

Consistent GMM Residuals-Based Tests of Functional Form

Appendix B: Omitted Proofs and Simulation Results

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In this appendix we characterize the data generating process and required limits under Assumption 3 (Section B.1), the proof of Theorem A.1 (Section B.2), and omitted simulation results (Section B.3). Citations used only here are referenced at the end.

B.1 ASSUMPTION 3 Let $\delta > 0$ be a small number, and let $\text{vec}(z)$ denote the vector that stacks columns of z . Write $\Gamma^{(*)} = \Gamma \times \dots \times \Gamma$ the space of matrices $\xi^{(-)} = [\xi_{(1)}^{(-)}, \dots, \xi_{(k)}^{(-)}]$, $\xi^{(+)} = [\xi_{(1)}^{(+)}, \dots, \xi_{(k)}^{(+)}]$ or $[\xi^{(-)}, \xi^{(+)}]$ (the dimension of $\Gamma^{(*)}$ depends on the case). Let $\hat{\xi}$ denote either $\hat{\xi}^{(-)}$, $\hat{\xi}^{(+)}$ or $[\hat{\xi}^{(-)}, \hat{\xi}^{(+)}]$. Similarly, $\mathfrak{Z} = \mathfrak{Z}_1 \otimes \dots \otimes \mathfrak{Z}_k$ where \mathfrak{Z}_i denote either $\mathfrak{Z}_i^{(+)} := \{\text{argsup}_{\gamma \in \Gamma} \{(\partial/\partial\gamma_i)E[\epsilon_t F(\gamma' x_{t-1})]\}\}$, $\mathfrak{Z}_i^{(-)} := \{\text{arginf}_{\gamma \in \Gamma} \{(\partial/\partial\gamma_i)E[\epsilon_t F(\gamma' x_{t-1})]\}\}$ or $\mathfrak{Z}_i^{(-)} \otimes \mathfrak{Z}_i^{(+)}$.

3.1: Each $z_t \in \{y_t, \tilde{x}_{1,t}, \dots, \tilde{x}_{k-1,t}\}$ is L_2 -Near Epoch Dependent [NED] on a strong mixing base $\{\epsilon_t\}$ with mixing coefficients $\alpha_i = O(i^{-\lambda})$ for some $\lambda > 1$, in the following uniform sense (see Hill 2008, cf. Gallant and White 1988):

$$\sup_{t \in \mathbb{Z}} \|z_t - E[z_t | \{\epsilon_{t-i}\}_{i=0}^m]\|_r = O(v(m)) \text{ where } v(m) \rightarrow 0 \text{ as } m \rightarrow \infty.$$

The error ϵ_t has an almost everywhere positive continuous density, and $\|\epsilon_t\|_{4+\delta} < \infty$.

Remark: We require $\{y_t, \tilde{x}_t\}$ to be L_2 -NED on a strong mixing base $\{\epsilon_t\}$ in order to exploit weak limit theory in Hill (2008), cf. Bierens (1991). Mixing errors allow for GARCH and stochastic volatility errors (e.g. Carrasco and Chen 2002), any strong mixing $\{y_t, \tilde{x}_t\}$ is automatically L_2 -NED on itself, and in general NED captures a broad array of linear and nonlinear time series with geometric or hyperbolic memory decay. Examples include threshold-type models (e.g. An and Huang 1996), and various nonlinear AR-GARCH (e.g. Meitz and Saikkonen 2008). See Hill (2011b) for a variety of examples and references in other contexts.

3.2: The parameter space Φ is a compact subset of \mathbb{R}^p . The known response function $f_t(\phi)$ is \mathfrak{S}_{t-1} -measurable, almost surely twice continuously differentiable on Φ .

3.3:

i. Estimating equations are $m_t(\phi) = (y_t - f_t(\phi)) \times c_t(\phi)$, $c_t : \Phi \rightarrow \mathbb{R}^q$, $q \geq p$, $c_t(\phi)$ is \mathfrak{S}_{t-1} -measurable and almost surely differentiable on Θ . Further $E[m_t(\phi)] = 0$ if and only if $\phi = \phi_0$, a unique interior point of Φ , hence ϵ_t is covariance orthogonal to $c_t(\phi_0)$ under either hypothesis. The GMM estimator is $\hat{\phi} = \arg \inf_{\phi \in \Phi} \{\bar{m}(\phi)' \hat{\Upsilon} \bar{m}(\phi)\}$ where $\bar{m}(\phi) = 1/n \sum_{t=1}^n m_t(\phi)$, $\hat{\Upsilon} \in \mathbb{R}^{q \times q}$ is positive definite, and $|\hat{\Upsilon} - \Upsilon| \xrightarrow{P} 0$ for some positive definite $\Upsilon \in \mathbb{R}^{q \times q}$.

ii. $\hat{A}(\phi, \gamma, \xi) \xrightarrow{P} A(\phi, \gamma, \xi) \in \mathbb{R}^{(k+1) \times q}$ uniformly on $\Phi \times \Gamma \times \Gamma^{(*)}$, where $A(\phi, \gamma, \xi) =$

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$-\text{plim}_{n \rightarrow \infty} \{(1/n \sum_{t=1}^n w_t(\gamma, \xi)(\partial/\partial\phi')f_t(\phi)) \times (H(\phi)^{-1}(\partial/\partial\phi')\bar{m}(\phi) \times \Upsilon)\}$ and $H(\phi) = -\text{plim}_{n \rightarrow \infty} \{(\partial/\partial\phi')\bar{m}(\phi) \times \Upsilon \times (\partial/\partial\phi)\bar{m}(\phi)\}$.

iii. There exists a unique element ξ^* of \mathfrak{Z} satisfying $|\text{vec}(\hat{\xi}) - \text{vec}(\xi^*)| = O_p(1/\sqrt{n})$. Further,

$$\begin{aligned} \sup_{\gamma \in \Gamma} \sup_{1 \leq t \leq n} |w_t(\gamma, \hat{\xi}) - w_t(\gamma, \xi^*)| &= O_p(1/\sqrt{n}) \text{ and } \sup_{\phi \in \Phi, \gamma \in \Gamma} |\hat{A}(\phi, \gamma, \hat{\xi}) - A(\phi, \gamma, \xi^*)| = O_p(1/\sqrt{n}) \\ \sup_{\gamma \in \Gamma} \sup_{1 \leq t \leq n} |\hat{g}_t(\gamma, \hat{\xi}) - g_t(\gamma, \xi^*)| &= O_p(1/\sqrt{n}) \text{ and } \sup_{\phi \in \Phi} |H_n(\phi) - H(\phi)| = O_p(1/\sqrt{n}). \end{aligned}$$

3.4:

i. There exists a mapping $\mu : \Gamma \times \Gamma^{(*)} \rightarrow \mathbb{R}$ satisfying $(1/n) \sum_{t=1}^n u_t g_t(\gamma, \xi) \xrightarrow{p} \lim_{n \rightarrow \infty} (1/n) \sum_{t=1}^n E[u_t g_t(\gamma, \xi)] = \mu(\gamma, \xi)$ uniformly on $\Gamma \times \Gamma^{(*)}$.

ii. There exists a mapping $\Sigma : \Gamma \times \Gamma \rightarrow \mathbb{R}^{(k+1) \times (k+1)}$ satisfying $(1/n) \sum_{t=1}^n E[\epsilon_t^2 | \mathfrak{S}_{t-1}] \times g_t(\gamma_1, \xi) g_t(\gamma_2, \xi)' \xrightarrow{p} \Sigma(\gamma_1, \gamma_2)$, $(1/n) \sum_{t=1}^n \epsilon_t^2 g_t(\gamma_1, \xi) g_t(\gamma_2, \xi)' \xrightarrow{p} \Sigma(\gamma_1, \gamma_2)$ and $(1/n) \sum_{t=1}^n E[\epsilon_t^2 g_t(\gamma_1, \xi) g_t(\gamma_2, \xi)'] \rightarrow \Sigma(\gamma_1, \gamma_2)$ uniformly on $\Gamma \times \Gamma$.

iii. $\limsup_{n \rightarrow \infty} \sup_{\gamma \in \Gamma} 1/n \sum_{t=1}^n E|\epsilon_t^2 u_t g_t(\gamma, \xi)|^{2+\delta} < \infty$.

3.5: Uniformly on $1 \leq t \leq n$: $\|\sup_{\gamma \in \Gamma, \xi \in \Gamma^{(*)}} |g_t(\gamma, \xi)|\|_{4+\delta} < K$, $\|\sup_{\gamma \in \Gamma, \xi \in \Gamma^{(*)}} |(\partial/\partial\gamma)g_t(\gamma, \xi)|\|_{4+\delta} < K$, $\|\sup_{\phi \in \Phi} |c_t(\phi)|\|_{4+\delta} < K$, and $\|\sup_{\phi \in \Phi} |(\partial/\partial\phi)c_t(\phi)|\|_{4+\delta} < C$.

3.6: The local alternative random variable u_t is \mathfrak{S}_{t-1} -measurable, governed by a non-generate distribution.

B.2 PROOF OF THEOREM A.1

Recall the claim.

THEOREM A.1. Assume Assumptions 1 and 2, and $P(E[\epsilon_t | \mathfrak{S}_{t-1}] = 0) < 1$ hold. Then $S = \{0\}$ if and only if ϵ_t is L_2 -orthogonal to the closed linear span of $\{x_{i,t} F'(\xi'_{(i)} x_t)\}_{i=1}^k$, and otherwise S is empty.

We only treat $\xi_{(i)} = \xi_{(i)}^{(+)}$, and for the sake of convention assume

$$\frac{\partial}{\partial \gamma_i} E[\epsilon_t F(\gamma' x_t)] |_{\gamma = \xi_{(i)}} \geq 0.$$

All subsequent results carry over to the general case $(\partial/\partial\gamma_i)E[\epsilon_t F(\gamma' x_t)] |_{\gamma = \xi_{(i)}} \geq 0$, and to $\text{arginf}_{\gamma \in \Gamma} \{(\partial/\partial\gamma_i)E[\epsilon_t F(\gamma' x_t)]\}$.

The proof requires one supporting result. Define

$$\varpi(\gamma, \xi) := E \left[\epsilon_t \left(F(\gamma' x_t) - \sum_{i=1}^k \gamma_i x_{i,t} F'(\xi'_{(i)} x_t) \right) \right].$$

LEMMA B.1. Under Assumptions 1 and 2 if $P(E[\epsilon_t | \mathfrak{S}_{t-1}] = 0) < 1$ then $\varpi(\gamma, \xi) = 0$ for $\gamma \in \Gamma$ if and only if $\gamma = 0$.

Remark: The two structures $E[\epsilon_t F(\gamma' x_t)]$ and $\sum_{i=1}^k \gamma_i E[\epsilon_t x_{i,t} F'(\xi'_{(i)} x_t)]$ are equal only at the origin $\gamma = 0$. But this means if $E[\epsilon_t F(\gamma' x_t)] = 0$ under H_1 for any $\gamma \neq 0$ then $\sum_{i=1}^k \gamma_i E[\epsilon_t x_{i,t} F'(\xi'_{(i)} x_t)] \neq 0$ hence at least one moment condition $E[\epsilon_t x_{i,t} F'(\xi'_{(i)} x_t)] \neq 0$. Similarly, if all moments $E[\epsilon_t x_{i,t} F'(\xi'_{(i)} x_t)] = 0$ then $E[\epsilon_t F(\gamma' x_t)] \neq 0$ for all $\gamma \neq 0$.

PROOF OF THEOREM A.1. Assume $P[E(\epsilon_t | \mathfrak{S}_{t-1}) = 0] < 1$. By Lemma B.1 we know for each $\gamma \neq 0$

$$\varpi(\gamma, \xi) = E[\epsilon_t F(\gamma' x_t)] - \sum_{i=1}^k \gamma_i E[\epsilon_t x_{i,t} F'(\xi'_{(i)} x_t)] \neq 0.$$

Trivially, therefore, at least one moment condition $E[\epsilon_t F(\gamma' x_t)]$, $E[\epsilon_t x_{1,t} F'(\xi'_{(1)} x_t)]$, ..., or $E[\epsilon_t x_{k,t} F'(\xi'_{(k)} x_t)]$ must be non-zero, hence $E[\epsilon_t w_t(\gamma, \xi)] \neq 0$ for every $\gamma \neq 0$.

Finally, under Assumption 1

$$E[[\epsilon_t w_t(0, \xi)] = \left[0, E \left[\epsilon_t x_{1,t} F' \left(\xi'_{(1)} x_t \right) \right], \dots, E \left[\epsilon_t x_{k,t} F' \left(\xi'_{(k)} x_t \right) \right] \right]',$$

hence $E[\epsilon_t w_t(0, \xi)] = 0$ if and only if ϵ_t is orthogonal to the closed linear span of $\{x_{i,t} F'(\xi'_{(i)} x_t)\}_{i=1}^k$. \mathcal{QED} .

In order to prove Lemma B.1 we require an easy extension of Theorem 1 of Bierens and Ploberger (1997) and Theorem 2.3 and Corollary 3.9 of Stinchcombe and White (1998). Let $h_t : D \rightarrow \mathbb{R}^k$ be an \mathfrak{S}_{t-1} -measurable, uniformly bounded function, where D is an arbitrary subset of \mathbb{R}^l for some $l \geq 0$. Write $h_t = h_t(\delta)$ by convention when $l = 0$. Examples of $h_t(\delta)$ include x_t , $|x_t|^\delta \times \text{sign}(x_t)$ and $(\partial/\partial\phi)f_t(\phi)$ where $\delta = \phi$ provided x_t and $(\partial/\partial\phi)f_t(\phi)$ are bounded with probability one.

LEMMA B.2. *Let weight F satisfy Assumption 1. If $P(E[\epsilon_t | \mathfrak{S}_{t-1}] = 0) < 1$ then for each $\delta \in D$ the set $S = \bigcap_{i=1}^k \{\gamma \in \mathbb{R}^k : E[\epsilon_t h_{i,t}(\delta) F(\gamma' x_t)] = 0\}$ and $P(\gamma' x_t \in R_0) = 1\}$, has Lebesgue measure zero and is nowhere dense in \mathbb{R}^k .*

Remark: By Assumption 1 and Corollary 3.9 of Stinchcombe and White (1998), Lemma B.2 holds with $F(\cdot)$ replaced by $F'(\cdot)$. If x_t is not bounded with probability one then replace it with any Borel measurable, bounded one-to-one mapping $\Psi(x_t)$ (Bierens 1990).

PROOF OF LEMMA B.1. Recall

$$\xi_{(i)} \in \mathfrak{Z}_i := \left\{ \text{argsup}_{\gamma \in \Gamma} \left\{ \frac{\partial}{\partial \gamma_i} \mu(\gamma) \right\} \right\} = \left\{ \text{argsup}_{\gamma \in \Gamma} \left\{ E \left[\epsilon_t \frac{\partial}{\partial \gamma_i} F(\gamma' x_t) \right] \right\} \right\}.$$

The construction of $\xi_{(i)}$ implies for all $\gamma \in \Gamma$

$$(1) \quad E[\epsilon_t x_{i,t} F'(\gamma' x_t)] \leq E \left[\epsilon_t x_{i,t} F' \left(\xi'_{(i)} x_t \right) \right] = \sup_{\gamma \in \Gamma} E[\epsilon_t x_{i,t} F'(\gamma' x_t)]$$

and Assumptions 1 and 3 imply $\varpi(0, \xi) = E[\epsilon_t F(0' x_t)] = 0$. Differentiate $\varpi(\gamma, \xi)$ with respect to γ_j , and add and subtract $E[\epsilon_t x_{j,t} F'(0' x_t)]$:

$$(2) \quad \frac{\partial}{\partial \gamma_j} \varpi(\gamma, \xi) = E[\epsilon_t x_{j,t} \{F'(\gamma' x_t) - F'(0' x_t)\}] \\ - E \left[\epsilon_t x_{j,t} \{F'(\xi'_{(j)} x_t) - F'(0' x_t)\} \right] \leq 0.$$

Thus $\varpi(\gamma, \xi)$ is zero at $\gamma = 0$ and is weakly decreasing in γ , and (1) implies $E[\epsilon_t x_{j,t} \{F'(\xi'_{(j)} x_t) - F'(0' x_t)\}] \geq 0 \forall j = 1 \dots k$.

In order to prove weak inequality (2) is in fact strict, there are two cases.

Case 1 ($E[\epsilon_t x_{j,t} \{F'(\xi'_{(j)} x_t) - F'(0' x_t)\}] = 0$): Trivially

$$E[\epsilon_t x_{j,t} \{F'(\gamma' x_t) - F'(0' x_t)\}] |_{\gamma=0} = 0.$$

Lemma B.2 therefore implies there exists an open neighborhood $N(0) \subset \Gamma$ of zero satisfying

$$E[\epsilon_t x_{j,t} \{F'(\gamma' x_t) - F'(0' x_t)\}] \neq 0 \quad \forall \gamma \in N(0)/0.$$

Since by assumption $E[\epsilon_t x_{j,t} \{F'(\xi'_{(j)} x_t) - F'(0' x_t)\}] = 0$ we deduce from (2)

$$E[\epsilon_t x_{j,t} \{F'(\gamma' x_t) - F'(0' x_t)\}] < 0 \quad \forall \gamma \in N(0)/0.$$

Thus $\varpi(\gamma, \xi)$ is zero at $\gamma = 0$, strictly decreasing arbitrarily close to $\gamma = 0$, and weakly decreasing everywhere else. This implies $\varpi(\gamma, \xi) \neq 0$ for every $\gamma \neq 0$.

Case 2 ($E[\epsilon_t x_{j,t} \{F'(\xi'_{(j)} x_t) - F'(0' x_t)\}] > 0$): Use (2) to deduce for each $j = 1 \dots k$

$$\frac{\partial}{\partial \gamma_j} \varpi(\gamma, \xi)|_{\gamma=0} = 0 - E \left[\epsilon_t x_{j,t} \left\{ F'(\xi'_{(j)} x_t) - F'(0' x_t) \right\} \right] < 0.$$

Again, $\varpi(\gamma, \xi)$ is zero at $\gamma = 0$, strictly decreasing at $\gamma = 0$ and weakly decreasing everywhere else. *QED*.

B.3 OMITTED SIMULATION RESULTS Finally, we present omitted simulation results for fixed CM tests.

Table B.1: Fixed CM Tests Rejection Frequencies^a

n = 200					
	AR	LSTAR	ESTAR	SETAR	BILIN
L- $\mathcal{T}_n(\gamma_0)$.055 ^b	.079	.749	.455	.058
E- $\mathcal{T}_n(\gamma_0)$.054	.205	.546	.279	.035
L- $\mathcal{B}_n(\gamma_0)$ ^c	.039	.080	.115	.115	.060
E- $\mathcal{B}_n(\gamma_0)$.049	.109	.197	.153	.031
L- $\mathcal{H}_n(\gamma_*)$ ^d	.054	.061	.087	.132	.050
E- $\mathcal{H}_n(\gamma_*)$.042	.085	.174	.151	.025
UMP(γ_0) ^e	-	.987	1.00	1.00	.299
n = 500					
	AR	LSTAR	ESTAR	SETAR	BILIN
L- $\mathcal{T}_n(\gamma_0)$.046	.370	1.00	.913	.128
E- $\mathcal{T}_n(\gamma_0)$.049	.691	.949	.827	.041
L- $\mathcal{B}_n(\gamma_0)$.057	.212	.428	.357	.061
E- $\mathcal{B}_n(\gamma_0)$.046	.290	.632	.454	.018
L- $\mathcal{H}_n(\gamma_*)$.039	.215	.433	.339	.062
E- $\mathcal{H}_n(\gamma_*)$.051	.299	.647	.442	.015
UMP(γ_0)	-	1.00	1.00	1.00	.483
n = 2000					
	AR	LSTAR	ESTAR	SETAR	BILIN
L- $\mathcal{T}_n(\gamma_0)$.045	.968	1.00	1.00	.419
E- $\mathcal{T}_n(\gamma_0)$.042	1.00	1.00	1.00	.029
L- $\mathcal{B}_n(\gamma_0)$.053	.742	.967	.891	.075
E- $\mathcal{B}_n(\gamma_0)$.053	.893	1.00	.974	.005
L- $\mathcal{H}_n(\gamma_*)$.062	.732	.933	.901	.064
E- $\mathcal{H}_n(\gamma_*)$.061	.891	.984	.974	.004
UMP(γ_0)	-	1.00	1.00	1.00	.751

- The nuisance parameter is fixed at the mid-point $\gamma_0 = [5.25, \dots, 5.25]'$ of $\Gamma = [.5, 10]^{p+1}$.
- L = logistic; E = exponential. Values are rejection frequency at the 5% level.
- Bierens' (1990) and Lee et al's (1996) CM test with logistic or exponential weight $F(\gamma' \psi_t)$.
- Hill's (2008) STAR test with logistic or exponential weight $x_t F(\gamma' \psi_t)$.
- The Uniformly Most Powerful test. Each test statistic is designed to be UMP for the particular H_1 .
In simulations not reported here, each statistic obtains empirical size roughly equal to nominal size.

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