. $(y \neq x)$.

wlog Suppose $x_i = y_i$ for some $i < k$ and ~~[illegible]~~ $y_k > x_k$

for $1 \leq k \leq n$.

Now $\alpha x = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$

$\alpha y = (\alpha y_1, \alpha y_2, \ldots, \alpha y_n)$

} <u>LG is homothetic</u>

Then $\alpha x_i = \alpha y_i$ for for $i < k$, and ~~[illegible]~~ $\alpha y_k > \alpha x_k$

$\Rightarrow$ ~~[illegible]~~ $\alpha y \succ \alpha x$ $\rightarrow$ LG pref. are homothetic.

$z = \alpha x + (1-\alpha) y = (\alpha x_1 + (1-\alpha) y_1, \ldots \ldots \alpha x_k + (1-\alpha) y_k + \ldots \alpha x_n + (1-\alpha) y_n)$

then for $i < k$ $z_i = x_i$ since $\alpha x_i + (1-\alpha) y_i = \alpha x_i + (1-\alpha) x_i = x_i$

for $i = k$ $z_k \succ x_k$ since $z_k = \alpha x_k + (1-\alpha) y_k$

$> \alpha x_k + (1-\alpha) x_k$ (by assumpt.)

$= x_k$

hence $z = \alpha x + (1-\alpha) y \succ x \rightarrow$ L.G pref. are S-CONV.

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similar to show if $z \succ x$ and $y \succ x$ then

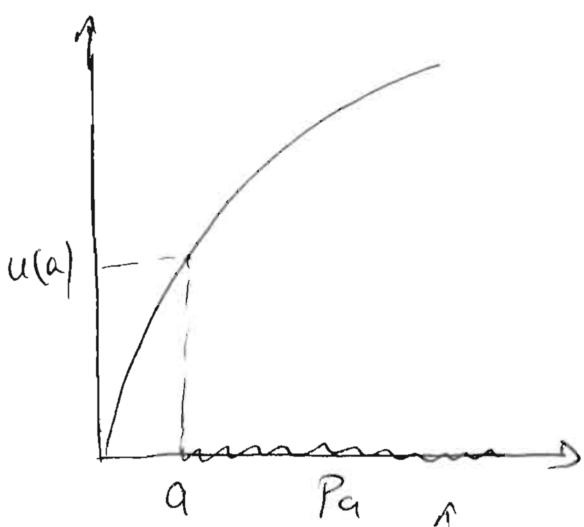
$\alpha z + (1-\alpha) y \succ x$.

Def The function $u: X \to \mathbb{R}$ is quasiconcave if $\underbrace{\{x : u(x) \geq u(y)\}}_{\equiv P_y}$ is convex for each $y \in X$.

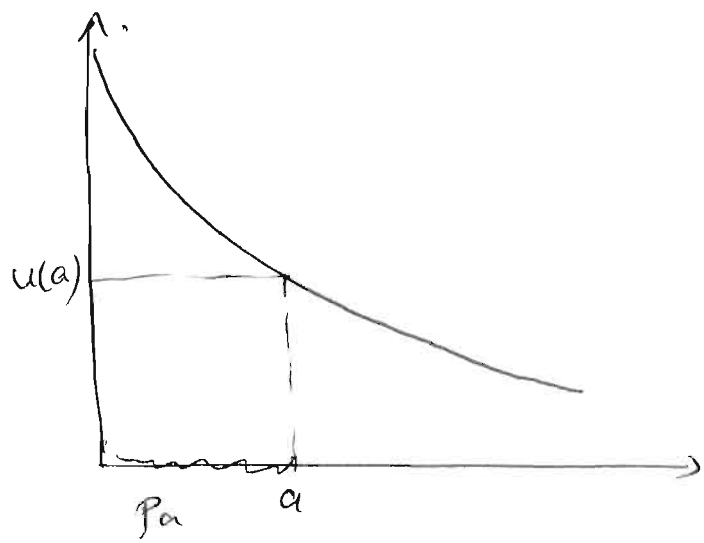
Def $u: X \to \mathbb{R}$ is q. concave if for all $x, y \in X$

$u(\alpha x + (1-\alpha)y) \geq \min\{u(x), u(y)\}$ for any $\alpha \in [0,1]$.

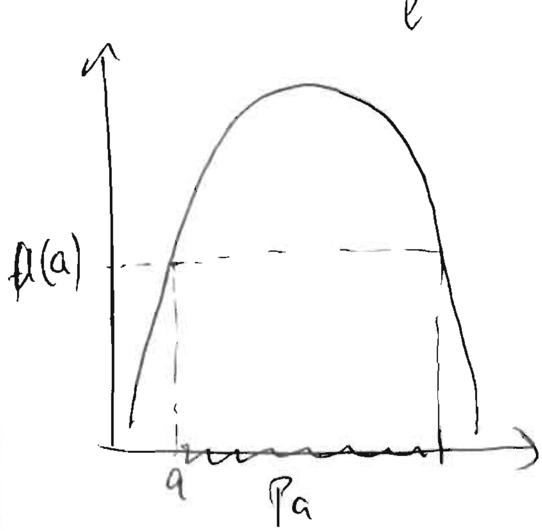
Examples

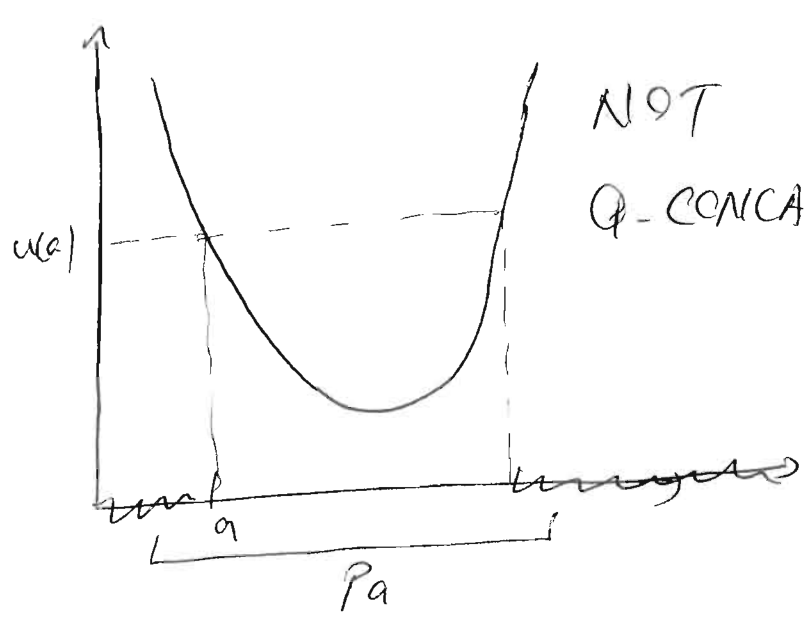


Q- CONCAVE



Q-CONCAVE





NOT Q-CONCAVE

NOTE If $\succsim$ is CCNV $\Rightarrow u(\cdot)$ is Q.-concave.

16 Def HOM $\succsim$ is homothetic if $x \succsim y$ implies $\alpha x \succsim \alpha y$ for all $\alpha \geq 0$

17 Def A func. $u \in \mathbb{R}^X$ is positively homogeneous of degree λ if $u(\alpha x) = \alpha^\lambda u(x)$ for all $\alpha \geq 0$.

Lemma If u represents $\succsim$ and is p.h.o.d λ, then $\succsim$ is HOM.

Proof $x \succsim y$ iff $u(x) \geq u(y)$ iff $\alpha^\lambda u(x) \geq \alpha^\lambda u(y)$

iff $u(\alpha x) \geq u(\alpha y)$ iff $\alpha x \succsim \alpha y$.

Lemma If $\succsim$ is HOM, and CTS satisfies MON, then it has a utility representation that is h.o.d 1.

Proof Let $u(x) = \lambda = u(\lambda e)$ so that $x \sim \lambda e$.

Then $\alpha x \sim \alpha \lambda e$, i.e. $u(\alpha x) = u(\alpha \lambda e) = \alpha \lambda u(e)$

$= \alpha u(\lambda e) = \alpha u(x)$

Concave functions

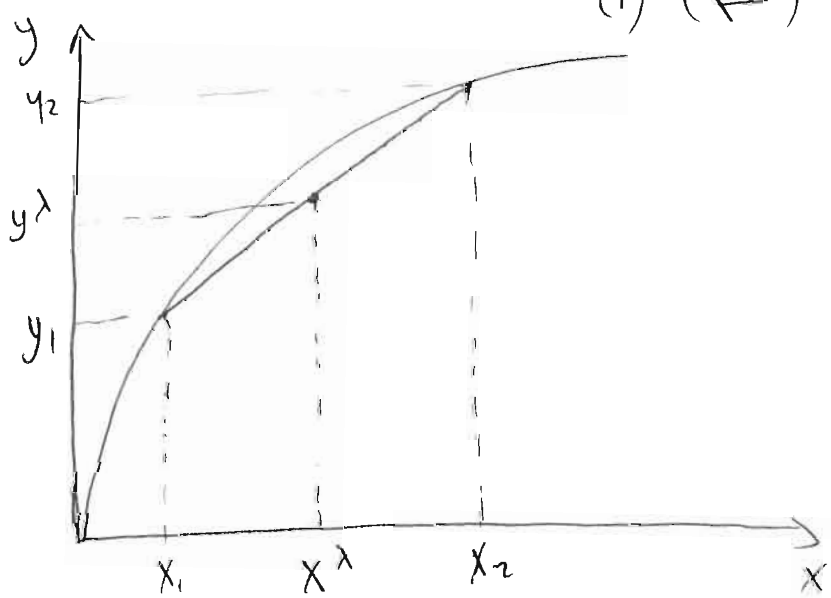
Def 1 $f: X \to \mathbb{R}$ is concave if for any $x, y \in X$ and $\alpha \in [0,1]$ $\quad f(\alpha x + (1-\alpha)y) \geq \alpha f(x) + (1-\alpha) f(y)$

Def 2 $f: X \to \mathbb{R}$ is concave if $hypo(f)$ is convex.

where

$$hypo(f) = \{ (x, r) \in X \times \mathbb{R} : r \leq f(x) \}$$

Claim Def 1 $\Leftrightarrow$ Def 2



(i) $(\Leftarrow)$ $\quad x^\lambda = \lambda x_1 + (1-\lambda) x_2$

$y^\lambda = \lambda . y_1 + (1-\lambda) y_2$

$= \lambda . f(x_1) + (1-\lambda) f(x_2)$

Since $hypo(f)$ is convex

$(x^\lambda, y^\lambda) \in hypo(f)$

$\Rightarrow f(x^\lambda) \geq y^\lambda$ by def

i.e $f(\lambda x_1 + (1-\lambda) x_2)$

$\geq \lambda . f(x_1) + (1-\lambda) f(x_2)$

(ii) ($\Rightarrow$) i.e. Def 1 $\Rightarrow$ Def 2. Suppose f is conc. per Def 1.

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let $z = (x_1, y_1)$ and $t = (x_2, y_2) \in hypo(f)$.

Then $f(x_1) \geq y_1$ and $f(x_2) \geq y_2$ by def. of hypo(f)

Let $\lambda \in [0,1]$. Then by def I.

$$f(\lambda x_1 + (1-\lambda) x_2) \geq \lambda f(x_1) + (1-\lambda) f(x_2)$$
$$\geq \lambda y_1 + (1-\lambda) y_2$$

then

$(\lambda z, (1-\lambda) t) = (\lambda x_1 + (1-\lambda) x_2, \lambda y_1 + (1-\lambda) y_2) \in hypo(f)$

Thus hypo(f) is convex. //