The Kernel Self-Normalized, Tail-Trimmed Sum for Dependent, Heterogeneous Data, with an Application to Robust Least Squares

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MOTIVATION : Robust Model Estimation of Heavy-Tailed Data

Financial and macroeconomic time series:

Data : *equity markets*, *option prices*, *exchange rates*, insurance claims,...

Stylized Traits : *Asymmetric*, persistent, heteroskedastic, *heavy-tailed*

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Models : 1. Value-at-Risk, CAPM, Price/Volume, Volatility Spillover

2. Linear GARCH, Asymmetric GARCH (EGARCH, TGARCH)

Implications : 1. QML, GMM, NLLS *may not* be asymptotically *normal*.

2. Robust methods *cannot handle traits* simultaneously.

MOTIVATION: Tail Trimmed GMM (Hill and Renault 2008)

Asymmetric GARCH without parameter restrictions : QARCH (Sentana 1995)

$$\%\Delta\text{NASDAQ} \;:\; x_t = \sigma_t \varepsilon_t, \;\; \varepsilon_t \overset{\text{iid}}{\sim} (0,1), \;\; \sigma_t^2 = (\alpha + \beta x_{t-1})^2 \;:\; \theta = [\alpha, \beta]'$$

$$\textbf{Estim. Eq.'s} \;:\; m_t(\theta) = \{x_t^2 - (\alpha + \beta x_{t-1})^2\} \times z_{t-1}^2 \;:\; z_{t-1}^2 = [1, x_{t-1}, x_{t-1}^2, ...]'$$

$$\textbf{Identification} \;:\; E\left[m_t(\theta)\right] = E[\{x_t^2 - (\alpha + \beta x_{t-1})^2\} \times z_{t-1}^2] = 0 \;\; iff \;\; \theta = \theta_0$$

$$\textbf{GMM} \;:\; \hat{\theta} \text{ solves } \min_{\theta \in \Theta} \left\{ \sum m_t(\theta)' \times \hat{\Omega} \times \sum m_t(\theta) \right\} \text{ for some p.s.d. } \hat{\Omega} \overset{p}{\to} \Omega.$$

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If $E(\varepsilon_t^4) = \infty \;\Rightarrow$ **GMM** and **QML** fail.

L. Hansen (1982) - **GMM**.

Hall and Yao (2003) - **QML** : Covariance stationary GARCH, $E[\varepsilon^2] < \infty$.

Jensen and Rahbek (2004) - **QML** : ARCH(1) with non-stationary solution.

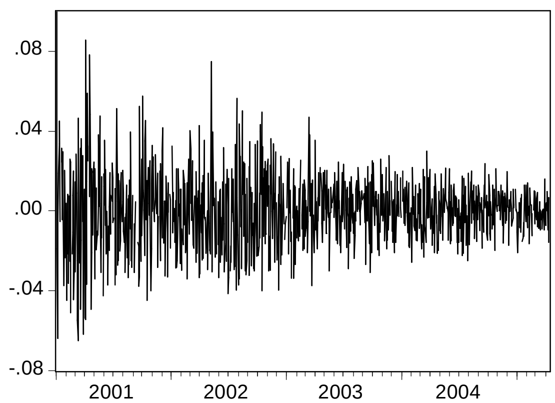
Franc and Zakoïan (2004) – **QML** : ARMA-GARCH, $E[\varepsilon^4] < \infty$ yet $\boldsymbol{E|x_t|^p = \infty}$.

Ling (2007) - $\boldsymbol{E(\varepsilon_t^2) < \infty}$ and $\boldsymbol{E|x_t|^p = \infty}$, *symmetrically* trimmed QMLE.

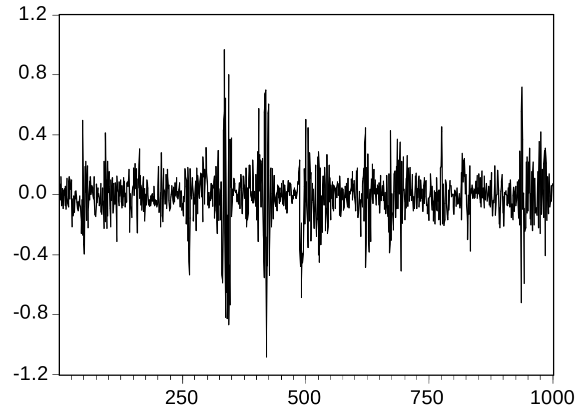
Linear GARCH.

MOTIVATION: Tail Trimmed GMM (Hill and Renault 2008)

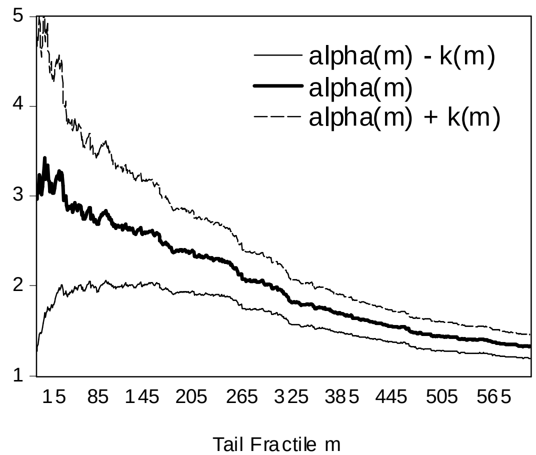
NASDAQ Daily Log Returns
skew : .477 (.000)
kurt : 6.48 (.000)



IGARCH(1,1) with iid N(0,1) shocks



NASDAQ Daily Log Returns
B. Hill Estimator wth Robust Kernel 95% Bands



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Asymmetric GARCH without parameter restrictions : QARCH

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$$m_t(\theta) = \{x_t^2 - (\alpha + \beta x_{t-1})^2\} \times z_{t-1}^2 \ : \ z_{t-1}^2 = [1, x_{t-1}, x_{t-1}^2, ,...]' \in \Re^q, \ q \geq 3$$

Solution : Trim $\{k_{1,n}, k_{2,n}\}$ left/right-tail ob.'s of $m_t(\theta) \Rightarrow m_{k_n,t}^{(tr)}(\theta)$

Intermediate Order Sequences : $k_{i,n} \to \infty$ and $\dfrac{k_{i,n}}{n} \to 0.$

Tail-Trimmed GMM $\hat{\theta}^{tr}$ solves $\min_{\theta \in \Theta} \left\{ \sum_{t=1}^{n} m_{k_n,t}^{(tr)}(\theta)' \times \hat{\Omega} \times \sum_{t=1}^{n} m_{k_n,t}^{(tr)}(\theta) \right\}$

MOTIVATION: Tail Trimmed GMM (Hill and Renault 2008)

Large sample inference (Gaussian):

$$V_n^{1/2}\left(\hat{\theta}^{tr} - \theta_0\right) = \underline{\underline{\Sigma_n^{-1/2}\sum_{t=1}^{n} m_{k_n,t}^{(tr)}(\theta_0)}} + o_p(1) \xrightarrow{d} N\left(0, I_q\right), \quad \left\|V_n\right\| \leq Kn^{1/2}$$

$$\text{where } \Sigma_n = E\left[\sum_{t=1}^{n} m_{k_n,t}^{(tr)}(\theta) \times \sum_{t=1}^{n} m_{k_n,t}^{(tr)}(\theta)'\right]$$

Requires a *general* CLT for (asymmetrically) *tail-trimmed sums*.

TAIL TRIMMED SUMS FOR NON-IID DATA

Let $\{y_t\}$ be a sequence of random variables with distributions $\{F_t\}$.

Tail processes: $y_t^{(-)} \equiv y_t \times I(y_t < 0)$ and $y_t^{(+)} \equiv y_t \times I(y_t \geq 0)$

Order statistics: $y_{(1)}^{(-)} \leq y_{(2)}^{(-)} \leq \cdots$ and $y_{(1)}^{(+)} \geq y_{(2)}^{(+)} \geq \cdots$

$\{k_{1,n}, k_{2,n}\}$ = # trimmed tail observations, $k_{i,n} \to \infty$, $\dfrac{k_{i,n}}{n} \to 0.$

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TAIL TRIMMED SUMS FOR NON-IID DATA

Under "**general conditions**":

$$\frac{\sum_{t=1}^{n}\left\{\hat{y}_{k_n,t}-E\left[\hat{y}_{k_n,t}\right]\right\}}{\sqrt{\sum_{s,t=1}^{n} w(|s-t|/\gamma_n)\left(\hat{y}_{k_n,s}-\bar{y}_{k_n}^{tr}\right)\left(\hat{y}_{k_n,t}-\bar{y}_{k_n}^{tr}\right)}} \xrightarrow{d} N(0,1)$$

$$\hat{y}_{k_n,t} \equiv y_t \times I\left(y_{(k_{1,n})}^{(-)} \leq y_t \leq y_{(k_{1,n})}^{(+)}\right) \text{ and } \bar{y}_{k_n}^{tr} := \frac{1}{n}\sum_{t=1}^{n}\hat{y}_{k_n,t}$$

$w(\text{-})$ = kernel function (Bartlett, Parzen, and Tukey-Hanning, …)

γ_n = bandwidth.

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Mixingales, **mixing, mds** : linear and nonlinear ARFIMA (switching, …)

: linear and nonlinear GARCH (IGARCH, QGARCH)

: Stochastic volatility

TAIL TRIMMED SUMS FOR NON-IID DATA

General Conditions :

1. Thin or thick probability tails :

$$P(|y_t| > y) \leq y^{-\kappa} L(y) \text{ as } y \to \infty \text{ for slowly varying } L(y),\ \kappa > 0$$

2. Trimmed $y_t \times I\left(y^{(-)}_{(k_{1,n})} \leq y_t \leq y^{(+)}_{(k_{2,n})}\right)$ is predictable (mixingale).

Mixingales, **mixing, mds** : linear and nonlinear ARFIMA (switching, …)

: linear and nonlinear GARCH (IGARCH, TGARCH)

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LITERATURE

- **Trimmed or truncated, *tail* or *fixed*-quantile trimmed sums**

$$\frac{1}{\sigma_n}\sum_{t=l_n}^{n-k_n} y_{(i)} \stackrel{d}{\rightarrow} N(0,1) \quad \text{where } \{k_n, l_n\} \rightarrow \infty$$

IID : vast majority (Bickel 1965, Stigler 1973,…)

Mixing : Finite variance (Hahn, Kuelbs and Samur 1987)

Linear distributed lags : fixed quantile (Wu 2005)

Domains of attraction : iid (Griffin and Pruitt 1987; Hahn and Weiner 1992)

***Nothing* on non-parametric variance σ_n^2 estimation…**

Csörgő, Horváhth and Mason (1986), Pruitt (1985),

Csörgő, Haeusler, and Mason (1988), Pozdnyakov (2003, 2004),

Hahn and Weiner (1992).

EXAMPLE : Tail-Trimmed Least Squares : AR(1)

$$x_t = .9x_{t-1} + \varepsilon_t \quad : \quad \varepsilon_t \overset{iid}{\sim} \text{Pareto}, \ \alpha = 1.75, \ \ \text{trim} \ \ k_n \approx n^{\delta}, \ \delta \in (0,1)$$

$$m_t(\theta) = (x_t - \theta x_{t-1})x_{t-1} \ \text{ and } \ m_{k_n,t}^{(tr)}(\theta) = m_t(\theta) \times I\left(\left|m_t(\theta)\right| \le m_{(k_n+1)}^{(a)}(\theta)\right)$$

$$\text{TTLS} : \hat{\theta}^{tr} = \arg\min_{\theta \in \Theta}\left(\sum_{t=1}^{n} m_{k_n,t}^{(tr)}(\theta)\right)^2$$

$$\text{Trivial special case of TTGMM} : V_n^{1/2}(\hat{\theta}^{tr} - \theta_0) \overset{d}{\rightarrow} N(0,1) \quad V_n \le Kn^{1/2}.$$

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Tail Trimmed Least Squares : Kolmogorov Smirnov Tests of Normality