Ex 4.10 Find the optimal bundles given (p, w)

i) $U(x) = x_1^{\alpha} x_2^{\beta}$

$$\mathcal{L} = x_1^{\alpha} x_2^{\beta} + \lambda \left[w - p_1 x_1 - p_2 x_2 \right]$$

FOC

$$\frac{\partial \mathcal{L}}{\partial x_1} = \alpha x_1^{\alpha-1} x_2^{\beta} - \lambda p_1 = 0 \qquad (1)$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \beta x_1^{\alpha} x_2^{\beta-1} - \lambda p_2 = 0 \qquad (2)$$

$$\frac{\partial \mathcal{L}}{\partial x} = w - p_1 x_1 - p_2 x_2 = 0 \qquad (3)$$

combine (1) and (2) to get

$$\frac{\alpha x_1^{\alpha-1} x_2^{\beta}}{\beta x_1^{\alpha} x_2^{\beta-1}} = \frac{p_1}{p_2} \Rightarrow \frac{\alpha x_2}{\beta x_1} = \frac{p_1}{p_2}$$

then $x_2 = \frac{p_1}{p_2} \cdot \frac{\beta}{\alpha} x_1$

substitute this into the BC: (3)

$$p_1 x_1 + \cancel{p_2} \cdot \underbrace{\frac{p_1}{\cancel{p_2}} \cdot \frac{\beta}{\alpha} \cdot x_1}_{x_2} = w \Rightarrow x_1 = \frac{\alpha \cdot w}{(\alpha+\beta) p_1}$$

Then

$$x_2 = \frac{\cancel{p_1}}{p_2} \cdot \frac{\beta}{\cancel{\alpha}} \cdot \frac{\cancel{\alpha} \cdot w}{(\alpha+\beta)\cancel{p_1}}$$

$$= \frac{\beta \cdot w}{(\alpha+\beta) p_2}$$

Which is the same demand function as the log-utility case.

This is not surprising, because ~~logarith~~ these functions are monotone transformations of each other, hence represent the same preferences. i.e

if $u(x) = x_1^{\alpha} x_2^{\beta}$

define $\tilde{u}(x) = \ln[u(x)]$ then

$$\tilde{u}(x) = \ln[x_1^{\alpha} x_2^{\beta}]$$

$$= \ln(x_1^{\alpha}) + \ln(x_2^{\beta})$$

$$= \alpha \ln(x_1) + \beta \ln(x_2) \quad //$$

Ex. 4.10 ii) $u(x) = x_1 - e^{-x_2}$

$$\mathcal{L} = x_1 - e^{-x_2} + \lambda(w - p_1 x_1 - p_2 x_2)$$

FOC

$$\frac{\partial \mathcal{L}}{\partial x_1} = 1 - \lambda p_1 = 0 \qquad (1)$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = e^{-x_2} - \lambda p_2 = 0 \qquad (2)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = w - p_1 x_1 - p_2 x_2 = 0 \qquad (3)$$

combining (1) and (2) gives

$$\frac{1}{e^{-x_2}} = \frac{p_1}{p_2} \quad \text{or} \quad e^{x_2} = \frac{p_1}{p_2}$$

take log of both sides

$$x_2 = \ln(p_1/p_2)$$

: note demand for good 2 is independent of wealth!

Substitute into the BC:

$$p_1 x_1 + p_2 \cdot \ln(p_1/p_2) = w$$

$$x_1 = \frac{w}{p_1} - \frac{p_2 \cdot \ln(p_1/p_2)}{p_1}$$

Consumer Maximization for Leontief Utility

$$\max_{x_1, x_2} \; u(x) = \min\{a x_1, b x_2\} \quad \text{s.t} \quad p_1 x_1 + p_2 x_2 \le w$$

Note at the optimum consumer will set

$$a x_1 = b x_2 \quad \Rightarrow \quad x_2 = \frac{a}{b} x_1$$

substitute this into the BC

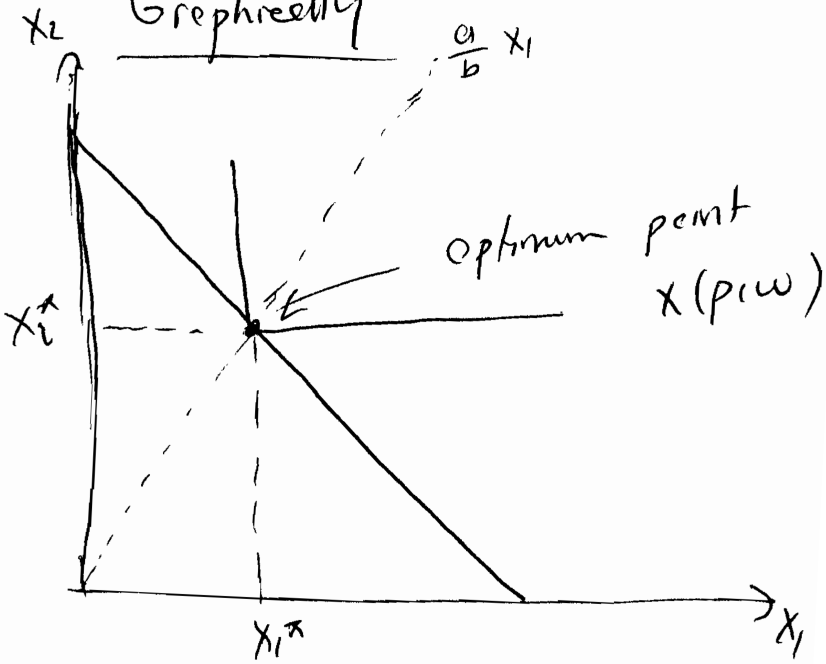
$$p_1 x_1 + p_2 \cdot \frac{a}{b} x_1 = w$$

$$x_1 \cdot \left(p_1 + p_2 \cdot \frac{a}{b}\right) = w$$

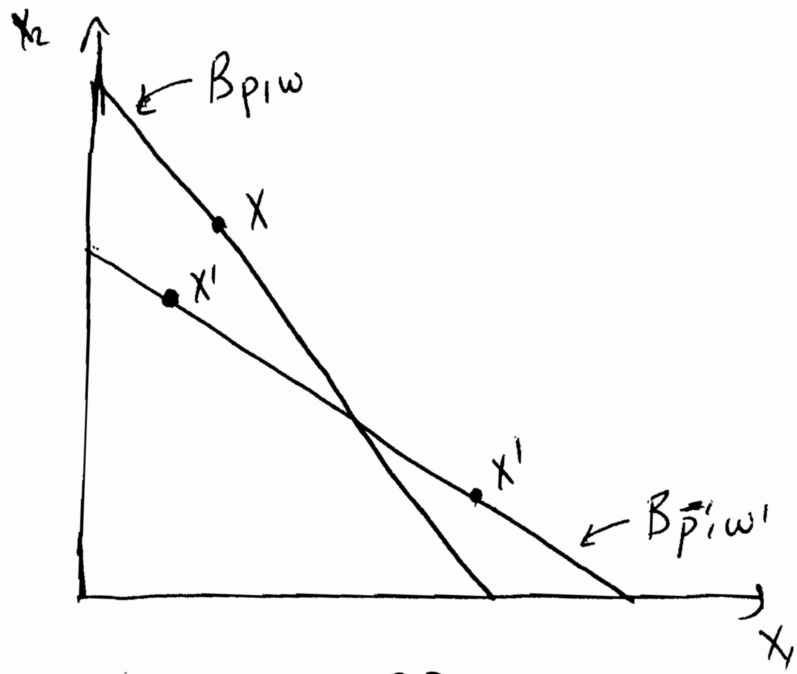
$$x_1 \cdot \left(\frac{b \cdot p_1 + a \cdot p_2}{b}\right) = w \quad \Rightarrow x_1 = \frac{b \cdot w}{b \cdot p_1 + a \cdot p_2}$$

Then $x_2 = \dfrac{a w}{b p_1 + a p_2}$

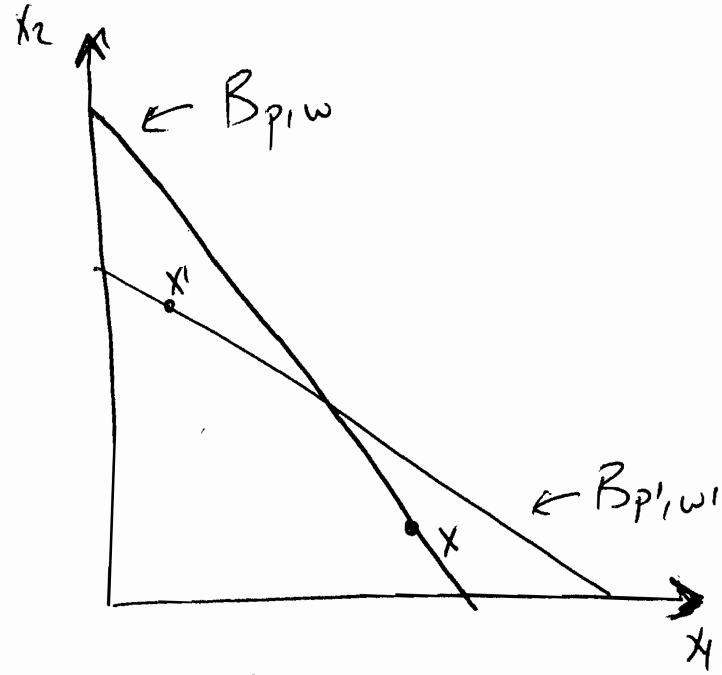
Graphically



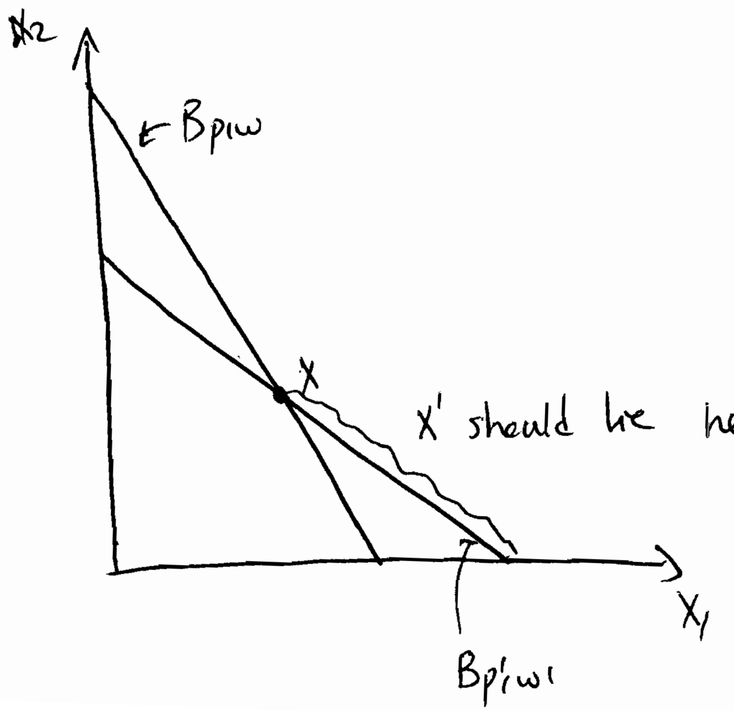
13 Def A function $x(p, w)$ satisfies WARP if $\langle p, x(p', w') \rangle \leq w$ and $x(p, w) \neq x(p', w')$ implies $\langle p', x(p, w) \rangle > w'$.



satisfies WARP



violates WARP!



x' should be here to satisfy WARP!

Ex 5.1 Let $A, B \subset \mathbb{R}^N$ be convex. Show that $\alpha A + (1-\alpha) B$ is also convex for $\alpha \in (0,1)$.

let $x, y \in \alpha A + (1-\alpha) B$

WTS that $\lambda x + (1-\lambda) y \in \alpha A + (1-\alpha) B$ for $\lambda \in [0,1]$.

$x \in \alpha A + (1-\alpha) B$ implies by definition that $\exists\, x_a \in A$ and $x_b \in B$ s.t. $x = \alpha x_a + (1-\alpha) x_b$

similarly for $y \in \alpha A + (1-\alpha) B$, $\exists\, y_a \in A$ and $y_b \in B$ s.t. $y = \alpha y_a + (1-\alpha) y_b$. Therefore:

$$\lambda x + (1-\lambda) y = \lambda [\alpha x_a + (1-\alpha) x_b] + (1-\lambda)[\alpha y_a + (1-\alpha) y_b]$$

$$= \alpha \underbrace{[\lambda x_a + (1-\lambda) y_a]}_{\substack{\in A \text{ since } A \text{ is convex} \\ \equiv a}} + (1-\alpha) \underbrace{[\lambda x_b + (1-\lambda) y_b]}_{\substack{\in B \text{ since } B \text{ is convex} \\ \equiv b}}$$

$$= \alpha \cdot a + (1-\alpha) b \in \alpha A + (1-\alpha) B$$

by definition. //

Exs.2

let $w = 12$ $\quad p_1 = 3$ $\quad p_2 = 4$ $\quad B(p, w) = B$

$w' = 18$ $\quad p_1' = 6$ $\quad p_2' = 3.$ $\quad B(p', w') = B'$

let $x = \left(0, \frac{w}{p_2}\right) = (0, 3)$

$y = \left(0, \frac{w'}{p_2'}\right) = (0, 6)$

This is another simple counter example showing that these two sets are not equal!

let $\alpha = 0.5$.

$\rightarrow \alpha x + (1-\alpha) y = 0.5x + 0.5y \in \alpha B + (1-\alpha) B'$

$= (0, 1.5) + (0, 3) = \underline{\underline{(0, 4.5)}}$ the highest pt.

$\alpha w + (1-\alpha) w' = 15$

$\alpha p + (1-\alpha) p' = 0.5(3, 4) + 0.5(6, 3) = (4.5, 3.5)$

highest pt on x_2 axis $\left(0, \frac{15}{3.5}\right) = (0, 4.28)$
$\neq (0, 4.5)$

So the statement is false!