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Thinking About Mathematical Thinking

A review of

Mathematical Reasoning: Patterns, Problems, Conjectures, and Proofs

by Raymond S. Nickerson

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Reviewed by
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Perhaps no academic subject elicits stronger or more varied reactions than mathematics. For some people, there is no greater joy than discovering the beauty of an elegant proof or a solution to a complex puzzle. For others, the subject produces only fear and loathing. In *Mathematical Reasoning: Patterns, Problems, Conjectures, and Proofs*, Raymond Nickerson explores the history and foundations of mathematics, reviews most of what is known about how mathematicians approach their subject matter, and explores the properties that give rise to such diverse responses.

His unifying concepts—pattern discovery, problem solving, conjecture, and proof—are central to many areas of psychology. His book, therefore, contains much of value to those who are interested in thought processes, as well as to mathematicians who want to know more about the psychological background of their own discipline.

In recent years Nickerson has devoted his time and energy to reviewing a number of important topics that bear on psychology—chance, rationality, and now mathematical reasoning (Nickerson, 2004, 2008; see also *PsycCRITIQUES* reviews of these two books, Pitz, 2005, 2008). As with the earlier books, he demonstrates mastery of a complex area and reviews a range of issues that have generated research and controversy. Again the range of his scholarship is impressive.

Mathematics has been an important tool in the development of psychological theory, as it has with other sciences. Some of the contributions of mathematics to psychology are discussed in the book, but they are not its primary focus. Rather, Nickerson is concerned with the nature of mathematics itself, and he explores ways in which psychology and other social sciences might contribute to a better understanding of the field.

Nickerson begins with a survey of its historical development. He traces the development of ideas from the earliest evidence of number systems through recent struggles to provide a coherent foundation for mathematical theory. Much of the history has been covered by others, sometimes more completely. It is worth including here, though, as a background for topics that he discusses later. Knowing the history helps the reader gain perspective on the subsequent topics, many of which should be of direct interest to psychologists.

One of the recent developments in mathematics that has had a significant impact on some areas of psychology is Gödel's incompleteness theorems, which prove the fundamental incompleteness of mathematics. In any axiomatic system there are always propositions that are true but that cannot be proved to be true. Some writers have asserted that Gödel's proof demonstrates the impossibility of such enterprises as artificial intelligence. Nickerson's discussion of the topic is thoughtful and well reasoned, and makes clear that it does not limit the power of mathematics as a theoretical tool.

Exploring the Minds of Mathematicians

Although mathematics might appear to some people to be the epitome of precise thinking, several unresolved ambiguities have pervaded its history. Do numbers exist independently of human knowledge? What makes a proof convincing or otherwise? How can the paradoxes of infinity and the infinitesimal be seen as compelling by some and absurd by others?

Nickerson provides an elegant analysis of these points and notes that psychological investigation might shed light on some of them. To date, though, no psychologist seems to have taken on that task. Mathematics is a cognitive activity and as such is of interest to anyone who wants to understand how the mind works. A common theme that emerges in the history of mathematics is that established habits of mind create serious barriers to progress and development.

For example, the aversion toward irrational numbers evidenced by the early Greeks placed limits on the kind of mathematics they could consider. Before the invention of place notation for representing numbers, many methods we now take for granted were impossible. It is remarkable how the cognitive difficulties encountered by one generation of mathematicians may readily be overcome by later generations, once the breakthrough has occurred. One wonders what barriers might exist today that inhibit our progress.

Advances in mathematics occur because of the problem-solving activity of its experts. Aids such as computers notwithstanding, mathematics is a discipline that relies on cognitive activity alone. Yet surprisingly little is known about the process by which mathematicians carry out their intellectual work. Many are reluctant to examine their own thought processes. The British mathematician G. H. Hardy was famously dismissive of mathematicians who waste their time describing what they do rather than getting on with the job itself, and his attitude is probably common. Yet a cognitive psychologist must assume that an examination of these thought processes would be productive.

A small number of mathematicians, notably George Polya and Imre Lakatos, have written about their own thinking, and Nickerson makes extensive use of their accounts. However, it is not clear to what degree one can trust first-person accounts. Retrospective assessments of a person's thought processes are notoriously subject to distortion. Ericsson and Simon (1984) reviewed the use of verbal protocols as a means for studying problem-solving processes. They noted a difference between protocols recorded at the time of the problem solving and protocols that are recorded later. The latter, especially if recorded long after the problem-solving activity, can be inaccurate accounts of the cognitive processes.

Nickerson devotes one chapter to the "predilections, presumptions, and personalities" of mathematicians. As he says, "mathematicians represent as variable and colorful an assortment . . . [as] any disciplinary group" (p. 249). He has little to say, though, about the unusual personality traits of some mathematicians. Broad generalizations are, of course, unjustified.

Nevertheless, the peculiar traits of some (Grigory Perelman, for example, who recently refused to accept the Fields medal and other prizes) have inspired a number of writers. Fictional accounts and biographies may fuel beliefs in a connection between the reclusive character of many mathematicians and some aspects of autism. Is there any reality to this common belief? There is indeed some evidence to support it (e.g., Baron-Cohen, Wheelwright, Skinner, Martin, & Clubley, 2001), raising interesting research questions. Nickerson, though, avoids the issue.

Research on the Teaching of Mathematics

Although the minds and personalities of expert mathematicians have not attracted much study by psychologists, the education of beginners has been a prolific topic of research. In spite of the large number of publications in this area, however, it proves difficult to offer well-established conclusions that might help the educator.

Nickerson begins with a review of what is known about young children. It is clear that some understanding of numbers is present in infants. The foundation for this understanding must be innate, but early experience is obviously important also. Culture and language seem to play an important role in the later development of mathematical skills. For example, Nickerson cites research suggesting that language differences can affect the ease with which children from different cultures acquire the ability to perform basic arithmetic.

Nickerson perpetuates one unfortunate misunderstanding when discussing the role of innate and experiential factors by asking about the relative importance of these factors. Such a question is meaningless when one is considering development in a single person. (It is akin to asking whether hardware or software is more important in explaining the performance of a computer.) The question does make sense, though, when one is addressing differences among individuals or groups.

It is clear that cultural and experiential factors are very influential in accounting for differences. Nickerson points out, for example, that different beliefs across cultures concerning the roles of nature and nurture may be significant sources of differences. He provides a summary of the contentious issues surrounding sex differences in mathematical abilities and touches quickly on the many explanations that have been proposed for observed results. A thorough discussion of this topic would require a book to itself.

Regardless of group and individual differences, there seems to be a consensus that achievements by students in the United States give cause for concern. Unfortunately, while it is easy to claim that psychology has a vital role to play in improving education in mathematics, there are not many practical conclusions to be derived from the existing research. A task force established by the American Psychological Association has reviewed similar topics (Newcombe et al., 2009). That report presumably was published too late to be included in Nickerson's review, but it provides a useful supplement.

There are many directions that researchers might take in looking for practical solutions. Nickerson suggests that a study of the history of mathematics might provide insights into the difficulties faced by students, thus providing an effective bridge to his earlier chapters. He also points out that we must determine the purpose to be served by mathematics before we can determine how best to design an educational program. In that regard, he emphasizes how important is the playful nature of the subject. Indeed, since motivational factors are so critical, it is appropriate that educators and researchers alike consider how students might discover the joy that some people find in mathematics.

Those who take delight in mathematics know how difficult it can be to convey that delight to others. Yet one would like to believe that it is possible and that it is not restricted to experts. A charming novel by Yoko Ogawa (2009), *The Housekeeper and the Professor*, describes the reaction of an untutored housekeeper when she learns of amicable numbers—two numbers such that each is equal to the sum of the proper divisors of the other. The housekeeper finds something magical about this affinity between 220 (her date of birth) and 284 (a number on the professor's watch). What is the source of her sense of magic? Is the spontaneous discovery of mathematics found only in fiction? There has been too little exploration of these questions, by mathematicians or by psychologists. Nickerson's chapters devoted to the beauty of mathematics might point the way.

I found little to quarrel with in Nickerson's presentation. The only weakness, I thought, was that the book does not cite much research conducted since 2000, especially in the area of education. It is not clear that any breakthroughs have been achieved in the last 10 years, but one would like to know where we stand now on some of the issues. For example, the role of metacognition in learning has been studied extensively in recent years (see Wateres & Schneider, 2010, for example), so the short section on metacognition seems inadequate.

Nevertheless, there is a wealth of useful information here. The book offers more questions than answers, but the most important contribution that it can make is to point to the potential for future research that lies in the unanswered questions. I hope that readers take it as an invitation and inspiration to address the connections between mathematical reasoning and psychology.

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