Tail and Non-Tail Memory with Applications to Random Volatility

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OBJECTIVES

APPLICATIONS: ROBUST INFERENCE

Develop tail dependence properties for …

Tail shape inference (linear and nonlinear GARCH, stochastic vol.)

Non-parametric tail dependence inference

Develop tail-trimmed properties for…

Robust GMM, NLLS, QMLE for heavy-tailed random volatility models

(robust = asymptotically Gaussian for persistent heavy-tailed data)
OBJECTIVES

- THEORY

New notions of tail dependence, persistence, decay.

Tail Dependence \( \{ x_{t-h}, y_t \} : |x_{t-h}| > b_x \ and \ |y_t| > b_y \) as \( \{ b_x, b_y \} \to \infty \)

Tail Events : \( I(|x_t| > b_x) \) and \( I(|y_t| > b_y) \) as \( b_x, b_y \to \infty \)

Tail Exceedances : \( \max \{ \ln |x_t| - \ln b_x, 0 \} \) and \( \max \{ \ln |y_t| - \ln b_y, 0 \} \)
OBJECTIVES

NASDAQ Daily Log Returns and Upper 5th percentile: \( n = 1044 \)

“Tail” events and exceedances cluster: *how long? why?*
OBJECTIVES

□ THEORY

New notions of tail dependence, persistence, decay.

Tail Dependence $\{x_{t-h}, y_t\} : |x_{t-h}| > b_x \text{ and } |y_t| > b_y$ as $\{b_x, b_y\} \rightarrow \infty$

New notions of tail-trimmed dependence, persistence, decay.

Non-Tail $\{x_{t-h}, y_t\} : |x_{t-h}| \leq b_x \text{ and } |y_t| \leq b_y$ as $\{b_x, b_y\} \rightarrow \infty$
Tail and Non-Tail Memory with Applications to Random Volatility

LITERATURE

TAIL DEPENDENCE

Joint Distribution Tail Exponents for Bivariate \{X_t,Y_t\}

\[
P(W_{x,t-h} > \varepsilon, W_{y,t} > \varepsilon) = \varepsilon^{-1/\gamma_h} L_h(\varepsilon) \quad \text{where} \quad \gamma \in [0,1]
\]

\{W_{x,t}, W_{y,t}\} \text{ are Unit Frechet transforms; } L_h(\varepsilon) \text{ is slowly varying}

\[\gamma_h < 1/2 \text{ (neg. dep.), } \gamma_h = 1/2 \text{ (indep.), } \gamma_h > 1/2 \text{ (pos. dep.)}\]


- Must know marginal distributions; \{X_t\} and \{Y_t\} are iid.
- Ad hoc models of \(L_h\) (Weibull, Logistic,…).
LITERATURE

TAIL DEPENDENCE

Joint Distribution Tail Exponents for Bivariate \( \{ X_t, Y_t \} \)

\[
P(W_{x,t-h} > \varepsilon, W_{y,t} > \varepsilon) = \varepsilon^{-1/\gamma_h} L_h(\varepsilon) \quad \text{where} \quad \gamma \in [0,1]
\]

\( \{ W_{x,t}, W_{y,t} \} \) are Unit Frechet transforms; \( L_h(\varepsilon) \) is slowly varying

\( \gamma_h < 1/2 \) (neg. dep.), \( \gamma_h = 1/2 \) (indep.), \( \gamma_h > 1/2 \) (pos. dep.)

- Assumes exponent \( \gamma_h \) denotes core expression of dependence.
- Ignores bivariate dependence decay as \( h \to \infty \).
- Cannot capture degenerate cases (stochastic volatility).
LITERATURE

TAIL DEPENDENCE

Tail Copula

\[ \Lambda(h) = \lim_{b_x, b_y \to \infty} \frac{P(|X_{t-h}| > b_x, |Y_t| > b_y)}{\sqrt{P(|X_{t-h}| > b_x) \times P(|Y_t| > b_y)}} \] (two-tailed)

Schmidt and Stadtmüller (2006); Klüppleberg, Kuhn, Peng (2007)

Ignores memory decay; \( \{X_t\} \) and \( \{Y_t\} \) are iid, and \( h = 0 \).

Cannot capture degenerate cases (stochastic volatility).
LITERATURE

- **TRIMMING**: $x_t \times I(|x_t| \leq b_x)$ or **TRUNCATION**: $x_t \times I(|x_t| \leq b_x) + b_x \times I(|x_t| > b_x)$

  Inference for mean, or least squares for heavy tailed $\{x_t\}$

  1. IID
  2. Weakly Dep. + finite variance (e.g. covariance stationary GARCH)
  3. Linear distributed lags and fixed quantile trimming
  4. Mean/location; Least Squares (not GMM; not heavy-tail indifferent)

Bickel (1965), Stigler (1973b), Csörgő, Horváth and Mason (1986)
Griffin and Pruitt (1987), Hahn, Kuelbs, Weiner (1990)
Hahn and Weiner (1992)
Wu (2005), Agulló, Croux and Van Aelst (2008)
TAIL MEMORY

\{x_t, y_t\} is a bivariate time series, \(x_t, y_t > 0\), sample size = \(n\).

Tail-specific data:

Two Tailed: \(|x_t|\)

Left Tailed: \(-x_t \times I(x_t < 0)\)

Right Tailed: \(x_t \times I(x_t \geq 0)\)
TAIL MEMORY

**Extreme value** within sample \( \{x_t\}_{t=1}^n : x_t > b_{x,n} \to \infty. \)

\( k_n = \) number of extremes : \( k_n \to \infty \) and \( k_n/n \to 0 \) as \( n \to \infty. \)

\( b_{x,n} \to \infty \) is the \( k_n/n^{th} \) upper quantile of \( x_t : \)

\[
P(x_t > b_{x,n}) \sim \frac{k_n}{n} \to 0
\]

Leadbetter, Rootzen, Lindgren (1983), Resnick (1987),
Embrechts, Klüppleberg, Mikosch (2003)
TAIL MEMORY : Estimable

Coefficient of Tail Dependence:

Joint / marginal tail probability discrepancy:

\[
r_n(h) = \frac{n}{k_n} \left[ P(x_{t-h} > b_{x,n}, y_t > b_{y,n}) - P(x_{t-h} > b_{x,n}) \times P(y_t > b_{y,n}) \right]
\]

\[
= \frac{n}{k_n} \left[ P_{x,y,h}(b_{x,n}, b_{y,n}) - P_x(b_{x,n}) \times P_y(b_{y,n}) \right]
\]

\[
\lim_{n \to \infty} r_n(h) \in [0, 1]
\]
TAIL MEMORY : Estimable

Coefficient of Tail Dependence: \( r_n(h) = \frac{n}{k_n} \left[ P_{x,y,h}(b_{x,n}, b_{y,n}) - P_x(b_{x,n})P_y(b_{y,n}) \right] \)

Local tail dep. : \( \frac{n}{k_n} r_n(h) \sim \frac{P(x_{t-h} > b_{x,n}, y_t > b_{y,n})}{P(x_{t-h} > b_{x,n}) \times P(y_t > b_{y,n})} - 1 \rightarrow 0 \)

Distant tail dep. : \( r_n(h) \sim \Lambda(h) = \frac{P(x_{t-h} > b_{x,n}, y_t > b_{y,n})}{\sqrt{P(x_{t-h} > b_{x,n}) \times P(y_t > b_{y,n})}} \rightarrow 0 \)

Tail index & copula: distant tail dependence.

Stochastic Volatility : local tail dependent.
TAIL MEMORY : Estimable

Coefficient of Tail Dependence:

Easy to estimate:

$$\hat{r}_n(h) = \frac{1}{k_n} \sum_{t=1}^{n} I(x_{t-h} > x_{(k_n+1)}, y_t > y_{(k_n+1)}) - \frac{k_n}{n}$$

Order statistics $x_{(1)} \geq x_{(2)} \geq ... \geq x_{(n)}$

Asymptotic theory requires more abstract notions of tail dependence.

Do not want to restrict joint tail, non-extremes, extremal dependence.
TAIL MEMORY : Theoretical

Tail Dependence Definitions

Extreme events \( x_t > b_{x,n} \) can be predicted using information from "near epoch" of \( \{\varepsilon_{t-l_n}, ..., \varepsilon_t\} \), where \( l_n \to \infty \) as \( n \to \infty \).

Example: \( x_t = \) log returns to NASDAQ

\( \varepsilon_t = \) vector of market specific news/innovations.
TAIL MEMORY: Theoretical


\[ \{x_t\} \text{ is } L_0 \text{-E-APP if there exists a function } h_{n,t}^{(l_n)} \text{ of } \{\varepsilon_{t-l_n}, \ldots, \varepsilon_t\}: \]

\[ |I(x_t > b_{x,n}) - h_{n,t}^{(l_n)}| \rightarrow 0 \text{ in probability as } n \rightarrow \infty. \]

As $n \rightarrow \infty$, extremes $|x_t| > b_n$ can be perfectly predicted from $\varepsilon_t$.

Says nothing about joint distribution, non-extremes, density smoothness.
TAIL MEMORY : Theoretical

☐ Coverage and Applications


: GARCH, IGARCH, Explosive GARCH, stochastic volatility

: linear and nonlinear GARCH, Threshold AR’s

$L_0$-E-APP implies : Asymptotically normal tail shape estimator.

Asymptotically normal tail dependence estimator.
TAIL MEMORY : Theoretical

GARCH(1,1)

\[ x_t = h_t \varepsilon_t, \quad \varepsilon_t \sim N(0,1), \quad E[\varepsilon_t] = 0, \quad h_t^2 = \alpha + \beta x_{t-1}^2 + \gamma h_{t-1}^2, \quad \alpha > 0, \quad \beta, \gamma \geq 0 \]

Tails are regularly varying: \( P(\{x_t\} > \varepsilon) = \varepsilon^{-\kappa} L(\varepsilon) \) (Basrak et al 2006)

\{x_t\} is \( L_0 - E - \text{APP} \) for all \( \alpha + \beta \leq 1 \) and many \( \alpha + \beta \geq 1 \).
TAIL MEMORY : Theoretical

**GARCH(1,1)**

\[
x_t = h_t \epsilon_t, \quad \epsilon_t \sim (0,1), \quad E[\epsilon_t] = 0, \quad h_t^2 = \alpha + \beta x_{t-1}^2 + \gamma h_{t-1}^2, \quad \alpha > 0, \quad \beta, \gamma \geq 0
\]

Tails are regularly varying: \( P(|x_t| > \epsilon) = \epsilon^{-\kappa} L(\epsilon) \) (Basrak et al 2006)

B. Hill's (1975): \( \sqrt{k_n} \{ \hat{\kappa}_n - \kappa \} \xrightarrow{d} N(0, \nu_1^2) \)

Coeff. of Tail Dep. : \( \sqrt{k_n} \{ \hat{r}_n(h) - r_n(h) \} \xrightarrow{d} N(0, \nu_2^2) \)

Both cases: kernel variance estimators \( \hat{\nu}_{i,n}^2 \xrightarrow{p} \nu_i^2 \)
TAIL MEMORY : Theoretical

$L_0$-E-APP in general (nonlinear GARCH; regime switching; ...)

\[
\sqrt{k_n \{\hat{\kappa}_n - \kappa\}} \xrightarrow{d} N(0, \nu_1^2)
\]

\[
\sqrt{k_n \{\hat{r}_n(h) - r_n(h)\}} \xrightarrow{d} N(0, \nu_2^2)
\]

Do not need joint tail specification.

Do not need to restrict non-extremes.

Extremes allowed substantial dependence and heterogeneity.
TAIL MEMORY : Theoretical

NASDAQ Daily Log Returns and Upper 5th percentile : \( n = 1044 \)

"Tail" events and exceedances cluster : how long?
TAIL MEMORY : Theoretical

\[ r_n(h) = \frac{n}{k_n} \left[ P_{x,y,h}(b_{x,n},b_{y,n}) - P_x(b_{x,n})P_y(b_{y,n}) \right] \]

NASDAQ Median Two Tailed Serial
Tail Dependence Coefficient \( r(h) \) and 95\% Band

"Long" serial extremal memory.
NON-TAIL (TAIL-TRIMMED) MEMORY

TAIL-TRIMMED PROCESS

\[ X^{(tr)}_{n,t} := x_t \times I(\mid x_t \mid \leq b_n) \]

Inherently bounded with infinitely many moments:

\[ \mid X^{(tr)}_{n,t} \mid \leq b_{x,n} \quad \text{and} \quad E \mid X^{(tr)}_{n,t} \mid^p < \infty \text{ for all } p > 0. \]
NON-TAIL (TAIL-TRIMMED) MEMORY

☐ POPULATION AND TAIL-TRIMMED MEMORY

If \( \{x_t\} \) is \( L_0 - \text{Approximable} \) (Pötscher and Prucha 1991)

\[
|x_t - g_t^{(l)}| \to 0 \text{ in probability as } l \to \infty
\]

for some function \( g_t^{(l)} \) of near each \( \{\varepsilon_{t-i}, \ldots, \varepsilon_t\} \)

then tail-trimmed \( X_{n,t}^{(tr)} := x_t \times I(|x_t| \leq b_n) \) satisfies LLN and CLT.
NON-TAIL (TAIL-TRIMMED) MEMORY

- \( L_0 \)-APP : Near Epoch Dependence, mixingale, mixing.
  - GARCH, IGARCH, Explosive GARCH, stochastic volatility
  - linear and nonlinear GARCH, Threshold AR’s

Tail-trimmed \( X_{n,t}^{(tr)} := x_t \times I(|x_t| \leq b_n) \) satisfies CLT.

Tail-trimming ensures standard asymptotics for NLLS, GMM, QMLE.
NON-TAIL (TAIL-TRIMMED) MEMORY

EXAMPLE: Tail-Trimmed GMM (Hill and Renault)

\[
\text{GARCH}(1,1): x_t = \sigma_t \varepsilon_t, \quad \varepsilon_t \sim \text{iid}, \quad \sigma_t^2 = \alpha + \beta x_{t-1}^2 + \gamma \sigma_{t-1}^2 : \theta = [\alpha, \beta, \gamma]
\]

\[
m_t(\theta) = (x_t^2 - \alpha - \beta x_{t-1}^2 - \gamma \sigma_{t-1}^2) \times z_{t-1}^2 : z_{t-1}^2 = [1, x_{t-1}^2, \sigma_{t-1}^2, \ldots]'
\]

Identification: \( E[m_t(\theta)] = 0 \) iff \( \theta = \theta_0 \)

\[
m_{n,t}^{(tr)}(\theta) \equiv m_t(\theta) \times I\left(m_t^{(a)}(\theta) \leq m_{(k_n+1)}^{(a)}(\theta)\right) : \text{where } m_t^{(a)}(\theta) := |m_t(\theta)|
\]
NON-TAIL (TAIL-TRIMMED) MEMORY

EXAMPLE: Tail-Trimmed GMM (Hill and Renault)

GARCH(1,1) : \( x_t = \sigma_t \varepsilon_t, \ varepsilon_t \sim \text{iid}, \ \sigma_t^2 = \alpha + \beta x_{t-1}^2 + \gamma \sigma_{t-1}^2 : \theta = [\alpha, \beta, \gamma] \)

\[
m_{n,t}^{(tr)} (\theta) \equiv m_t (\theta) \times I \left( m_t^{(a)} (\theta) \leq m_{(k_n+1)}^{(a)} (\theta) \right) : \text{where } m_t^{(a)} (\theta) := |m_t (\theta)|
\]

\( \hat{\Theta}^{tr} \) solves \( \min \left\{ \frac{1}{n} \sum_{t=1}^{n} m_{n,t}^{(tr)} (\theta) \times \hat{\Omega} \times \frac{1}{n} \sum_{t=1}^{n} m_{n,t}^{(tr)} (\theta) \right\} : \hat{\Omega} \to \Omega \ \text{(p.d.)} \)
NON-TAIL (TAIL-TRIMMED) MEMORY

- EXAMPLE: Tail-Trimmed GMM (Hill and Renault)

\[
\text{GARCH}(1,1) : \quad x_t = \sigma_t \varepsilon_t, \quad \varepsilon_t \sim \text{iid}, \quad \sigma_t^2 = \alpha + \beta x_{t-1}^2 + \gamma \sigma_{t-1}^2 \quad : \quad \theta = [\alpha, \beta, \gamma]
\]

\[
\hat{\theta}^{tr} \quad \text{solves} \quad \min \left\{ \frac{1}{n} \sum_{t=1}^{n} m_{n,t}^{(tr)} (\theta) \times \hat{\Omega} \times \frac{1}{n} \sum_{t=1}^{n} m_{n,t}^{(tr)} (\theta) \right\}
\]

\[
V_n^{1/2} (\hat{\theta}^{tr} - \theta_0) \overset{d}{\to} N(0, I_3) \quad \text{for some matrix} \quad V_n \in \mathbb{R}^{3 \times 3}
\]

Rate of convergence \( \leq n^{1/2} \).
EXAMPLE : Tail-Trimmed Least Squares : AR(1)

\[ x_t = 0.9x_{t-1} + \varepsilon_t \quad \varepsilon_t \sim \text{Pareto } (\alpha = 1.75), \quad \text{trim } k_n \approx n^\delta, \quad \delta \in (0, 1) \]

Tail rimmed Least Squares :
Kolmogorov Smirnov Tests of Normality

\[ x_t = \theta x_{t-1} + \varepsilon_t, \quad n = 500, \quad \text{rep} = 500 \]

Least Squares $\hat{\theta}$ $\text{KS} = .108$

KS 1\% = .061

TTLS $\hat{\theta}^{tr}$
SUMMARY

As long as $x_t$ is predictable in some sense using near epoch $\{\varepsilon_{t-l}, \ldots, \varepsilon_t\}$:

- **Extreme value estimators** are asymptotically normal:

  \[
  \sqrt{k_n} \{\hat{\kappa}_n - \kappa\} \xrightarrow{d} N(0, \nu_1^2)
  \]

  *Do need joint tail specification.\*

  *Non-extremes can be anything.\*

  \[
  \sqrt{k_n} \{\hat{r}_n (h) - r_n (h)\} \xrightarrow{d} N(0, \nu_2^2)
  \]

  *Extremes can be highly dependent.\*

- **Tail Trimmed estimator(s)** are asymptotically normal (GMM)

- Covers linear and nonlinear GARCH, linear and nonlinear ARFIMA…