Mathematics Review for Public Finance

Econ. 440
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The mathematics portion of this documents is merely a review: it is assumed that you have had differential calculus and know what a derivative is (its definition and basic properties and formulas).

A. CALCULUS REVIEW

Denote by $f'(x)$ or $\partial f(x)/\partial x$ or $(\partial/\partial x)f(x)$ the first derivative of function f with respect to its argument.

1. PARTIAL DERIVATIVE

1.1 $f(x) = ax^b$ implies $f'(x) = bax^{b-1}$.

If $b < 1$, then $f'(x)$ is defined for all real x accept 0.

1.2 $f(x) = ln(x)$ implies $f'(x) = 1/x$ for $x > 0$.

1.3 $f(x) = exp\{ax\} = e^{ax}$ implies $f'(x) = ae^{ax}$.

1.4 $f(x) = a^x$ implies $f'(x) = ln(a)\times f(x)$.

1.5 If $f(x_1,\ldots,x_k)$ has more than one argument then $f_j(x_1,\ldots,x_k) = \partial f(x_1,\ldots,x_k)/\partial x_j$.

If

$$f(x_1, x_2) = \ln(x_1) + ax_2^b, b > 0, x > 0$$

then

$$f_1(x_1, x_2) = 1/x_1 \qquad f_2(x_1, x_2) = bax_2^b$$

If

$$f(x_1, x_2) = Kx_1^\alpha x_2^\beta, \quad \alpha, \beta > 0$$

then

$$f_1(x_1, x_2) = \alpha K x_1^{\alpha-1} x_2^\beta \qquad f_2(x_1, x_2) = \beta K x_1^\alpha x_2^{\beta-1}$$

2. CHAIN RULE

The derivative of $f(g(x))$ where g is a function is simply

$$\partial f(g(x))/\partial x = f'(g(x)) \times g'(x)$$

If

$$f(x) = \frac{1}{1+\ln(x)} \quad \text{(i.e. } f(g(x)) \text{ where } g(x) = \ln(x) \text{ and } f(u) = (1+u)^{-1}\text{)}$$

then

$$f'(x) = -\frac{1}{(1+\ln(x))^2} \times \frac{1}{x}$$

If

$$f(x_1, x_2) = g(h(x_1, x_2))$$

then

$$f_1(x_1, x_2) = g'(h(x_1, x_2)) \times h_1(x_1, x_2)$$

3. SECOND DERIVATIVE

The second derivative of f with respect to x is denoted $f''(\mathrm{x})$ or $\partial^2 f(x)/\partial x \partial x$ or $(\partial/\partial x)^2 f(x)$.

If $f(x) = ln(x)$ then $f''(x) = -1/x^2$.

4. CROSS DERIVATIVE AND YOUNG'S THEOREM

The cross-derivative is a specific second derivative of a multivariate function:

$$f_{1,2}(x_1, x_2) = \frac{\partial^2}{\partial x_1 \partial x_2} f(x_1, x_2)$$

If

$$f(x_1, x_2) = K x_1^{\alpha} x_2^{\beta}, \quad \alpha, \beta > 0$$

then

$$f_{1,2}(x_1, x_2) = \alpha\beta K x_1^{\alpha-1} x_2^{\beta-1}$$

Young's Theorem states

$$f_{1,2}(x_1, x_2) = f_{2,1}(x_1, x_2)$$

If

$$f(x_1, x_2) = K x_1^{\alpha} x_2^{\beta}, \quad \alpha, \beta > 0$$

then simple inspection from the previous examples shows (compare to $f_{1,2}$ above…)

$$f_{2,1}(x_1, x_2) = \alpha\beta K x_1^{\alpha-1} x_2^{\beta-1}$$

5. TOTAL DIFFERENTIATION

The total derivative of f is

$$df(x_1,...,x_k)=\frac{\partial}{\partial x_1}f(x_1,...,x_k)dx_1+...+\frac{\partial}{\partial x_k}f(x_1,...,x_k)dx_k$$

Literally, the total amount of change of f, magnified by the amount of change in each argument.

If

$$f(x_1,x_2)=Kx_1^{\alpha}x_2^{\beta},\quad \alpha,\beta>0$$

then

$$df_1(x_1,x_2)=\alpha Kx_1^{\alpha-1}x_2^{\beta}dx_1+\beta Kx_1^{\alpha}x_2^{\beta-1}dx_2$$

We can solve for relative magnitude of change in the arguments x_1 and x_2 required to make the total change in f zero:

$$df(x_1,x_2)=f_1(x_1,x_2)dx_1+.f_2(x_1,x_2)dx_2=0\quad\rightarrow\quad\frac{dx_2}{dx_1}=-\frac{f_1(x_1,x_2)}{f_2(x_1,x_2)}\quad\forall\{x_1,x\}:f_2(x_1,x_2)\neq 0$$

If

$$f(x_1,x_2)=Kx_1^{\alpha}x_2^{\beta},\quad \alpha,\beta>0$$

then

$$\frac{dx_2}{dx_1}=-\frac{\alpha Kx_1^{\alpha-1}x_2^{\beta}}{\beta Kx_1^{\alpha}x_2^{\beta-1}}=-\frac{\alpha x_2}{\beta x_1}$$

B. GROWTH

1. COMPOUNDING ONCE PER PERIOD AND LOGARITHMS

Suppose A_t is a discretely valued variable, indexed by time t (e.g. aggregate savings). If A_t grows at rate a and is compounded once per time period t then

$$A_t=(1+a)^t A_0$$

Notice

$$\%\Delta A_{t+1}=\frac{A_{t+1}-A_t}{A_t}=\frac{(1+a)^{t+1}-(1+a)^t}{(1+a)^t}=\frac{\frac{(1+a)^{t+1}}{(1+a)^t}-\frac{(1+a)^t}{(1+a)^t}}{\frac{(1+a)^t}{(1+a)^t}}$$

$$=1+a-1=a$$

$$\ln A_t=t\times\ln(1+a)+\ln(A_0)\quad\rightarrow\quad\ln A_t\approx\beta_0+\beta_1 t\ \text{ where }\beta_1=a\text{ denotes growth}$$

$$\begin{aligned}\ln A_{t+1}-\ln A_t&=(t+1)\times\ln(1+a)-t\times\ln(1+a)\\&\approx(t+1)a-t\times a=a\end{aligned}$$

where we use the approximation $ln(1+a) \approx a$ when a is small, due to $ln(1) = 0$. This follows from an exact Taylor expansion, invoking the mean-value-theorem:

$$f(x) = f(a) + f'(a)|_{a=a^*} (x-a)$$

for some $a^* \in [a, x]$. Now look at the natural log:

$$\ln(1+a) = \ln(1) + \frac{1}{1+a^*}a = \frac{1}{1+a^*}a \approx a \text{ for small } a \text{ because } a^* \in [0,a]$$

Thus, the log-difference $ln(X_{t+1}) - ln(X_t)$ of a time series X_t is synonymously its growth.

And, $ln(X_t)$ should be close to a linear function of the growth rate of X_t.

2. CONSTANT GROWTH AS LINEAR TREND

If growth is a constant then $ln(X_t)$ will appear to increase linearly. Suppose

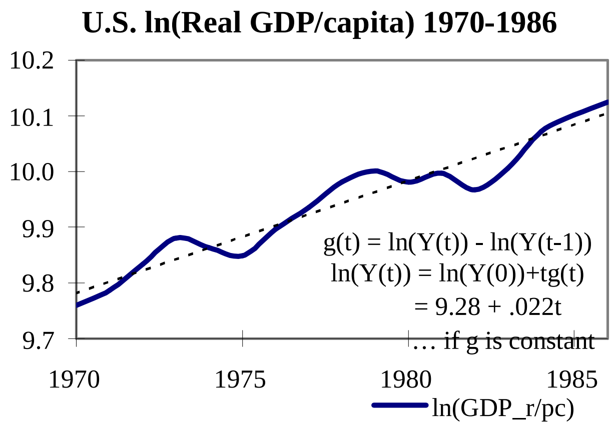
$$g = \ln(X_t) - \ln(X_{t-1})$$

a constant for all periods. Then iterate to get

$$\begin{aligned} \ln(X_t) &= g + \ln(X_{t-1}) \\ &= g + g + \ln(X_{t-2}) \\ &= \ldots \\ &= g + \ldots + g + \ln(X_0) \\ &= g \times t + \ln(X_0) \\ &= \ln(X_0) + g \times t \\ &= a + b \times t \end{aligned}$$

This is simply a linear function of time (i.e. a **Linear Trend**), with intercept $ln(X_0)$ and slope g. In other words, *if growth is constant than a macroeconomic time series will increase linearly*, with subtle deviations due to business cycles.

Example: The plot below shows real $ln(\text{GDP}_t)$ and plots a linear trend $a + bt$. Notice how well $ln(\text{GDP}_t)$ cycles up and down the trend line.



The small peaks and troughs above and below trend are the business cycles.