Prove that $x(p, w)$ is continous in p, w.

Proof. Here, we assume that $\succsim$ is strictly convex, so $x(p,w)$ is a function. (For the general case see Mas-Colell. Chapter 3, Appendix A.)

For contradiction, suppose $x(p,w)$ is not continous in p, w.

Let $(p_n, w_n) \longrightarrow (p^*, w^*)$ s.t. $x_n = x(p_n, w_n)$ and $x^* = x(p^*, w^*)$ but $x_n \longrightarrow y^* \neq x^*$.

Therefore, there exists $\varepsilon > 0$ s.t. $\|x_n - x^*\| > \varepsilon$ for all n.

- Showing that (x_n) has a convergent subsequence.

Since $(p_n), (w_n)$ are convergent sequences, the sets $\{p_n : n \in \mathbb{N}\}$ and $\{w_n : n \in \mathbb{N}\}$ are bounded. Therefore, $\exists$ $\bar{w}$ s.t. $w_n \leq \bar{w}$ for all n, and since $p_n \gg 0$ $\forall n$ and $\exists$ $m > 0$ s.t. $p_n \gg (m, \ldots, m)$ for all n. Therefore $x_n \in [0, \bar{w}/m]^N$. Since $[0, \bar{w}/m]$ is compact, the sequence (x_n) has a convergent subsequence. Rename subsequence, say $x_n \longrightarrow y^* \neq x^*$.

- Recall the following properties of inner product spaces:

If (x_n) and (y_n) are sequences in $\mathbb{R}^N$ that converge to x and y respectively, then $\langle x_n, y_n \rangle \longrightarrow \langle x, y \rangle$

Note $\langle p_n, x_n \rangle \leq w_n$ for all n, taking the limits of both sides as $n \longrightarrow \infty$ gives $\langle p^*, y^* \rangle \leq w^*$ i.e. $y^* \in B(p^*, w^*)$. But since x^* is the unique optimal bundle in $B(p^*, w^*)$ we have that $x^* \succ y^*$.

Let $N_\varepsilon(z)$ denote the ε ball around $z \in X$. Since $\succsim$ is continuous, there exists an $\varepsilon > 0$ s.t. $N_\varepsilon(x^*) \cap N_\varepsilon(y^*) = \emptyset$ and for any $z \in N_\varepsilon(x^*)$ and $z' \in N_\varepsilon(y^*)$, $z \succ z'$.
Moreover, $\exists\, M \in \mathbb{N}$ s.t. for any $n > M$ $x_n \in N_\varepsilon(y^*)$. (Since $x_n \longrightarrow y^*$).

(Continued in the next page!)

Let $\delta > 0$ be s.t $z = x^* - \delta(1, \ldots 1) \in N_\varepsilon(x^*)$

Then $\langle p^*, z \rangle = \langle p^*, x^* \rangle - \delta \sum p_i < w^*$

(since $\langle p^*, x^* \rangle \leq w^*$)

Let $\gamma > 0$ s.t. $\langle p^*, z \rangle < w^* - \gamma$

Since $p_n \longrightarrow p^*$, $\exists\, \mathcal{M}_2$ s.t for all $n > \mathcal{M}_2$

$\langle p^n, z \rangle < w^* - \gamma$ (Since $\langle p^n, z \rangle \longrightarrow \langle p^*, z \rangle < w^* - \gamma$)

Also since $w^n \longrightarrow w^*$ $\exists\, \mathcal{M}_3$ s.t for all $n > \mathcal{M}_3$

$w^n \in N_\eta(w^*)$ where $\eta < \gamma$. (such an η clearly exists)

Let $M = \max\{\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3\}$.

Then for all $n > M$ we have that

$\langle p^n, z \rangle < w^* - \gamma < w^n < w^*$ or

$\langle p^n, z \rangle < w^* - \gamma < w^* < w^n$ for all such n.

i.e $\langle p^n, z \rangle \in B(p^n, w^n)$ for these n which

But we also have $z \succ x^n$ for these n. Hence $x(p, w)$

contradicts to the optimality of x^n.

is continous in (p, w).

<u>Ex 5.2</u> Let $B(p,w)$ and $B(p',w')$ be budget sets. Consider the claim that, for any $\alpha \in (0,1)$

$$\alpha B(p,w) + (1-\alpha) B(p',w') = B(\alpha p + (1-\alpha) p', \alpha w + (1-\alpha) w')$$

Is this true? Prove or provide a counter example.

- This statement is not true. Here's a counter example.

For convenience let $B \equiv B(p,w)$, $B' \equiv B(p',w')$

$B^{\alpha} = \alpha B + (1-\alpha) B'$ and $B^{\lambda} = B(\alpha p + (1-\alpha) p', \alpha w + (1-\alpha) w')$

i) Show that $B^{\lambda} \subset B \cup B'$ (i.e $x \in B^{\lambda} \Rightarrow x \in B$ or $x \in B'$)

We'll prove the contrapositive statement. i.e.

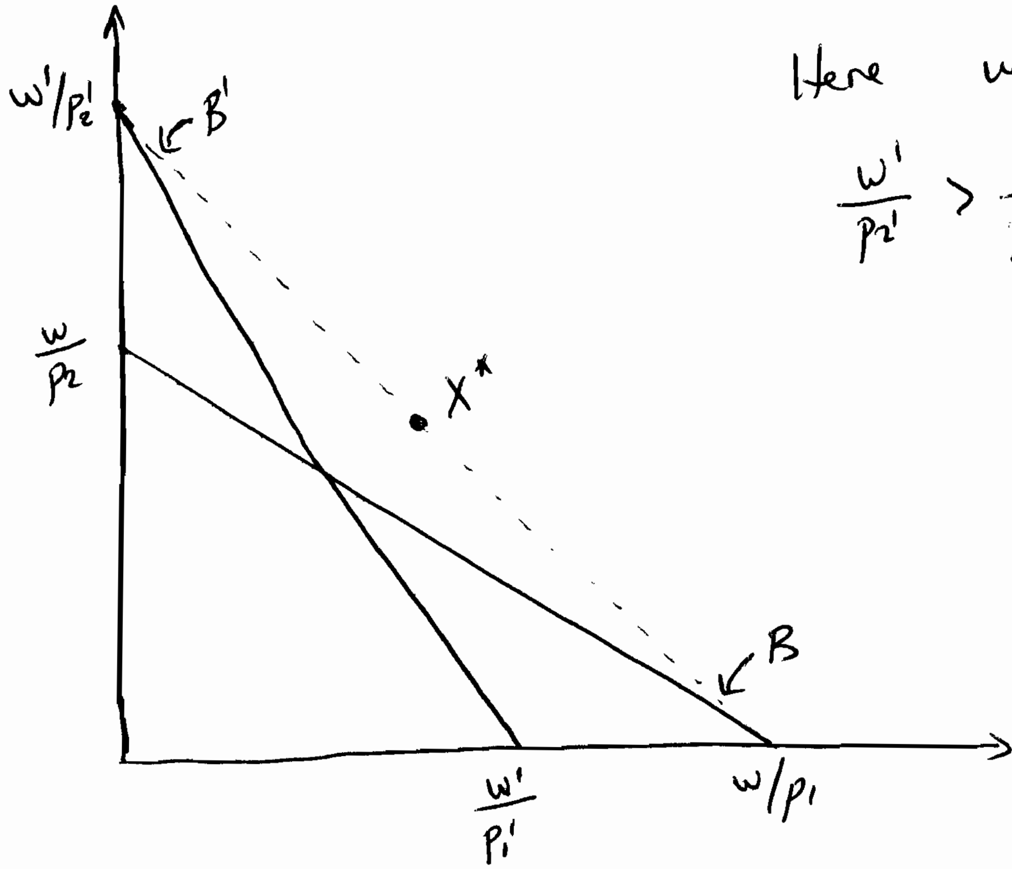
if $x \notin B$ and $x \notin B'$ then $x \notin B^{\lambda}$.

Let $x \notin B$ and $x \notin B'$ then by def. $x \cdot p > w$ and $x \cdot p' > w'$ which implies ~~[illegible]~~ $x \cdot \alpha p + x \cdot (1-\alpha) p' > \alpha w + (1-\alpha) w'$

i.e $x(\alpha p + (1-\alpha) p') > \alpha w + (1-\alpha) w' \Rightarrow x \notin B^{\lambda}$.

hence $B^{\lambda} \subset B \cup B'$.

Now look at the following example:



Here we have

$$\frac{w'}{p_2'} > \frac{w}{p_2} \quad \text{and} \quad \frac{w}{p_1} > \frac{w'}{p_1'}$$

$\left(\frac{w}{p_1}, 0\right) \in B$

Let $\alpha \in (0,1)$. Since $\left(0, w'/p_2'\right) \in B'$ and ~~[illegible]~~ and x^α lies on the line that connects this points s.t.

$$X^\alpha = \alpha \cdot \left(\frac{w}{p_1}, 0\right) + (1-\alpha)\left(0, \frac{w'}{p_2'}\right) = \left(\alpha \frac{w}{p_1}, (1-\alpha)\frac{w'}{p_2'}\right)$$

hence $X^\alpha \in B^\alpha$ by construction.

WTS that $X^\alpha \notin B^\lambda$ or equivalently $X^\alpha \notin B$ and $X^\alpha \notin B'$

using our 1st claim.

i) Suppose ~~x^*~~ $x^* \in B$

Then $p \cdot x^* \leq w$

$$\Leftrightarrow \quad \cancel{p_1}\alpha \frac{w}{\cancel{p_1}} + p_2 \cdot (1-\alpha) \frac{w'}{p_2'} \leq w$$

$$\cancel{(1-\alpha)} p_2 \cdot \frac{w'}{p_2'} \leq \cancel{(1-\alpha)} w$$

$$\frac{w'}{p_2'} \leq \frac{w}{p_2}$$ but this contradicts to $\frac{w'}{p_2'} > \frac{w}{p_2}$ (our assumption)

Hence $x^* \notin B$.

ii) Suppose $x^* \in B'$. then $p' \cdot x^* \leq w'$

$$\Leftrightarrow \quad p_1' \cdot \alpha \cdot \frac{w}{p_1} + \cancel{p_2'} \cdot (1-\alpha) \frac{w'}{\cancel{p_2'}} \leq w'$$

$$\cancel{\alpha} \cdot p_1' \cdot \frac{w}{p_1} \leq \cancel{\alpha} w'$$

$$\frac{w}{p_1} \leq \frac{w'}{p_1'}$$ which contradicts to our assumption $\frac{w}{p_1} > \frac{w'}{p_1'}$.

Hence we have $x \notin B$ and $x \notin B'$ which implies $x \notin B^\lambda$.

Therefore $B^\lambda \neq B^\alpha$!