Supplemental Material for
"Heavy Tail Robust Frequency Domain Estimation"

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A Introduction

This appendix presents the theory for minimum mean-squared-error selection of the trimming fractile $k_{T,h}$ in the special case where $y_t y_{t-h}$ has a symmetric distribution for $h \neq 0$ (Section B). It also contains details on the robust Whittle estimator (Section C), the omitted proofs of Theorems 2.3 and 3.3 (Section D), and omitted tables (Section E).

Let $\tilde{\gamma}_{T,h}$ be the quantity used in practice for centering:

$$\tilde{\gamma}_{T,h} \equiv \frac{1}{T} \sum_{t=h+1}^{T} y_t y_{t-h} \text{ if } P(y_t y_{t-h} > 0) < 1, \text{ else } \tilde{\gamma}_{T,h} \equiv 0,$$

and define its probability limit:

$$\tilde{\gamma}_h \equiv E[y_t y_{t-h}] \text{ if } P(y_t y_{t-h} > 0) < 1, \text{ else } \tilde{\gamma}_h \equiv 0.$$

Usable lags are

$$h = \{0, 1, ..., b_T\}$$

for a sequence of bandwidths $\{b_T\}$.

Define $\mathcal{Z}_s := \sigma(y_{\tau} : s \leq \tau \leq t)$ and mixing coefficients $\alpha_h := \sup_{\mathcal{A} \subset \mathcal{Z}_{-\infty}, \mathcal{B} \subset \mathcal{Z}_{t+h}} |P(\mathcal{A} \cap \mathcal{B}) - P(\mathcal{A})P(\mathcal{B})|$. Let $L_2(\mathcal{F}) := L_2(\Omega, \mathcal{F}, \mathcal{P})$ be the space of $\mathcal{F}$-measurable $L_2$-bounded random variables, define $\rho(\mathcal{A}, \mathcal{B}) := \sup_{f \in L_2(\mathcal{A}), g \in L_2(\mathcal{B})} |\text{corr}(f, g)|$ and let $\mathcal{G}_h$ and $\mathcal{T}_h$ be non-empty subsets of $\mathbb{N}$ with

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inf_{s \in \mathcal{G}_h, t \in \mathcal{T}_h} \{|s - t| \geq h\}. Define the interlaced maximal correlation coefficient \( \rho_h^* := \sup_{\mathcal{G}_h, \mathcal{T}_h} \rho(y_t : t \in \mathcal{G}_h, s \in \mathcal{T}_h) \) where the supremum is taken over all \( \mathcal{G}_h \) and \( \mathcal{T}_h \).

We use the following assumptions.

**Assumption A** (data generating process).

1. \( \{y_t y_{t-h}\} \) is a stationary, \( L_p \)-bounded process, \( p > 1 \), with an absolutely continuous non-degenerate distribution with unbounded support. Further, \( y_t \) has absolutely summable covariances, and is \( \alpha \)-mixing \( \alpha_h = O(h^{-\rho/(p-2)}) \) with \( \rho_1^* < 1 \).

2. \( y_t \) has spectrum \( f(\lambda, \theta_0) \) for unique \( \theta_0 \) in the interior of compact \( \Theta \subset \mathbb{R}^k \) with properties:

   (i) \( \theta_0 \neq \theta \) then \( f(\lambda, \theta) \neq f(\lambda, \theta_0) \);

   (ii) \( 0 < f(\lambda, \theta) \leq K < \infty \) for each \( \lambda \in [-\pi, \pi] \) and \( \theta \in \Theta \);

   (iii) \( f(\lambda, \theta) \) is twice continuously differentiable in \( \theta \), with derivatives \( (\partial/\partial \theta)^i f(\lambda, \theta) \) for \( i = 1, 2 \) uniformly bounded on \( [-\pi, \pi] \times \Theta \);

   (iv) \( h(\lambda, \theta) \in \{f(\lambda, \theta), (\partial/\partial \theta)f(\lambda, \theta)\} \) are uniformly Hölder continuous of degree \( \alpha \in (1/2, 1] \) in \( \lambda \): \( \sup_{\theta \in \Theta} |h(\lambda, \theta) - h(\omega, \theta)| \leq K|\lambda - \omega|^{\alpha} \) for all \( \lambda, \omega \in [-\pi, \pi] \) and some \( K > 0 \).

3. \( \inf_{\theta \in \Theta} |\sigma(\theta)| > 0 \).

**Assumption B** (regularly varying tails). \( P(y_t y_{t-h} - \hat{\gamma}_h \leq -c) = L_{h,1}(c)c^{-\kappa_{h,1}} \) and \( P(y_t y_{t-h} - \hat{\gamma}_h \geq c) = L_{h,2}(c)c^{-\kappa_{h,2}} \) where \( L_{h,i}(c) \) are slowly varying, and \( \kappa_{h,i} > 1 \).

**Assumption B’** (second order power law and fractile rates). \( P(y_t y_{t-h} - \hat{\gamma}_h \leq -c) = d_{h,1}c^{-\kappa_{h,1}}(1 + O(r_1(c))) \) and \( P(y_t y_{t-h} - \hat{\gamma}_h \geq c) = d_{h,2}c^{-\kappa_{h,2}}(1 + O(r_2(c))) \), where \( d_{h,i} > 0, \kappa_{h,i} > 1 \), and \( r_1 \) are measurable functions. Let \( e_{h,i} > 0, e_{h,0} \equiv \min\{e_{h,1}, e_{h,2}\} \) and \( \kappa_{h,0} \equiv \min\{\kappa_{h,1}, \kappa_{h,2}\} \). Then \( m_{T,h} \in \{1, ..., T-h\} \) and \( m_{T,h} \to \infty \), and either \( r_1(c) = c^{-\kappa_{h,1}} \) and \( m_{T,h} = o((T-h)^{2\kappa_{h,0}/(2\kappa_{h,0} + \kappa_{h,1})}) \), or \( r_1(c) = \ln(c)^{-\kappa_{h,1}} \) and \( m_{T,h} = o(\ln(T-h)^{2\kappa_{h,0}}) \). Finally, \( m_{T,h}/k_{T,h} \to \infty \).

**Assumption C** (trimming and bandwidth rates).

1. In general \( k_{T,h} \to \infty, k_{T,h}/(T-h) \to 0 \), and \( k_{T,h} \sim K_{h,h}k_{T,h} \) for some \( K_{h,h} > 0 \) and each \( h \in \{0, ..., b_T\} \). If mean-centering at lag \( h \) is used then \( k_{T,h} \to \infty \) at most at a slowly varying rate.

2. Let \( b_T \leq T - 1, b_T \to \infty \), and \( T - b_T \to \infty \). Further \( b_T/T^{1/(2\alpha)} \to \infty \) where \( \alpha \in (1/2, 1] \) is the Hölder continuity degree in Assumption A.2.iv.

## B  Mean-Squared-Error Minimization

Write the bias and scale as \( B_T = [B_{i,T}]_{i=1}^k \) and \( V_T = [V_{i,j,T}]_{i,j=1}^k \). By Theorem 2.2 \( T^{1/2}V_T^{-1/2}(\hat{\theta}_T^* - \theta_0 + B_T) \overset{d}{\to} N(0, I_k) \), hence the first order mean-squared-error \( \text{[mse]} \) \( M_{i,T} \) of \( \hat{\theta}_{i,T}^* \) is

\[
M_{i,T} \equiv B_{i,T}^2 + V_{i,i,T}/T.
\]
The bias of $\hat{\theta}_T^*$ is complicated unless $y_t y_{T-h}$ is symmetrically distributed for $h > 1$, since then $B_T = \Omega^{-1}(2\pi)^{-1} \omega \{ E[y_t^2 I(y_t^2 < c_{T,0})] - E[y_t^2] \}$ only depends on $E[y_t^2 I(y_t^2 < c_{T,0})]$. In this case, the fmse minimizing $k_{T,h}$ satisfies $k_{T,h} \to [0, \infty)$, a constant that depends on $\kappa \equiv \arg\sup \{ \alpha > 0 : E|y|^\alpha < \infty \} > 2$.

**Theorem B.1** Let Assumptions A-C hold, and assume $y_t y_{T-h}$ for $h \neq 0$ have symmetric distributions. The fmse minimizing $k_{T,h} \sim K > 0$ if $\kappa \in (2, 4) \cup (4, \infty)$, and $k_{T,h} = 0$ if $\kappa = 4$.

**Proof.** By distribution symmetry of $y_t y_{T-h}$ for $h \neq 0$, Theorem 2.2 and Karamata’s Theorem $B_i,T = K \{ E[y_i^2 I(y_i^2 < c_{T,0})] - E[y_i^2] \} \sim K(k_T/T) c_{T,0} \sim K(T/k_T)^{2/\kappa-1}$, hence squared bias is $B_{i,T} \sim K(T/k_T)^{4/\kappa-2}$.

Next, by Theorem 2.2 $V_{i,i,T} \sim K E[y_i^4 I(y_i^2 < c_{T,0})]$. If $\kappa > 4$ then $V_{i,i,T} \sim K \{ E[y_i^4] - E[y_i^4 I(y_i^2 \geq c_{T,0})] \}$, while by Karamata’s Theorem $E[y_i^4 I(y_i^2 \geq c_{T,0})] \sim K(k_T/T) c_{T,0}^2 = K ((k_T/T))^{1-4/\kappa}$. Now use Theorem 2.2 to deduce by case that if $\kappa > 4$ then $V_{i,i,T} \sim K \{ E[y_i^4] - K(k_T/T)^{1-4/\kappa} \}$, if $\kappa = 4$ then $V_{i,i,T} \sim L(T) \to \infty$ for some slowly varying $L(T)$, and if $\kappa < 4$ then $V_{i,i,T} \sim K(T/k_T)^{4/\kappa-1}$.

Therefore $M_{i,T} \sim \tilde{M}_{i,T}(k_T)$, where by case:

$$\kappa > 4 : \tilde{M}_{i,T}(a) = K \left( \frac{a}{T} \right)^{2-4/\kappa} + K \frac{1}{T} \left\{ E[y_i^4] - K \left( \frac{a}{T} \right)^{1-4/\kappa} \right\}$$

$$\kappa = 4 : \tilde{M}_{i,T}(a) = K \left( \frac{a}{T} \right)^{2-4/\kappa} + K \frac{1}{T} L(T) = K \frac{a}{T} + K \frac{1}{T} L(T)$$

$$\kappa < 4 : \tilde{M}_{i,T}(a) = K \left( \frac{a}{T} \right)^{2-4/\kappa} + K \frac{1}{T} \left( \frac{T}{a} \right)^{4/\kappa-1}.$$ 

Now let $a \in [0, \infty)$ be real valued and differentiate $\tilde{M}_{i,T}(a)$ to deduce $\tilde{a} \equiv \arg\min_{a \geq 0} \tilde{M}_{i,T}(a)$ satisfies:

$$\kappa > 4 : \frac{\partial}{\partial a} \tilde{M}_{i,T} (\tilde{a}) = K \left( 2 - \frac{4}{\kappa} \right) \frac{\tilde{a}^{1-4/\kappa}}{T^{2-4/\kappa}} - K \left( 1 - \frac{4}{\kappa} \right) \frac{\tilde{a}^{-4/\kappa}}{T^{2-4/\kappa}} = 0$$

for some $\tilde{a} \sim K$.

$$\kappa = 4 : \frac{\partial}{\partial a} \tilde{M}_{i,T} (\tilde{a}) = K \frac{1}{T} > 0$$

hence $\tilde{a} = 0$ 

$$\kappa < 4 : \frac{\partial}{\partial a} \tilde{M}_{i,T} (\tilde{a}) = K \left( 2 - \frac{4}{\kappa} \right) \frac{\tilde{a}^{1-4/\kappa}}{T^{2-4/\kappa}} - K \left( \frac{4}{\kappa} - 1 \right) \frac{T^{4/\kappa-2}}{\tilde{a}^{4/\kappa}} = 0$$

for some $\tilde{a} \sim K$.

This completes the proof. QED.
C Robust Whittle Estimator

Now suppose the spectral density has the form
\[ f(\lambda, \theta) = \frac{\sigma^2}{2\pi} \times \frac{|B(e^{-i\lambda}, \beta)|^2}{|A(e^{-i\lambda}, \beta)|^2} = \frac{\sigma^2}{2\pi} \times \omega(\lambda, \beta) \text{ where } \theta = [\beta', \sigma^2]' , \]  
\hspace{1cm} (C.1)

where \( A \) and \( B \) are continuously differentiable functions of \((\lambda, \beta) \in [-\pi, \pi] \times B\) and \( B \) is a compact subset of \( \mathbb{R}^k \). Autoregressions and squared GARCH processes, for example, satisfy (C.1). Notice Assumption A.2 imposes smoothness and boundedness properties on \( f(\lambda, \theta) \) which instantly apply to \( \omega(\lambda, \beta) \) under obvious restrictions on \( A \) and \( B \).

Define:
\[ \tilde{\gamma}^{(0)}_{h,t} \equiv |y_t y_{t-h} - \hat{\gamma}_{T,h}| \text{ and } \tilde{\gamma}^{(0)}_{h,(1)} \geq \tilde{\gamma}^{(0)}_{h,(2)} \geq \cdots \tilde{\gamma}^{(0)}_{h,(T-h)}. \]

The negligibly transformed optimal bias-corrected Whittle estimator solves
\[ \tilde{\beta}_T^{(obc)} = \arg \min_{\beta \in B} \sum_{j \in F} \tilde{I}_T^{(obc)}(\lambda_j) \omega(\lambda_j, \beta) \]
where
\[ \tilde{I}_T^{(obc)}(\lambda) = \frac{1}{2\pi} \left( \tilde{\gamma}^{(obc)}_{T,0} (\tilde{\gamma}^{(0)}_{0,(kT,h)}) + 2 \sum_{h=1}^{b_T} \tilde{\gamma}^{(obc)}_{T,h} (\tilde{\gamma}^{(0)}_{h,(kT,h)}) \times \cos(\lambda h) \right). \]

See Section 3 in the main paper for details on \( \tilde{\gamma}^{(obc)}_{T,h} (\tilde{\gamma}^{(0)}_{h,(kT,h)}) \) and the bandwidth \( b_T \).

Define \( \tilde{\nu}_T := \tilde{\Omega}^{-1} \tilde{S}_T \tilde{\Omega}^{-1} \) where
\[ \tilde{\Omega} := \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\partial \ln \omega(\lambda)}{\partial \theta} \frac{\partial \ln \omega(\lambda)}{\partial \theta'} d\lambda \]
\[ \tilde{S}_T := T \times E \left[ \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{I_T(\lambda)}{\omega(\lambda)} \frac{\partial \omega(\lambda)}{\partial \beta'} d\lambda \right) \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{I_T(\lambda)}{\omega(\lambda)} \frac{\partial \omega(\lambda)}{\partial \beta'} d\lambda \right) \right]. \]

The following theorem can be proved by exploiting our proof of Theorems 2.2 and 3.1 and following Hannan (1973)'s proof of his Theorem 2. We therefore omit the proof.

**Theorem C.1** Under Assumptions A, B' and C we have \( T^{1/2} \tilde{\nu}_T^{1/2} (\tilde{\beta}_T^{(obc)} - \beta_0) \xrightarrow{d} N \left(0, I_k\right)\).

The asymptotic variance has a classic structure if \( y_t \) is linear with a finite fourth moment. Consider
\[ y_t = \sum_{i=0}^{\infty} \xi_i (\theta_0) \epsilon_{t-i} \text{ where } \sum_{i=0}^{\infty} \xi_i^2 (\theta_0) < \infty, \xi_0 (\theta_0) = 1 \]  
\hspace{1cm} (C.2)
\[ \sigma^2 (\theta_0) := E[\epsilon_t^2] < \infty \text{ and } E[\epsilon_s \epsilon_t] = 0 \forall s \neq t, \]
where \( \epsilon_t \) is a homoscedastic martingale difference. Define the \( \sigma \)-field \( \mathcal{F}_t := \sigma(y_{\tau} : \tau \leq t) \). Theorem C.1, negligibility, dominated convergence, and arguments in Dunsmuir (1979, proof of Theorem 2.1) imply the next result. Define the \( \sigma \)-field \( \mathcal{F}_t := \sigma(y_{\tau} : \tau \leq t) \).

**Corollary C.2** In addition to Assumption A, let \( y_t \) satisfy (C.2), and assume \( E[\epsilon_t | \mathcal{F}_{t-1}] = 0 \) a.s., \( E[\epsilon_t^2 | \mathcal{F}_{t-1}] = \sigma^2 \) a.s., \( E[\epsilon_t^3 | \mathcal{F}_{t-1}] = s \) a.s., and \( E[\epsilon_t^4] = K < \infty \). Let \( k_{T,h} \to \infty \) and \( k_{T,h}/(T - h) = o(1) \). Then \( T^{1/2}(\beta_T^{(obs)} - \beta_0) \xrightarrow{d} N(0, \Omega^{-1}) \).

### D. Omitted Proofs

#### D.1 Proof of Theorem 2.3 (FD-QML Bias)

**Theorem 2.3.** Under Assumptions A-D:

\( i. \quad T^{1/2}V_T^{-1/2}(\hat{\theta}_T^* - \theta_0) \xrightarrow{d} N(0, I_k) \) if either \( \kappa > 4 \) or \( \kappa = 4, k_{T,h} = o(\ln(T)) \) and \( P(|y_{T}y_{T-h} - \gamma_h| \geq c) = d_{h,0} e^{-\kappa h,0}(1 + o(1)) \) where \( \kappa_{0,0} = 2 \leq \kappa_{h,0} \);

\( ii. \quad T^{1/2}V_T^{-1/2}(\hat{\theta}_T^* - \theta_0 + B_T) \xrightarrow{d} N(0, I_k) \) if \( \kappa \in (2, 4) \), where \( T^{1/2}\|V_T^{-1/2}B_T\| \sim Kk_{T,h}^{1/2} \to \infty \) if \( y_{T}y_{T-h} \) has a symmetric distribution, else \( \lim_{T \to \infty} T^{1/2}\|V_T^{-1/2}B_T\|/k_{T,h}^{1/2} \in [0, \infty) \).

**Proof.** We first characterize \( \lim_{T \to \infty} (T/\|V_T\|)^{1/2}\{E[y_T^2 I(y_T^2 < c_T,0)] - E[y_T^2]\} \). By negligibility and Theorem 2.2, \( \|V_T\| \sim K[E(y_T^2 I(y_T^2 < c_T,0))] \), and trivially

\[
E[y_T^2 I(y_T^2 < c_T,0)] - E[y_T^2] = E[(y_T^2 - E[y_T^2]) I(y_T^2 \geq c_T,0)],
\]

hence

\[
\left( \frac{T}{\|V_T\|} \right)^{1/2} \{E[y_T^2 I(y_T^2 < c_T,0)] - E[y_T^2]\} \sim -K \frac{T^{1/2}E[y_T^2 I(y_T^2 \geq c_T,0)]}{(E[y_T^2 I(y_T^2 < c_T,0)])^{1/2}}. \tag{D.3}
\]

Recall the tail components \( \mathcal{L}_{h,0}(c) \equiv \mathcal{L}_{h,1}(c) + \mathcal{L}_{h,2}(c) \) and \( \kappa_{h,0} \equiv \max \{ \kappa_{h,1}, \kappa_{h,2} \} \), where \( \kappa_{0,0} = \kappa/2 > 1 \). By Karamata’s Theorem (Resnick 1987, Theorem 0.6):

\[
E[y_T^2 I(y_T^2 \geq c_T,0)] \sim \frac{2}{\kappa - 2} \frac{k_{T,0}}{T} c_{T,0}.
\]

The remaining \( E[(y_T^2 I(y_T^2 < c_T,0)) \in (D.3) \) is \( O(1) \) if \( \kappa > 4 \), and is evaluated for \( \kappa \in (2, 4) \) by Karamata’s Theorem: for some slowly varying \( \tilde{\mathcal{L}}_4(c) \):

\[
\kappa \in (2, 4) : \quad E[y_T^2 I(y_T^2 < c_T,0)] \sim \frac{4}{4 - \kappa} \frac{k_{T,0}}{T} c_{T,0} \tag{D.4}
\]

\[
\kappa = 4 : \quad E[y_T^2 I(y_T^2 < c_T,0)] = \tilde{\mathcal{L}}_4(T).
\]

We now deduce conditions for asymptotic bias \( \lim_{T \to \infty} (T/\|V_T\|)^{1/2}\{E[y_T^2 I(y_T^2 < c_T,0)] - E[y_T^2]\} \neq 0 \). Use \( c_{T,0} \sim \mathcal{L}_{0,0}(T^{2/k}(T/k_{T,0})^{2/k}) \) to deduce if \( \kappa > 4 \) then \( (T/\|V_T\|)^{1/2}\{E[y_T^2 I(y_T^2 < c_T,0)] - E[y_T^2]\} \)
$E[y_t^2] \to 0$ only if $T^{1/2}L_{0,0}(T)^{2/\kappa}(k_{T,0}/T)^{1-4/\kappa} \to 0$. The latter holds by slow variation for $L_{0,0}(T)$, and slow variation for $k_{T,0}$ by Assumption D.

If $\kappa = 4$ then $c_{T,0} \sim L_{0,0}(T)^{1/2}(T/k_{T,0})^{1/2}$ hence $(T/||V_T||)^{1/2}\{E[y_t^4(y_t^2 < c_{T,0})] - E[y_t^2]\} \to 0$ only if

$$T^{1/2}\frac{E[y_t^2 I(y_t^2 \geq c_{T,0})]}{(E[y_t^4 I(y_t^2 < c_{T,0})])^{1/2}} \sim K T^{1/2}(k_{T,0}/T) c_{T,0} \sim K \frac{k_{T,0}^{1/2} L_{0,0}(T)^{1/2}}{L_4(T)^{1/2}} \to 0. \quad (D.5)$$

If $P(|y_t y_{t-h}| - E[y_t y_{t-h}]| \geq c) = d_{h,0}c^{-2}(1 + o(1))$ with $d_{h,0} > 0$, then $L_{0,0}(T) \sim d$, and by direct integration $L_4(T) = d_{0,0} \ln(T)$, hence $k_{T,h} \to \infty$ and $k_{T,h} = o(\ln(T))$ ensure (D.5).

If $\kappa \in (2, 4)$ then we need:

$$T^{1/2}E[y_t^2 I(y_t^2 \geq c_{T,0})] \sim K T^{1/2}(k_{T,0}/T) c_{T,0} = K k_{T,0}^{1/2} \to 0,$$

which is ruled out by $k_{T,h} \to \infty$.

A similar set of derivations applies to any $\gamma^{*}_{T,h}(c_{T,h}) \equiv (T - h)(T - h - k_{T-h})^{-1}E[y_t y_{t-h}] - E[y_t y_{t-h}]$ where $h \neq 0$, although there are two differences. First, when $y_t y_{t-h}$ has a symmetric distribution then $\gamma^{*}_{T,h}(c_{T,h}) = E[y_t y_{t-h}]$ for $h \neq 0$. Second, if $y_t$ is independent then $y_t y_{t-h}$ for $h \neq 0$ has tail index $\kappa > 2$ and therefore $T^{1/2}V_T^{1/2}|\gamma^{*}_{T,h}(c_{T,h}) - E[y_t y_{t-h}]| = o(1)$ if $k_{T,h}/\ln(T) \to 0$ by repeating the above arguments. Otherwise since we trim symmetrically there is potential bias with each $\gamma^{*}_{T,h}(c_{T,h})$: $T^{1/2}V_T^{1/2}|\gamma^{*}_{T,h}(c_{T,h}) - E[y_t y_{t-h}]| = O(k_{T,h}^{1/2})$ when $\kappa \in (2, 4)$, and $T^{1/2}V_T^{1/2}|\gamma^{*}_{T,h}(c_{T,h}) - E[y_t y_{t-h}]| = o(1)$ when $\kappa \geq 4$ as long as $k_{T,h}/\ln(T) \to 0$. Hence by covariance summability Assumption B.1

$$\sum_{h=0}^{\infty} T^{1/2}V_T^{1/2}|\gamma^{*}_{T,h}(c_{T,h}) - E[y_t y_{t-h}]| = O(k_{T,h}^{1/2})$$

and $\sum_{h=0}^{\infty} T^{1/2}V_T^{1/2}|\gamma^{*}_{T,h}(c_{T,h}) - E[y_t y_{t-h}]| = o(1)$ when $\kappa \geq 4$ if $k_{T,h}/\ln(T) \to 0$. The claim now follows from the bias form $B_T = \Omega^{-1}(2\pi)^{-1}\sum_{h=-\infty}^{\infty} \omega_h \gamma^{*}_{T,h}(c_{T,h}) - E[y_t y_{t-h}]$ by Theorem 2.2. $QED$.

D.2 Proof of Theorem 3.3 (Bias-Correction: Thin Tails)

Recall for Theorem 3.3 we assume tails decay exponentially fast for the sake of discussion:

$$P(|y_t y_{t-h} - \gamma^{*}_{T,h}| \geq c) = \vartheta_h \exp\{-\zeta_h e^{\delta_h}\} \text{ where } \vartheta_h, \zeta_h, \delta_h > 0. \quad (D.6)$$

Recall

$$\Omega \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\partial \ln f(\lambda; \theta_0)}{\partial \theta} \frac{\partial \ln f(\lambda'; \theta_0)}{\partial \theta'} d\lambda \quad \text{and} \quad \Pi \equiv \frac{(K - 3\sigma^4)}{\sigma^8(\theta_0)} \frac{\partial \sigma^2(\theta_0)}{\partial \theta} \frac{\partial \sigma^2(\theta_0)}{\partial \theta'}. \quad (D.7)$$

**Theorem 3.3.** Let Assumption A hold, assume (D.6) applies for each $h$, and let $k_{T,h} \to \infty$, $k_{T,h}/\ln(T) \to 0$, $m_{T,h} \to \infty$, and $m_{T,h}/T \to 0$. Then $T^{1/2}R_{T,h} \overset{p}{\to} 0$ hence $T^{1/2}V_T^{1/2}(\tilde{\theta}_T^{\text{oob}} - \theta_0) \overset{d}{\to} N(0, \Sigma)$.

Further, if the conditions of Corollary 2.5 hold then $T^{1/2}(\tilde{\theta}_T^{\text{oob}} - \theta_0) \overset{d}{\to} N(0, \Sigma)$ where

$$\Sigma = \Omega^{-1}(2\Omega + \Pi)\Omega^{-1}$$

and $\Omega$ and $\Pi$ are defined in (D.7).
Proof. We will only show $T^{1/2} \hat{R}_{T,0} \xrightarrow{p} 0$, the remaining $T^{1/2} \hat{R}_{T,h} \xrightarrow{p} 0$ being similar. Drop the lag 0 subscript everywhere, e.g. $c_T = c_{T,0}$, $m_T = m_{T,0}$, $\kappa_{m_T} = \kappa_{0,0,m_T}$, etc. In order to ease the notational burden, we assume $\gamma_h = 0$. Write

$$\mathcal{Y}_t := y_t^2$$

and define order statistics $\mathcal{Y}(1) \geq \mathcal{Y}(2) \geq \cdots \geq \mathcal{Y}(T)$.

For any tiny $\varepsilon > 0$,

$$T^{1/2} \hat{R}_{T,0} = \frac{1}{\kappa_{m_T} - 1} \times \frac{k_T}{T^{1/2-\varepsilon}} \times \frac{c_T}{T} \times \frac{\mathcal{Y}(k_T)}{c_T}.$$ 

We will show $c_T/T^\varepsilon \to 0$, $\mathcal{Y}(k_T)/c_T \xrightarrow{p} 1$, and $1/\kappa_{m_T} \xrightarrow{p} 0$ in three steps. In view of $k_T/T^{1/2-\varepsilon} \to 0$ by the supposition $k_T/\ln(T) \to 0$ the proof is then complete.

Step 1 ($c_T$). By (D.6), and the definition of $c_T$:

$$c_T = \left( \frac{\ln \theta + \frac{1}{\zeta} \ln \frac{T}{k_T} \right)^{1/\delta} \sim \frac{1}{\zeta^{1/\delta}} \left( \ln (T) \right)^{1/\delta} = o(T^\varepsilon). \quad (D.8)$$

Step 2 ($\mathcal{Y}(k_T)$). We will prove $k_T^{1/2} (\ln \mathcal{Y}(k_T) - \ln c_T) = O_p(1)$ hence $\mathcal{Y}(k_T)/c_T = 1 + O_p(1/k_T^{1/2})$. Define, $I_{T,t}(u) := (T/k_T)I(|y_t y_{t-h} > c_T e^u|)$ for any $u \in \mathbb{R}$, and $I_T(u) := 1/T \sum_{t=1}^T I_{T,t}(u)$. By construction $k_T^{1/2} (\ln \mathcal{Y}(k_T) - \ln c_T) \leq u$ for $u \in \mathbb{R}$ if and only if $I_T(u/k_T^{1/2}) \leq 1$, if and only if $\nu_{T} > c_T e^{u/k_T^{1/2}}$.

$$I_T(u/k_T^{1/2}) - E[I_T(u/k_T^{1/2})] \leq 1 - \frac{1}{k_T} P(\mathcal{Y} > c_T e^{u/k_T^{1/2}}) = 1 - \frac{P(\mathcal{Y} > c_T e^{u/k_T^{1/2}})}{P(\mathcal{Y} > c_T)}.$$ 

Now expand $P(\mathcal{Y} > c_T e^{u/k_T^{1/2}})$ around $u = 0$. Use $c_T \sim T^{-1/\delta} (\ln(T))^{1/\delta}$ and $(\partial/\partial c) P(\mathcal{Y} > c) = -\zeta \delta \exp\{-\zeta c\}$, and the mean-value-theorem, to deduce $k_T^{1/2} (\ln \mathcal{Y}(k_T) - \ln c_T) \leq u$ if and only if for some $|u^*| \leq |u|$:

$$k_T^{1/2} \left( I_T(u/k_T^{1/2}) - E[I_T(u/k_T^{1/2})] \right) \leq -\frac{1}{P(\mathcal{Y} > c_T)} \frac{\partial}{\partial c} P(\mathcal{Y} > c) \big|_{c=c_T e^{u^*/k_T^{1/2}}} \times c_T e^{u^*/k_T^{1/2}} \times u$$

$$= \delta \theta \zeta \frac{T}{k_T} \exp\{-\zeta \left(c_T e^{u^*/k_T^{1/2}}\right)^{\delta}\} \left(c_T e^{u^*/k_T^{1/2}}\right)^{\delta-1} \times c_T e^{u^*/k_T^{1/2}} u$$

$$= \delta \theta \ln(T) u \times (1 + o(1)).$$

Therefore $k_T^{1/2} (\ln \mathcal{Y}(k_T) - \ln c_T)$ and $\left(k_T^{3/2}/(\delta \theta \ln(T))\right) \times (I_T(u/k_T^{1/2}) - E[I_T(u/k_T^{1/2})])$ have the same limit distribution. But in view of the Assumption A mixing properties and the zero mean
Therefore \( k_T \) we have by Theorem 1.6 and Lemma 2.1 in McLeish (1975):

\[
k_T^{1/2} \left( T(u/k_T^{1/2}) - E \left[ T(u/k_T^{1/2}) \right] \right) = \frac{1}{T^{1/2}} \sum_{t=1}^{T} \left( \frac{T}{k_T} \right)^{1/2} \left\{ I \left( \mathcal{Y}_t > c_T \right) - E \left[ I \left( \mathcal{Y}_t > c_T \right) \right] \right\} = O_p \left( 1 \right).
\]

By the supposition \( k_T = O(\log(T)) \) it therefore follows

\[
\frac{k_T^{3/2}}{\ln(T)} \left( T(u/k_T^{1/2}) - E \left[ T(u/k_T^{1/2}) \right] \right) = \frac{k_T}{\ln(T)} k_T^{1/2} \left( T(u/k_T^{1/2}) - E \left[ T(u/k_T^{1/2}) \right] \right) = O_p \left( 1 \right).
\]

Therefore \( k_T^{1/2}(\ln \mathcal{Y}(k_T) - \ln c_T) = O_p(1) \).

**Step 3 (\( \kappa_{m_T} \)).** In view of distribution continuity we can define a sequence of positive numbers \( \{\tilde{c}_T\} \) that satisfies

\[
P \left( \sum_{i=1}^{T} \mathcal{Y}_i > \tilde{c}_T \right) = \frac{m_T}{T}.
\]

We have

\[
\hat{\kappa}_{m_T}^{-1} = \frac{1}{m_T} \sum_{j=1}^{m_T} \ln \left( \mathcal{Y}_j / \mathcal{Y}_{(m_T+1)} \right) = \frac{1}{m_T} \sum_{i=1}^{T} \ln \left( \mathcal{Y}_i / \mathcal{Y}_{(m_T+1)} \right) \times I \left( \mathcal{Y}_i > \mathcal{Y}_{(m_T+1)} \right)
\]

\[
= \frac{1}{m_T} \sum_{i=1}^{T} \ln \left( \mathcal{Y}_i / \mathcal{Y}_{(m_T+1)} \right) \times I \left( \mathcal{Y}_i > \mathcal{Y}_{(m_T+1)} \right) + o_p(1)
\]

\[
= \frac{1}{m_T} \sum_{i=1}^{T} \ln \left( \mathcal{Y}_i / \tilde{c}_T \right) \times I \left( \mathcal{Y}_i > \tilde{c}_T \right) + o_p(1) = \frac{1}{T} \sum_{i=1}^{T} \mathcal{Y}_{i,\tilde{i}} + o_p(1),
\]

say. The third equality follows from \( \mathcal{Y}_{(m_T+1)} / \tilde{c}_T = 1 + O_p \left( 1/m_T^{1/2} \right) \) by an application of Step 2 since

\[
\frac{1}{m_T} \sum_{i=1}^{T} \ln \left( \mathcal{Y}_i / \mathcal{Y}_{(m_T+1)} \right) \times I \left( \mathcal{Y}_i > \mathcal{Y}_{(m_T+1)} \right) - \frac{1}{m_T} \sum_{i=1}^{T} \ln \left( \mathcal{Y}_i / \tilde{c}_T \right) \times I \left( \mathcal{Y}_i > \mathcal{Y}_{(m_T+1)} \right)
\]

\[
= \left( \ln \left( \mathcal{Y}_{(m_T+1)} \right) - \ln \left( \tilde{c}_T \right) \right) \times \frac{1}{m_T} \sum_{i=1}^{T} I \left( \mathcal{Y}_i > \mathcal{Y}_{(m_T+1)} \right)
\]

\[
= O_p \left( \frac{1}{m_T^{1/2}} \times \sum_{i=1}^{T} I \left( \mathcal{Y}_i > \mathcal{Y}_{(m_T+1)} \right) \right) = O_p \left( 1/m_T^{1/2} \right).
\]
Notice this exploits the construction of $\mathcal{Y}_{(mT+1)}$ and distribution continuity: $1/mT \sum_{i=1}^{T} I(Y_i > \mathcal{Y}_{(mT+1)}) = 1$ a.s. The fourth equality in (D.9) can be verified under very general conditions in view of the stationary mixing property and the fact that $\mathcal{Y}_{(mT+1)}/\tilde{c}_T = 1 + O_p(1/mT^{1/2})$. The method of proof is identical to Lemmas A.2 and A.3 in Hill (2012).

In order to prove $\hat{\kappa}_{mT}^{-1} \xrightarrow{p} 0$ it remains to verify $1/T \sum_{i=1}^{T} \mathcal{Y}_{T,i} \xrightarrow{p} 0$. The variable $\mathcal{Y}_{T,i} = (T/mT) \ln(Y_i/\tilde{c}_T) I(Y_i > \tilde{c}_T)$ has a positive finite mean that decays to zero. This follows by invoking exponential tail bound (D.6) and $P(y_i^2 > \tilde{c}_T) = mT/T$ to obtain

$$E[\mathcal{Y}_{T,i}] = \frac{T}{mT - 1} \int_0^\infty P(\ln(Y_i/\tilde{c}_T) > u) du = \frac{T}{mT - 1} \int_0^\infty P(Y_i > \tilde{c}_Te^u) du$$

$$= \frac{T}{mT - 1} P(Y_i > \tilde{c}_T) \int_0^\infty \frac{P(Y_i > \tilde{c}_Te^u)}{P(Y_i > \tilde{c}_T)} du$$

$$= \frac{mT}{mT - 1} \int_0^\infty \exp \left\{-\delta \tilde{c}_T \left(e^{\delta u} - 1\right)\right\} du > 0.$$ 

Seeing that $\zeta > 0$, and (D.8) implies $\tilde{c}_T \to \infty$, it follows by dominated convergence $E[\mathcal{Y}_{T,i}] \xrightarrow{a.s.} 0$. Hence the Cesàro sum $1/T \sum_{i=1}^{T} E[\mathcal{Y}_{T,i}] \xrightarrow{a.s.} 0$. Now use Markov’s inequality and $1/T \sum_{i=1}^{T} E[\mathcal{Y}_{T,i}] \xrightarrow{a.s.} 0$ to deduce $1/T \sum_{i=1}^{T} \mathcal{Y}_{T,i} \xrightarrow{p} 0$. QED.
E Omitted Tables

This section contains omitted simulation results for AR models with $\phi_0 \in \{0, .75\}$, and for GARCH models with $T = 2000$. 
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<th>RMSE</th>
<th>KS$_{0.05}$</th>
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a. The model is $y_t = \phi_0 y_{t-1} + \epsilon_t$ where $\epsilon_t$ is iid Pareto or Normal, $E[\epsilon_t] = 0$ and $\sigma_0^2 = E[\epsilon_t^2]$. b. “Med” is the median, and “RMSE” is the root-mean-squared-error.

c. “KS$_{0.05}$” is the ratio of Kolmogorov-Smirnov statistic divided by its 5% critical value. Values greater than one suggest non-normality at the 5% level.
d. “no-trim” is standard FD-QML. “trim-obc” is the optimal bias corrected tail-trimmed FD-QML.
\( \phi_0 = .00^p \) and \( \sigma_t^2 = 1 \)

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</table>

a. The model is \( y_t = \phi_0 y_{t-1} + \epsilon_t \) where \( \epsilon_t \) is iid Pareto or Normal, \( E[\epsilon_t] = 0 \) and \( \sigma_t^2 = E[\epsilon_t^2] \). b. “Med” is the median, and “RMSE” is the root-mean-squared-error. c. “KS_{.05}” is the ratio of Kolmogorov-Smirnov statistic divided by its 5% critical value. Values greater than one suggest non-normality at the 5% level. d. “no-trim” is standard FD-QML. “trim-obc” is the optimal bias corrected tail-trimmed FD-QML.
Table E.3: FD-QML for GARCH(1,1), T = 2000

\[ [\alpha_0, \beta_0] = [0.3, 0.6] : \kappa \approx 2.05 \]

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\[ [\alpha_0, \beta_0] = [0.2, 0.7] : \kappa \approx 3.02 \]

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\[ [\alpha_0, \beta_0] = [0.05, 0.9] : \kappa \approx 4.05 \]

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a. The model is \( y_t = x_t^2 = \sigma_t^2 \epsilon_t^2 \) where \( \sigma_t^2 = 1 + \alpha_0 x_{t-1}^2 + \beta_0 \sigma_{t-2}^2 \) where \( \epsilon_t \) is iid standard normal.

b. “Med” is the median, and “RMSE” is the root-mean-squared-error.

c. “KS_{0.05}” is the ratio of Kolmogorov-Smirnov statistic divided by its 5% critical value. Values greater than one suggest non-normality at the 5% level.

d. “no-trim” is standard FD-QML. “trim-obc” is the optimal bias corrected tail-trimmed FD-QML.
References


