

Tail and Non-Tail Memory with Applications to Extreme Value and Robust Statistics*

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Abstract

New notions of tail and non-tail dependence are used to characterize separately extremal and non-extremal information, including tail log-exceedances and events, and tail-trimmed levels. We prove Near Epoch Dependence (McLeish 1975, Gallant and White 1988) and L_0 -Approximability (Pötscher and Prucha 1991) are equivalent for tail events and tail-trimmed levels, ensuring a Gaussian central limit theory for important extreme value and robust statistics under general conditions. We apply the theory to characterize the extremal and non-extremal memory properties of possibly very heavy tailed GARCH processes and distributed lags. This in turn is used to verify Gaussian limits for tail index, tail dependence and tail trimmed sums of these data, allowing for Gaussian asymptotics for a new Tail-Trimmed Least Squares estimator for heavy tailed processes.

1. INTRODUCTION We analyze notions of dependence separately restricted to extremal and non-extremal information. If population dependence is measured over the real line $(-\infty, \infty)$, extremal dependence is measured on $(-\infty, -b_1) \cup (b_2, \infty)$ as each $b_i \rightarrow \infty$, and non-extremal dependence on $(-b_1, b_2)$ as $b_i \rightarrow \infty$. The results permit fundamentally new Gaussian limit theory for tail shape and tail dependence estimators, and tail-trimmed sums for a large array of heavy tailed time series, including linear and nonlinear GARCH with unit or explosive roots, and non-stationary distributed lags with hyperbolic or geometric memory and heavy tailed shocks. Gaussian asymptotics for tail-trimmed sums supports new robust estimators, including asymptotically normal tail-trimmed versions of least squares and Quasi-Maximum Likelihood [QML] for heavy tailed data.

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Tail shape and tail dependence are natural objects of study in extreme value theory, with applications to cost, catastrophe, damage and risk modeling in finance, meteorology, insurance and macroeconomics (Leadbetter et al 1983; Beirlant et al 1996; Embrechts et al 1997). Tail trimming, however, is used for robust inference in the presence of heavy tails (Stigler 1973, Csörgő et al 1986, Hahn et al 1987, Hill 2009b, Hill and Renault 2010). Although the motivations and uses of these statistical methods are quite disparate, the underlying theory is remarkably similar due simply to the mathematics behind choosing the threshold b and to the depiction of memory.

1.1 GARCH Dependence

Although our dependence measures are quite general, one motivating application involves GARCH processes. Denote lag polynomials $\alpha(L) = \sum_{i=1}^p \alpha_i L^i$ and $\beta(L) = 1 - \sum_{i=1}^q \beta_i L^i$ with L the lag operator, and let

$$\begin{aligned} X_t &= \sigma_t \epsilon_t, \quad \epsilon_t \text{ is iid, } E|\epsilon_t|^r < \infty \text{ for some } r > 0; \\ \beta(L) \sigma_t^2 &= \omega + \alpha(L) X_t^2, \quad \omega > 0, \text{ at least one } \alpha_i, \beta_i > 0; \\ &\text{the roots of } \beta(z) \text{ lie outside unit circle;} \end{aligned} \tag{1}$$

with Lyapunov exponent¹ $\gamma < 0$; and the density of ϵ_t is positive on \mathbb{R} -a.e. The processes $\{X_t, \sigma_t\}$ have regularly varying marginal distribution tails of the form

$$P(|X_t| > x) = cx^{-\kappa} (1 + o(1)), \quad c > 0, \quad \kappa > 0. \tag{2}$$

See Basrak et al (2002a: Theorem 3.1), cf. Davis and Mikosch (1998) and Mikosch and Stărică (2000). Recall the tail index κ is identically the moment supremum: $E|X_t|^p < \infty \forall p < \kappa$ and $E|X_t|^{\kappa+\delta} = \infty \forall \delta \geq 0$ (Ibragimov and Linnik 1971). Further, roughly speaking $1/\kappa$ measures the mean exceedance of $\ln(|X_t|)$ above a large threshold: smaller κ are associated with larger average threshold exceedances. See, e.g., Hsing (1991: eq. (1.5)).

Power law tails (2) naturally arise in random volatility processes (de Haan et al 1989, Davis and Mikosch 1998) and first-price auction bids (Hill and Schneyerov 2010), they coincide with a maximum domain of attraction, and the domain of attraction of a stable law when $\kappa < 2$, and accurately characterize the tail behavior of many time series, including financial asset returns, insurance claims, telecommunication network data, urban growth and meteorological events. See the compendia Leadbetter et al (1983), Resnick (1987), and Embrechts et al (1997), and see Gabaix (2008) and Ibragimov (2009) and their citations for encyclopedic treatments.

In the GARCH case κ also arises in a Lyapunov-type moment condition. A GARCH(1,1), for example, satisfies (Mikosch and Stărică 2000)

$$E \left[(\alpha_1 \epsilon_t^2 + \beta_1)^{\kappa/2} \right] = 1. \tag{3}$$

¹The Lyapunov exponent γ is associated with the first order difference equation form of $Z_t := [X_t^2, \dots, X_{t-p+2}^2; \sigma_{t+1}^2, \sigma_t^2, \dots, \sigma_{t-q+2}^2]'$. It is easy to show Z_t is a first-order Markov $Z_t = A_t Z_{t-1} + B_t$ for some iid sequences $\{A_t, B_t\}$ of $k \times k$ matrices A_t and k -vectors B_t , $k \geq 1$. Identically $\gamma = \lim_{n \rightarrow \infty} n^{-1} \ln \|\prod_{t=1}^n A_t\|_o$, where $\|A\|_o = \sup_{x \in \mathbb{R}^k, |x|=1} |Ax|$. If ϵ_t in (1) is iid with zero mean and unit variance then $\gamma < 0$ given the remaining properties. See Basrak et al (2002a), cf. Bougerol and Picard (1992).

Suppose $E[\epsilon_t^2] = 1$. Then X_t has a finite variance $\kappa > 2$ if $\alpha_1 + \beta_1 < 1$, a hairline infinite variance $\kappa = 2$ in the IGARCH case $\alpha_1 + \beta_1 = 1$, and an infinite variance in the explosive root case $\alpha_1 + \beta_1 > 1$. Thus, very roughly speaking knowledge of the moments of ϵ_t allows the use of κ to gauge GARCH memory.

In general the analyst may want to estimate κ as a check on required moment conditions for a minimum distance estimator (e.g. Hill and Renault 2010), or as a measure of market risk (Embrechts et al 1997, Drees et al 2004, Iglesias and Linton 2009, Hill 2010) or tail dependence as a measure of risk decay and risk spillover (Stărică 1999, Ledford and Tawn 1997, 2003, Hill 2008, 2009a); or estimate the lag polynomial coefficients $\{\alpha, \beta\}$ to forecast volatility in the presence of extremes (Hall and Yao 2003, Davis and Mikosch 2009a, Hill and Renault 2010, Linton et al 2010).

The population memory properties of GARCH are now well known. If $\{X_t\}$ is governed by (1) then it is geometrically α -mixing (Boussama 1998, cf. Basrak et al 2002a), and a variety of nonlinear GARCH processes like Asymmetric, Multiplicative, and Smooth Transition GARCH are geometrically ergodic hence β -mixing (Carrasco and Chen 2002, Meitz and Saikkonen 2008, cf. Doukhan 1994). The root condition in (1) ensures an ARCH(∞) representation for σ_t^2 :

$$\sigma_t^2 = \pi_0 + \sum_{i=1}^{\infty} \pi_i X_{t-i}^2, \quad \pi_0 > 0, \quad \pi_i \geq 0, \quad \text{at least one } \pi_i > 0, \quad S := \sum_{i=1}^{\infty} \pi_i. \quad (4)$$

Davidson (2004) uses (4) to analyze population memory when $\epsilon_t \stackrel{iid}{\sim} (0, 1)$ (i.e. ϵ_t is iid with mean zero and unit variance). In the covariance stationary case $S < 1$ in general X_t is L_1 - or L_2 -Near Epoch Dependent [NED], and L_0 -Approximable [APP] when there is a unit ($S = 1$) or explosive ($S > 1$) root. See Section 2 for dependence definitions.

Although empirical studies of extremal dependence in random volatility processes abound (e.g. Stărică 1999, Longin and Solnik 2001), very few results formally characterize tail memory in GARCH data. One approach exploits the fact that GARCH class (1) belongs to the maximum domain of attraction: for each $z \geq 0$ and suitable normalizing sequence $\{u_n\}$

$$\lim_{n \rightarrow \infty} P \left(\frac{1}{u_n} \max_{1 \leq t \leq n} |X_t| \leq z \right) = e^{-\theta z^{-\kappa}}, \quad z \geq 0, \quad \theta \in [0, 1].$$

See Basrak et al (2002a); cf. Chernick (1981), de Haan et al (1989), Mikosch and Stărică (2000), and Davis and Mikosch (2009a). The inverted extremal index $1/\theta$ roughly measures the number of high threshold exceedances, hence tail memory; while $1/\kappa$ reveals the mean distance above a high threshold, hence tail thickness (Leadbetter 1974, 1983, Leadbetter et al 1983). Intuitively $\theta = 1$ dictates extremal independence, $\theta \in (0, 1)$ short-range dependence and $\theta = 0$ long-range dependence (Leadbetter 1983).

The extremal index has been characterized for autoregressions, moving averages, GARCH and stochastic volatility [SV] (Rootzén 1978, Chernick et al 1991, Mikosch and Stărică 2000, Davis and Mikosch (2009a,b). See Section 6.1, below, for details.

Nevertheless, θ does not portray memory decay per se, so knowledge of θ cannot in general ensure a central limit property for tail arrays of X_t and therefore for estimators of tail exponents like κ and θ . Indeed, inference on estimators of κ and θ invariably requires more information like population mixing or NED extremes (e.g. Chernick et al 1991, Hsing 1991, 1993, Mikosch and Stărică 2000, Hill 2010).

Another approach to modeling tail dependence exploits bivariate power-law tail decay with index η (Ledford and Tawn 1997, 2003, Stărică 1999, cf. Resnick 1987, Basrak et al 2002b). Ledford and Tawn (2003) estimate η in a time series framework that implicitly requires a population mixing condition. Stărică (1999) estimates the tail empirical measure for Constant Conditional Correlation GARCH, which embeds bivariate tail dependence information from η . The CCC-GARCH class exhibits mixing at a geometric rate, yet tail memory decay measured by η and how that relates to a central limit property for dependent data are not available². See Section 6.1 for a definition of η and expanded discussion.

An arguably more robust approach is based on tail indicators $I(|X_t| > c)$ and their autocorrelation. Compactly $r(h, c) := [P(X_{t-h} > c, X_t > c) - P_x(c)^2]/P(X_{t-h} > c)$ is referred to as the tail event correlation in Hill (2008, 2009a) and its limit $r(h) = \lim_{c \rightarrow \infty} r(h, c)$ the extremogram in Davis and Mikosch (2009c). The coefficient has been studied for ARMA, GARCH and SV, while an estimator of $r(h)$, like κ , θ and η , requires more information on memory. Davis and Mikosch (2009c) impose population α -mixing and multivariate regular variation for the joint tail of $\{X_t, X_{t-1}, \dots, X_{t-h}\}$, and Hill (2008, 2009a) imposes Near Epoch Dependence on tail events without joint tail restrictions. See Section 5.2 for estimation details and Section 6 for further discussion.

Finally, apparently there are no explicit treatments on the non-tail memory of random volatility processes. Thus, how such properties relate to robust estimation of such models is entirely unexplored.

1.2 Heavy Tailed Nonlinear Distributed Lags

A second motivating example is a heavy tailed distributed lag,

$$X_t = \sum_{i=0}^{\infty} \psi_{t,i} \epsilon_{t-i}, \quad \epsilon_t \sim (2) \text{ with } \kappa < 2 \quad (\text{i.e. } E[\epsilon_t^2] = \infty), \quad (5)$$

where $\psi_{t,0} = 1$, and $\{\psi_{t,i}\}$ is for each $i > 0$ a measurable stochastic process. Class (5) covers bilinear, ARFIMA, random coefficient autoregressions, and threshold and nonlinear autoregressions, and under fairly general conditions X_t is L_0 -APP or L_p -NED. See Section 4.

Studies of the tail probabilities of stationary linear processes $\sum_{i=0}^{\infty} \psi_i \epsilon_{t-i}$, $\epsilon_t \stackrel{iid}{\sim} (2)$, have a rich history (e.g. Feller 1946, 1971; Cline 1983), with several weak extremal dependence properties like D-mixing and the extremal index θ (Leadbetter 1974, 1983, Rootzén 1978, Leadbetter et al 1983, Chernick et al 1991, Smith 1992, Smith and Weissman 1994, Hsing 1993), and the extremogram $r(h)$ (Hill 2008, Davis and

²By comparison the tail index κ for GARCH(1,1) satisfies (3) and can therefore be computed by monte-carlo (Mikosch and Stărică 2000, Basrak et al 2002a).

Mikosch 2009c). D-mixing does not necessarily carry over to functions of D-mixing random variables, and in general is known to hold for a very limited class of processes. Refer to Section 6 for details.

Only recently has a central limit theory for trimmed or truncated sums of stationary linear processes been developed (e.g. Wu 2005, Hill 2009b, Hill and Renault 2010). The vast majority of this literature concerns independent data (e.g. Stigler 1973, Csörgő et al 1986, Hahn and Weiner 1992) and in a rare case mixing data with finite variance (Hahn et al 1987).

1.3 Tail and Non-Tail Memory

Hill (2009a, 2010), by comparison, imposes α -mixing on measurable functional arrays of some process $\{X_t\}$, covering tail events, exceedances and trimmed levels defined below. If such a function has a mixing property $\{X_t\}$ is said to be F-mixing, and any α -mixing $\{X_t\}$ is F-mixing covering linear and nonlinear ARMA-GARCH processes with sufficiently smooth probability densities (An and Huang 1996, Giraitis et al 2000, Carrasco and Chen 2002, Meitz and Saikkonen 2008).

The shortcomings of F-mixing, however, are the same as population and D-mixing conditions: it is difficult to verify, density smoothness is required, and for extreme value and robust limit theory broader dependence properties are in demand (Iglesias and Linton 2009, Rootzén 2008, Hill 2009a,b, 2010). Further, little is known about the mixing characteristics of long memory processes like ARFIMA and FIGARCH (e.g. Baillie et al 1996, Guegan and Ladoucette 2001) and Hyperbolic GARCH (Davidson 2004).

Let $\{X_t\}_{t=1}^n$ be a sample of size $n \geq 1$ and $\{k_n\}$ an intermediate order sequence: $1 \leq k_n < n$, $k_n \rightarrow \infty$ and $k_n/n \rightarrow 0$ as $n \rightarrow \infty$. Construct a sequence of asymptotic k_n/n^{th} -quantiles b_n of $|X_t|$ (Leadbetter et al 1983, Galambos 1987):

$$\frac{n}{k_n} P(|X_t| > b_n) \rightarrow 1. \quad (6)$$

Thus k_n is approximately the number of sample exceedances above a high threshold b_n . We implicitly assume b_n does not depend on t , although we allow X_t to be otherwise non-stationary. Now define the b_n -event

$$\bar{I}_{n,t} := I(|X_t| > b_n)$$

and b_n -exceedance, or peak-over-threshold (e.g. Smith 1984),

$$E_{n,t} := (\ln |X_t| - \ln b_n)_+, \quad \text{where } (z)_+ := \max\{0, z\}. \quad (7)$$

In the sequel we generalize to one-tailed cases. Tail arrays are exploited in a variety of contexts, including tail index and tail dependence estimation (Leadbetter et al 1983; Davison and Smith 1990; Hsing 1991, 1993; Hill 2008, 2009a, 2010), tail trimming and robust estimation (Hahn et al 1987, Hill 2009b, Hill and Renault 2010), and regression estimator performance and breakdown point analysis based on tail behavior (Jurečková 1981, He et al 1990).

In this paper we extend the notions of L_0 -APP and L_p -NED to tail and non-tail information. Our first task is to prove if X_t is L_0 -APP, as in explosive GARCH, then the triangular arrays $\{\bar{I}_{n,t}, E_{n,t}\}$ are also L_0 -APP, and $\{\bar{I}_{n,t}, E_{n,t}\}$ are L_0 -APP *if and only if* they are L_2 -NED. See Section 2.

Second, we prove if $\{X_t\}$ is L_0 -APP then the trimmed level

$$\hat{X}_{n,t} := X_t \times I(|X_t| \leq b_n) \tag{8}$$

is always L_2 -NED, even as $n \rightarrow \infty$, no matter how heavy tailed X_t is (Section 3). The array $\{\hat{X}_{n,t}\}$ not only serves to approximate location for heavy tailed data, it forms the asymptotic foundation for a class of robust estimators, including self-normalized tail-trimmed sums (Pruitt 1985, Hahn et al 1990, Hahn and Weiner 1992, Hill 2009b) and Generalized Method of Tail-Trimmed Moments (Hill and Renault 2010).

The primary contribution of this paper is a set of new dependence notions that permit a broad Gaussian central limit theory for tail and tail-trimmed arrays of linear and nonlinear distributed lags and random volatility processes. We never need to restrict density smoothness to ensure NED or APP for extremes or non-extremes.

Moreover, other than Hill's (2009a, 2010) F-mixing for tail or non-tail arrays, we believe this to be the first study that directly explores memory in "non-extremes" as defined above, and to study rigorously dependence notions away from the tails for the sake of improved asymptotic theory for robust statistics. This is the second major contribution.

As a third contribution in Section 4 the main dependence results are applied to GARCH and distributed lags (1), (4) and (5), and generalized to larger classes of linear and nonlinear processes including Nonlinear GARCH and Nonlinear Autoregressions.

Since $\{\bar{I}_{n,t}, E_{n,t}\}$ are directly linked to extremal statistics and $\{\bar{I}_{n,t}, \hat{X}_{n,t}\}$ to robust estimation, in Section 5 we characterize Gaussian limit theory for tail index and tail dependence estimators, and a tail-trimmed sum of L_0 -APP data. Finally, we show how fusing theory for dependent tail arrays $\{\bar{I}_{n,t}\}$ and tail-trimmed sums $\{\hat{X}_{n,t}\}$ supports new robust least squares and maximum likelihood estimators. This is the fourth contribution.

We focus on L_0 -APP and L_p -NED due to their relative ease of verification for a massive array of time series. Similar weak dependence measures can undoubtedly be divided into tail and non-tail constructions (e.g. Doukhan and Louhichi 1999, Nze et al 2002, Wu and Min 2005).

Throughout $K > 0$ denotes an arbitrary finite constant whose value may change from line to line; similarly $\iota > 0$ is infinitesimal and may change with the context. Unless other stated always $\rho \in (0, 1)$ is arbitrary and may change. Compactly write the L_p -norm: $\|x\|_p := (\sum_{i,j} E|x_{i,j}|^p)^{1/p}$.

2. TAIL MEMORY: EVENTS AND EXCEEDANCES For the remainder of the paper assume $X_t \geq 0$ *a.s.*, capturing two-tailed $|X_t|$ or one-tailed cases $-X_t I(X_t < 0)$ or $X_t I(X_t > 0)$.

We assume X_t is L_p -bounded for some $p > 0$: $E|X_t|^p < \infty$. It is easy to show by Markov's inequality the survival probability $\bar{F}_t(x) := P(X_t > x)$ is boundedly regularly varying: there exist $\kappa \in (0, p)$ and slowly varying $L_t : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$\lim_{x \rightarrow \infty} \sup_{t \in \mathbb{Z}} \left\{ \frac{\bar{F}_t(x)}{x^{-\kappa} L_t(x)} \right\} \leq K < \infty. \quad (9)$$

Class (9) includes thin (exponential) and thick (regularly varying) tails, and implies

$$x^p \bar{F}_t(x) \rightarrow 0 \quad \forall p < \kappa \text{ for each } t \in \mathbb{Z}.$$

Although uniform L_p -boundedness $\sup_{t \in \mathbb{Z}} E|X_t|^p < \infty$ is not required, we assume κ is the largest index that allows (9) and does not depend on t . If $\{x_t\}$ is stationary and (9) holds exactly $\bar{F}_t(x) \sim x^{-\kappa} L(x)$ as $x \rightarrow \infty$ as in (2) then κ is the index of regular variation and therefore the moment supremum: $\kappa = \sup\{\alpha > 0 : E|X_t|^\alpha < \infty\}$. See Bingham et al (1987) and Resnick (1987).

A convenient way to characterize tail memory is to restrict Near Epoch Dependence to the tails as in Hill (2009a, 2010). The NED property dates in various forms to Ibragimov (1962) and McLeish (1975), with a myriad subsequent extensions in Bierens (1987), Gallant and White (1988), and recently Nze et al (2002) and Wu and Min (2005). See Nze and Doukhan (2004) for compendium details on NED and similar concepts, and their usage in econometrics.

Let $\{F_t\}$ be a sequence of σ -fields induced by some possibly vector-valued process $\{\epsilon_t\}$,

$$F_t = \sigma(\epsilon_\tau : \tau \leq t) \quad \text{and} \quad F_s^t = \sigma(\epsilon_\tau : s \leq \tau \leq t).$$

If X_t is generated by (1), (4) or (5), for example, then ϵ_t is the scalar innovation.

L_p -Near Epoch Dependence $\{X_t\}$ is L_p -NED on $\{F_t\}$ or on $\{\epsilon_t\}$ with size $\lambda > 0$ if there exist deterministic sequences $\{d_t, \vartheta_l\}$, $d_t \geq 0$, $\vartheta_l \in [0, 1)$, and $\vartheta_l = o(l^{-\lambda})$ such that

$$\left\| X_t - E \left[X_t | F_{t-l}^{t+l} \right] \right\|_p \leq d_t \times \vartheta_l. \quad (10)$$

Remark 1: In general we drop the addendum "on $\{F_t\}$ " unless the base is unclear.

Remark 2: The "constants" d_t absorb time-dependence of the norm and control for scale, where $d_t \rightarrow \infty$ as $t \rightarrow \infty$ is possible due to trend. The "coefficients" ϑ_l gauge hyperbolic memory decay according to "size" λ , and geometric memory $\vartheta_l = o(\rho^l)$ means size λ is arbitrarily large. The property characterizes linear and nonlinear distributed lags with hyperbolic or geometric memory, thin or thick tailed shocks, bilinear data, covariance stationary GARCH (Davidson 1994, 2004) and SV (Hill 2011).

Remark 3: If F_t is adapted to X_t then $\{X_t\}$ is trivially L_p -NED with constants $d_t = 0$ and arbitrary size $\lambda > 0$. For example, since ϵ_t can be anything it can be mixing, and $X_t = \epsilon_t$ is always possible with $d_t = 0$ and $\vartheta_l = o(l^{-\lambda})$ for any $\lambda > 0$.

Thus, a mixing process is NED on itself covering many linear and nonlinear ARMA, GARCH and ARMA-GARCH processes. See Section 4.3 for examples.

A tail version of NED requires the b_n -event functional

$$I_{n,t}(u) := I(X_t \leq b_n e^u) \quad \text{and} \quad \bar{I}_{n,t}(u) := I(X_t > b_n e^u)$$

where

$$I_{n,t} := I_{n,t}(0) \quad \text{and} \quad \bar{I}_{n,t} := \bar{I}_{n,t}(0).$$

Throughout $\{l_n\}$ denotes a sequence of positive integers with lower bound $\liminf_{n \rightarrow \infty} l_n = l \in \mathbb{N}$. Notice $l_n \rightarrow \infty$ is allowed.

L_p -Extremal Near Epoch Dependence $\{X_t\}$ is L_p -Extremal-NED on $\{F_t\}$ or on $\{\epsilon_t\}$ with size $\lambda > 0$ if for some $\{l_n\}$

$$\left\| \bar{I}_{n,t}(u) - E \left[\bar{I}_{n,t}(u) | F_{t-l_n}^{t+l_n} \right] \right\|_p \leq d_{n,t}(u) \times \psi_{l_n}, \quad \forall n \geq 1, \quad (11)$$

where $d_{n,t} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is Lebesgue measurable, $\sup_{u \geq 0} \sup_{1 \leq t \leq n} d_{n,t}(u) = O((k_n/n)^{1/p})$, $\psi_{l_n} \in [0, 1)$ and $l_n^\lambda \psi_{l_n} \rightarrow 0$.

Remark 1: Extremal-NED [E-NED] is simply NED applied to $\bar{I}_{n,t}(u)$, where (11) implies

$$\frac{n}{k_n} \times l_n^{p\lambda} \times \sup_{1 \leq t \leq n} E \left| \bar{I}_{n,t}(u) - E \left[\bar{I}_{n,t}(u) | F_{t-l_n}^{t+l_n} \right] \right|^p \rightarrow 0 \text{ as } n \rightarrow \infty.$$

The outer scale n/k_n controls for inherent degeneracy since $\|\bar{I}_{n,t}(u)\|_p^p = O(k_n/n) = o(1)$, and the inner scale $e^u \geq 1$ smoothly expands the tail threshold. The constants $d_{n,t}(u)$ are uniformly bounded in t since $|\bar{I}_{n,t}(u) - E[\bar{I}_{n,t}(u) | F_{t-l_n}^{t+l_n}]| \in [0, 1]$.

Remark 2: A useful distinction between NED (10) and E-NED (11) is the sample-size dependent displacement sequence l_n in (11). This permits far greater flexibility in analyzing dependence in extremes and non-extremes, and therefore deriving associated limit theory. In particular, since $l_n \rightarrow \infty$ arbitrarily fast is allowed it can be tailored to ensure E-NED and suit asymptotic arguments for non-stationary data.

Remark 3: Similar to the NED case, any mixing process is L_2 -E-NED on itself with trivial constants and arbitrary size.

The case where the base σ -field F_t is adapted to X_t leads to trivial results. Thus all subsequent claims implicitly assume F_t is not adapted to X_t , unless otherwise stated.

Population L_p -NED implies L_q -E-NED for any $q \geq 2$, and L_q -E-NED is equivalent to L_r -E-NED for any $r \gtrsim q$. The latter applies because $\bar{I}_{n,t}(u)$ is bounded.

THEOREM 2.1

(i). Let $\{X_t\}$ be L_p -NED, $p > 0$, with constants d_t and coefficients ϑ_l of size

$\lambda > 0$. Then $\{X_t\}$ is L_q -E-NED for any $q \geq 2$ with constants $d_{n,t}(u) = K(k_n/n)^{1/q}e^{-up/q}$ and coefficients ψ_{l_n} of size $\theta = \lambda \times \min\{p, 1\}/(2q)$.

(ii). Let $\{X_t\}$ be L_q -E-NED, $q > 0$, with constants $d_{n,t}(u)$ and coefficients ψ_{l_n} of size $\theta > 0$. Then $\{X_t\}$ is L_r -E-NED for any $r \gtrsim q$ with constants $d_{n,t}^{q/\max\{q,r\}}(u)$ and coefficients $\psi_{l_n}^{q/\max\{q,r\}}$ of size $\theta \times (q/\max\{q,r\})$.

Remark 1: Size is irrelevant in the geometric memory case, so geometric L_p -NED implies geometric L_q -E-NED for all $q > 0$.

Remark 2: Irrespective of the degree of non-stationarity characterized by the NED constants d_t , since $l_n \rightarrow \infty$ is otherwise arbitrary the E-NED constants $d_{n,t}(u) = K(k_n/n)^{1/q}e^{-up/q}$ are uniformly bounded in $1 \leq t \leq n$, $n \geq 1$ and $u \geq 0$, and Lebesgue integrable on \mathbb{R}_+ . Consult the proof.

Remark 3: Hill (2010: Lemma B.1) proves if $\{X_t\}$ is L_2 -E-NED with coefficients ψ_{l_n} and Lebesgue integrable constants $d_{n,t}(u)$ then the exceedance process $\{E_{n,t}\} = \{(\ln(X_t/b_n)_+)\}$ is L_2 -NED with the same coefficients ψ_{l_n} and constants $K \int_0^\infty d_{n,t}(u)du$. Use Theorem 2.1 to deduce population L_p -NED implies L_2 -NED exceedances $\{E_{n,t}\}$.

We cannot in general apply Theorem 2.1 to GARCH processes with unit or explosive roots without further information about the error distribution. Davidson (2004), for example, only shows covariance stationary GARCH are L_1 -NED. An alternative route is to prove $\{X_t\}$ is mixing and use the fact that a mixing process is trivially NED on itself. This, however, invariably requires a smooth error distribution (e.g. Boussama 1998, Basrak et al 2002a, Carrasco and Chen 2002).

In order to characterize the extremes of these processes without additional assumptions on the errors, we exploit probability-based L_0 -Approximability (L_0 -APP: Pötscher and Prucha 1991): there exists an F_{t-l}^{t+l} -measurable function $g_t^{(l)}$ and sequences of deterministic positive numbers $\{f_t, v_l\}$, $\inf_{t \in \mathbb{Z}} f_t \geq f > 0$ and $v_l \in [0, 1)$, such that

$$P\left(\left|X_t - g_t^{(l)}\right| > f_t \times \delta\right) \leq v_l = o\left(l^{-\lambda}\right) \text{ for some } \lambda > 0 \text{ and any } \delta > 0. \quad (12)$$

L_0 -APP is useful for heavy tailed processes that may not satisfy certain moment conditions, as opposed to moment-based Near Epoch Dependence, mixingale (McLeish 1975), and so-called θ -Weak Dependence (Nze et al 2002) and L_p -Weak Dependence (Wu and Min 2005). Further, L_p -NED implies L_0 -APP (Davidson 1994: p. 274), and as with NED and E-NED if F_t is adapted to X_t then $\{X_t\}$ is trivially L_0 -APP on itself.

Now repeat the logic of E-NED by replacing X_t with $\bar{I}_{n,t}(u) = I(X_t > b_n e^u)$.

L_0 -Extremal-Approximability $\{X_t\}$ is L_0 -Extremal-Approximable on $\{F_t\}$ with size $\lambda > 0$ if there exists an $F_{t-l_n}^{t+l_n}$ -measurable stochastic functional $h_t^{(l_n)}(u)$ and coefficients $\varphi_{l_n} \in [0, 1)$, $\varphi_{l_n} = o(l_n^{-\lambda})$, such that

$$P\left(\left|\bar{I}_{n,t}(u) - h_t^{(l_n)}(u)\right| > f_{n,t} \times \delta_n\right) \leq K \times e_{n,t}(u) \times \varphi_{l_n}$$

for some $\{l_n\}$, $\liminf_{n \rightarrow \infty} l_n \in \mathbb{N}$, some deterministic triangular array $\{f_{n,t}\}$, $\liminf_{n \rightarrow \infty} \inf_{1 \leq t \leq n} f_{n,t} = f > 0$, and any deterministic sequence $\{\delta_n\}$, $\inf_{n \geq 1} \delta_n = \delta > 0$. The constants $e_{n,t} : \mathbb{R}_+ \rightarrow [0, 1]$ are Lebesgue measurable and $\sup_{u \geq 0} \sup_{1 \leq t \leq n} e_{n,t}(u) = O(k_n/n)$. The array $\{h_t^{(l_n)}(u)\}$ is the L_0 -E-APP "approximator" of X_t .

L_p -E-NED implies L_0 -E-APP by Markov's inequality, just like NED implies APP, hence a proof is omitted.

LEMMA 2.2 *If $\{X_t\}$ is L_p -E-NED, $p > 0$, with size λ then it is L_0 -E-APP with size $p\lambda$ and approximator $\{P(X_t > b_n e^u | F_{t-l_n}^{t+l_n})\}$.*

Similarly, population L_0 -APP implies L_0 -E-APP, just like L_p -NED implies L_q -E-NED.

LEMMA 2.3 *Let $\{X_t\}$ be L_0 -APP with size λ , constants f_t and approximator $\{g_t^{(l)}\}$. Then $\{X_t\}$ is L_0 -E-APP with coefficients φ_{l_n} of size λ , constants $e_{n,t}(u) = (k_n/n)e^{-\iota u} \in [0, 1]$ for tiny $\iota > 0$, and approximator $\{I(g_t^{(l_n)} > b_n e^u)\}$.*

Remark: Under L_0 -APP the resulting L_0 -E-APP coefficients $e_{n,t}(u) = K(k_n/n)e^{-\iota u}$ are inherently Lebesgue integrable on $[0, \infty)$. Since the magnitude of $\iota > 0$ has no impact on integrability we simply write in the sequel

$$e_{n,t}(u) = K(k_n/n)e^{-u}.$$

L_0 -E-APP always implies L_2 -E-NED, contrary to the population case (e.g. Pötscher and Prucha 1991, Davidson 1994), a key link between approximability and Gaussian limit theory for tail arrays.

LEMMA 2.4 *Let $\{X_t\}$ be L_0 -E-APP with size λ and constants $e_{n,t}(u) = (k_n/n)e^{-u}$. If the approximator is either $\{I(g_t^{(l_n)} > b_n e^u)\}$ for some $F_{t-l_n}^{t+l_n}$ -measurable array $\{g_t^{(l_n)}\}$, or $\{P(X_t > b_n e^u | F_{t-l_n}^{t+l_n})\}$, then $\{X_t\}$ is L_2 -E-NED with size $\lambda/2$ and constants $d_{n,t}(u) = K(k_n/n)^{1/2}e^{-u/2}$.*

The following results are easy consequences of Lemmas 2.2-2.4: population L_0 -APP is sufficient for the tail L_2 -E-NED property, and NED and L_0 -APP are equivalent in the tails.

THEOREM 2.5 *If $\{X_t\}$ is L_0 -APP with size λ and constants f_t then it is L_2 -E-NED with size $\lambda/2$ and constants $d_{n,t}(u) = K(k_n/n)^{1/2}e^{-u/2}$.*

THEOREM 2.6 *$\{X_t\}$ is L_2 -E-NED with size $\lambda/2$ and constants $d_{n,t}(u) = K(k_n/n)^{1/2}e^{-u/2}$ if and only if it is L_0 -E-APP with size λ and constants $e_{n,t}(u) = (k_n/n)e^u$.*

Since L_2 -E-NED with Lebesgue integrable constants $d_{n,t}(u)$ implies the exceedance $\{E_{n,t}\} = \{(\ln |X_t|/b_n)_+\}$ is L_2 -NED (see Remark 3 of Theorem 2.1 above, cf. Hill 2010), the next claim follows instantly from Theorem 2.6.

COROLLARY 2.7 *If $\{X_t\}$ is L_0 -APP with size λ then $\{\bar{I}_{n,t}(u), E_{n,t}\}$ are L_2 -NED with size $\lambda/2$.*

These results are useful since heavy tailed data may only be L_0 -APP (e.g. GARCH with explosive roots), yet have L_2 -NED extremal information $\{\bar{I}_{n,t}(u)\}$ and $\{E_{n,t}\}$. The latter properties permit Gaussian limit theory for tail shape, quantile and dependence estimators under substantially general conditions. See Section 5 for applications.

3. NON-TAIL MEMORY Since trivially $\|I_{n,t}(u) - E[I_{n,t}(u)|\mathcal{F}_{t-l_n}^{t+l_n}]\|_p = \|\bar{I}_{n,t}(u) - E[\bar{I}_{n,t}(u)|\mathcal{F}_{t-l_n}^{t+l_n}]\|_p$, Theorem 2.1 instantly applies to $I_{n,t}(u)$: Near Epoch Dependence carries over to non-tail events.

COROLLARY 3.1 *If $\{X_t\}$ is L_p -NED, $p > 0$, with constants d_t and size λ then $\{I_{n,t}(u)\}$ is L_q -NED for any $q \geq 2$ with constants $d_{n,t}(u)$ and size $\theta = \lambda \times \min\{p, 1\}/(2q)$. Further, if $\{I_{n,t}(u)\}$ is L_q -NED, $q > 0$, with constants $d_{n,t}(u)$ and size $\theta > 0$ then it is L_r -E-NED for any $r \geq q$ with constants $d_{n,t}^{q/\max\{q,r\}}(u)$ and size $\theta \times (q/\max\{q,r\})$.*

The next result shows population L_0 -APP carries over to power transforms of tail-trimmed levels $X_t I_{n,t}(u) = X_t I(X_t \leq b_n e^u)$ and extreme levels $X_t \bar{I}_{n,t}(u) = X_t I(X_t > b_n e^u)$.

LEMMA 3.2 *Let $\{X_t\}$ be L_0 -APP with coefficients v_t of size λ and approximator $\{g_t^{(l)}\}$, and choose any $s > 0$. Then $\{X_t^s I_{n,t}(u)\}$ and $\{X_t^s \bar{I}_{n,t}(u)\}$ are L_0 -APP with coefficients w_{t_n} of size λ and approximators $\{(g_t^{(l_n)})^s I(g_t^{(l_n)} \leq b_n e^u)\}$ and $\{(g_t^{(l_n)})^s I(g_t^{(l_n)} > b_n e^u)\}$ respectively.*

Since central limit theory exists for NED trimmed levels $X_t I(X_t \leq b_n)$ we now link APP to NED in non-extremes. Consider a $[0, 1]$ -bounded version $b_n^{-1} X_t I(X_t \leq b_n)$. Theorem 2.6 characterizes a two-way relationship between tail-based E-NED and E-APP. Since the relationship for $b_n^{-1} X_t I(X_t \leq b_n)$ is complicated by its product convolution structure, we only have the following one-way result.

THEOREM 3.3 *Let $\{X_t\}$ be L_0 -APP with size λ and approximator $\{g_t^{(l)}\}$, and choose any $s > 0$. Then $\{b_n^{-s} X_t^s I_{n,t}\}$ is L_p -NED for any $p > 0$ with size $\lambda/\max\{p, 2\}$ and constants $d_{n,t} = d$.*

Remark 1: The NED constants d do not depend on n and t because $b_n^{-1} X_t I(X_t \leq b_n)$ is uniformly bounded.

Remark 2: Since the proof exploits $b_n^{-1} X_t I(X_t \leq b_n) \in [0, 1]$ a similar argument for a standardized extreme level $X_t I(X_t > b_n)$ is not available.

The scale construction ensures $b_n^{-1} X_t I(X_t \leq b_n)$ has infinitely many moments. See, for example, Hahn et al (1990) for central limit theory for a Winsorized $b_n^{-1} \min\{X_t, b_n\}$

under independence. Asymptotic theory for the tail-trimmed sums $\sum_{t=1}^n X_t I(X_t \leq b_n)$, however, requires a scale $(E(\sum_{t=1}^n X_t I(X_t \leq b_n))^2)^{1/2}$ which may deviate substantially from b_n or $\sqrt{nb_n}$ (e.g. Hill 2009b, Hill and Renault 2010).

THEOREM 3.4 *Let $\{X_t\}$ be L_0 -APP with size $\lambda > 0$ and approximator $\{g_t^{(l)}\}$, and choose any $s > 0$. Then $\{X_t^s I_{n,t}\}$ is L_p -NED for any $p > 0$ with constants $d_{n,t} = d$ and coefficients $\varpi_{l_n} = o(l_n^{-\theta})$ of size $\theta = \lambda / \max\{p, 2\}$.*

Remark: The result implies any L_0 -APP process $\{X_t\}$ with arbitrarily thick tails can be negligibly trimmed into an L_2 -NED process, guaranteeing a Gaussian central limit theory for $\{X_t I(X_t \leq b_n)\}$. This is immensely useful since it implies a versatile robust estimation theory. See Section 5.

Use the fact that L_p -NED with size λ implies L_0 -APP with size $p\lambda$ and approximator $\{E[X_t | F_{t-l}^{t+l}]\}$ (Davidson 1994: p. 274) to deduce the following easy extension of Theorems 3.3 and 3.4.

COROLLARY 3.5 *If $\{X_t\}$ is L_p -NED with size $\lambda > 0$ then for any $s > 0$, $\{b_n^{-s} X_t^s I(X_t \leq b_n)\}$ and $\{X_t^s I(X_t \leq b_n)\}$ are L_q -NED for any $q > 0$ with respective sizes $p\lambda / \max\{q, 2\}$ and $p\lambda / \max\{q, 2\}$. In each case the constants $d_{n,t} = d$ are time-invariant.*

4. TAIL AND NON-TAIL MEMORY: EXAMPLES We now characterize memory in ARCH(∞) class (4) and distributed lags (5), and discuss extensions to other nonlinear processes. In order to reduce repetitive claims, notice Sections 2 and 3 show L_2 -NED applied to tail events or exceedances $\{I(X_t > b_n e^u), (\ln(X_t/b_n))_+\}$ or tail-trimmed level power transforms $\{X_t^s I(X_t \leq b_n)\}$ for any $s > 0$ results in time-independent constants, where the constants of $I(X_t > b_n e^u)$ are $O((k_n/n)^{1/2} e^{-u/2})$.

4.1 Linear Distributed Lags

Consider X_t in (5), and assume the errors ϵ_t have tail (2) with index $\kappa > 0$. Regular variation (2) and a bound on the coefficients permit a simple proof of L_0 -APP. Notice we only require the innovations to behave *like* an independent sequence and only in the tails. Recall $F_t = \sigma(\epsilon_\tau : \tau \leq t)$.

ASSUMPTION A

1. X_t is the linear distributed lag (5) where ϵ_t has for each t tail (2) with index $\kappa > 0$. Further, ϵ_t is uniformly L_p -bounded, $p > 0$, and weakly tail orthogonal: $P(\sum_{i=0}^{\infty} |a_i \epsilon_{t-i}| > x) \sim \sum_{i=0}^{\infty} P(|a_i \epsilon_{t-i}| > x)$ for all real sequences $\{a_i\}_{i=0}^{\infty}$, $\sum |a_i|^\kappa < \infty$.
2. Let $\psi_{t,0} = 1$ for each t . The remaining coefficients $\psi_{t,i}$ are measurable with respect to F_{t-i-1} , and there exists a sequence of non-stochastic real numbers $\{\psi_i\}$ satisfying $|\psi_{t,i}| \leq |\psi_i|$ a.s., where $\psi_i = O(i^{-\mu})$ for some $\mu > 1/\kappa$, or $\psi_i = O(\rho^i)$.

Remark 1: Weak tail orthogonality is substantially weaker than independence while independence guarantees it (e.g. Feller 1971, Cline 1983). The abstraction is not empty since it, and not independence, captures the error dependence properties for a distributed lag representation of a bilinear process (Hill 2008, 2010). See Example 2, below.

Remark 2: The property $P(\sum_{i=0}^{\infty} |a_i \epsilon_{t-i}| > x) \sim \sum_{i=0}^{\infty} P(|a_i \epsilon_{t-i}| > x)$ is a special case of a larger set of probability and moment inequalities for an encompassing class of processes. See especially Nagev (1979, 1998) and de la Peña et al (2003) for deep results.

Use $|\psi_{t,i}| \leq |\psi_i|$ *a.s.*, weak tail orthogonality, properties of regularly varying tails (2) and summability $\sum_{i=0}^{\infty} |\psi_i|^\kappa < \infty$ to deduce for any sequence $\{f_t\}$, $\inf_{t \in \mathbb{Z}} f_t > 0$, any $\delta > 0$, and any $l \in \mathbb{N}$ (cf. Feller 1971, Resnick 1987)

$$\begin{aligned} P\left(\left|X_t - \sum_{i=0}^l \psi_{t,i} \epsilon_{t-i}\right| > f_t \delta\right) &\leq P\left(\sum_{i=l+1}^{\infty} |\psi_i| \times |\epsilon_{t-i}| > f_t \delta\right) \\ &\sim \sum_{i=l+1}^{\infty} |\psi_i|^\kappa P(|\epsilon_{t-i}| > f_t \delta) \leq \sum_{i=l+1}^{\infty} |\psi_i|^\kappa. \end{aligned}$$

The following claim is therefore straightforward to prove.

LEMMA 4.1 *Under Assumption A $\{X_t\}$ is L_0 -APP on $\{\epsilon_t\}$ with F_{t-l}^{t+l} -measurable approximator $\{\sum_{i=0}^l \psi_{t,i} \epsilon_{t-i}\}$ and arbitrary size $\lambda > 0$ if $\psi_i = O(\rho^i)$, or size $\lambda = \mu\kappa - \iota > 1$ if $\psi_i = O(i^{-\mu})$.*

Remark: Summability $\sum_{i=0}^{\infty} |\psi_i|^\kappa < \infty$ forces a restriction on hyperbolic memory since $\sum_{i=0}^{\infty} |\psi_i|^\kappa \leq K \sum_{i=0}^{\infty} i^{-\mu\kappa} < \infty$ requires $\mu\kappa > 1$. This reveals a standard memory-moment trade-off: as $\kappa \searrow 0$ for heavier tails we require monotonically weaker memory $\psi_i = O(i^{-\mu}) \searrow 0$ since $\mu \nearrow \infty$.

In lieu of Lemma 4.1 apply Remark 3 of Theorem 2.1, and Theorems 2.5 and 3.4 to deduce the nonlinear distributed lag X_t has L_2 -NED extreme events $I(X_t > b_n e^u)$, exceedances $(\ln(X_t/b_n))_+$ and non-extreme levels $X_t I(X_t \leq b_n)$.

THEOREM 4.2 *Let Assumption A hold and let $\mu\kappa > 1$.*

- (i). $\{X_t\}$ is L_2 -E-NED on $\{\epsilon_t\}$ with arbitrary size in the geometric case, or size $\mu\kappa/2 - \iota$ if $\psi_i = O(i^{-\mu})$.
- (ii). $\{X_t^s I(X_t \leq b_n)\}$ is L_2 -NED on $\{\epsilon_t\}$ with time-independent constants, and arbitrary size in the geometric case, or size $\mu\kappa/2 - \iota$ if $\psi_i = O(i^{-\mu})$.

Examples of processes that satisfy Assumption A include ARFIMA and bilinear.

EXAMPLE 1 (ARFIMA): Consider an ARFIMA(1, d , 1) process $\{X_t\}$, $d < 1$:

$$(1 - L)^d (1 - \phi L) X_t = (1 + \theta L) \epsilon_t, \quad |\phi| < 1, \quad \epsilon_t \stackrel{iid}{\sim} (2) \text{ with index } \kappa > 1/(1 - d).$$

Since X_t has representation $\sum_{i=0}^{\infty} \psi_i \epsilon_{t-i}$ for iid $\epsilon_t \sim (2)$ and $\psi_i = O(i^{d-1})$ (Hosking 1981), Assumption A is satisfied: independence implies ϵ_t is weakly tail orthogonal, and $\psi_i = O(i^{-\mu})$ for $\mu = 1 - d > 1/\kappa > 0$. Therefore $\{X_t\}$ is L_0 -APP on $\{\epsilon_t\}$ with size $(1 - d)\kappa - \iota > 1$ by Lemma 4.1.

If $d = -1/2$, for example, the tail index must satisfy $\kappa > 2/3$ for Assumption A to hold. In general the memory-moment trade-off implies as $d \nearrow 1$, where $I(1)$ is the limit, the smallest allowed tail index must increase $\kappa \nearrow \infty$. The long-memory range $d \in (1/2, 1)$ requires finite variance $\kappa \in (2, \infty)$. Nevertheless, when $d \in (0, 1)$ the first difference $\Delta X_t := X_t - X_{t-1}$ is ARFIMA(1, \tilde{d} , 1) for some $\tilde{d} = d - 1 < 0$ so $\{\Delta X_t\}$ satisfies Assumption A for any comparatively small $\kappa > 1/(2 - d)$. In this case, for example, if $\kappa \geq 1$ then all $d < 1$ are covered.

EXAMPLE 2 (Bilinear): Consider a simple bilinear process

$$X_t = \phi X_{t-1} u_{t-1} + u_t, \quad u_t \stackrel{iid}{\sim} (2) \text{ with index } \kappa > 0, \quad \phi > 0, \quad \phi^{\kappa/2} E[u_t^{\kappa/2}] < 1.$$

Then X_t has the representation $\sum_{i=0}^{\infty} \psi_i \epsilon_{t-i}$, $\psi_i = O(\rho^i)$, $\rho \in (0, 1)$, and $\epsilon_t \sim (2)$ with index $\kappa/2$ (Davis and Resnick 1996: Corollary 2.4). Further, ϵ_t is weakly tail orthogonal by the proof of Lemma 8 in Hill (2010).

We impose F_{t-i-1} -measurability of $\psi_{t,i}$ in Assumption A.2 to ensure the NED base is the weakly orthogonal $\{\epsilon_t\}$ and therefore a simple $F_{t-l_n}^{t+l_n}$ -measurable approximator is available. The latter are key to proving Lemma 4.1. But such measurability does not characterize many nonlinear autoregressions.

EXAMPLE 3 (STAR): Consider an Exponential Smooth Transition Autoregression: $X_t = \phi X_{t-1} \exp\{-\gamma X_{t-1}^2\} + \epsilon_t$, $\gamma > 0$, $|\phi| < 1$ with $\epsilon_t \stackrel{iid}{\sim} (2)$. Then $X_t = \sum_{i=0}^{\infty} \psi_{t,i} \epsilon_{t-i}$ where $\psi_{t,0} = 1$ and the remaining $\psi_{t,i} = \phi^i \prod_{j=1}^i \exp\{-\gamma X_{t-j}^2\}$ are F_{t-1} -measurable for each $i \geq 1$. Thus Assumption A.2 does not hold.

In order to allow such nonlinear feedback, further restrictions on the errors are evidently required. See Sections 4.3 and 4.4, below.

4.2 GARCH

Consider GARCH class (1).

EXAMPLE 4 (GARCH): Write the GARCH(p, q) process as

$$X_t = \sigma_t \epsilon_t \quad \text{and} \quad \beta(L) \sigma_t^2 = \omega + \{\delta(L) - \beta(L)\} X_t^2$$

with lag polynomials $\delta(L) = 1 - \sum_{i=1}^p \delta_i L^i$ and $\beta(L) = 1 - \sum_{i=1}^q \beta_i L^i$. If the roots of $\beta(z)$ lie outside the unit circle then (4) holds with $\pi(z) = 1 - \delta(z)/\beta(z)$ and $\pi_i = O(\rho^i)$. Under covariance stationarity $\epsilon_t \stackrel{iid}{\sim} (0, 1)$ and $S := \pi(1) < 1$ Davidson (2004: Theorem 3.2) proves $\{X_t\}$ is geometrically L_1 -NED on $\{\epsilon_t\}$. Apply Theorem 2.1 and Corollary 3.5 to deduce $\{I(X_t > b_n e^u), (\ln(X_t/b_n))_+, X_t^s I(X_t \leq b_n)\}$ are geometrically L_2 -NED.

EXAMPLE 5 (FIGARCH): Davidson's (2004: Theorems 3.1-3.2) covariance

stationarity condition $S < 1$ rules out NED for IGARCH and FIGARCH since (4) is then satisfied with

$$\pi(L) = 1 - \frac{\delta(L)}{\beta(L)}(1 - L)^d, \quad d \in (0, 1]$$

for some $\delta(L)$. Both cases correspond with $S = \pi(1) = 1$, where IGARCH ($d = 1$) exhibits geometric decay $\pi_i = O(\rho^i)$, and FIGARCH ($d < 1$) hyperbolic decay since $(1 - L)^d = 1 - \sum_{i=1}^{\infty} \gamma_i L^i$ with $\gamma_i = O(i^{-1-d})$.

In order to characterize the tail and non-tail memory properties of short-range memory GARCH processes with unit ($S = 1$) or explosive ($S > 1$) roots, Davidson (2004) suggests a different approach based on approximability.

ASSUMPTION B Let $X_t = \sigma_t \epsilon_t$ satisfy (4) with $\epsilon_t \stackrel{iid}{\sim} (0, 1)$, $0 \leq \pi_i \leq C\rho^i$ and $C \in [0, 1/\rho)$.

Under Assumption B $\{X_t\}$ is geometrically L_0 -APP on $\{\epsilon_t\}$ by Theorem 3.3 of Davidson (2004). Now invoke Theorems 2.1 and 3.4 above to deduce GARCH processes have geometrically NED extremes and non-extremes.

THEOREM 4.3 Under Assumption B $\{I(X_t > b_n e^u), (\ln(X_t/b_n))_+, X_t^s I(X_t \leq b_n)\}$ are geometrically L_2 -NED on $\{\epsilon_t\}$.

Remark: Since $S = \sum_{i=0}^{\infty} \pi_i \leq C\rho/(1 - \rho)$ clearly $S = 1$ and $S > 1$ are easily feasible, covering IGARCH and many Explosive GARCH cases.

Theorem 4.3 exploits L_0 -APP to get around the hairline non-covariance stationary case $S = 1$ under short-range memory. Davidson (2004: eqs. (5.1) and (5.4)), however, suggests a new class of Hyperbolic GARCH(p, d, γ, q) models [HYGARCH] to conquer the FIGARCH property $S = 1$. Assume $X_t = \sigma_t \epsilon_t$ is governed by (4) where

$$\pi(L) = 1 - \frac{\delta(L)}{\beta(L)} \left[1 + \gamma \left((1 - L)^d - 1 \right) \right], \quad d \in (0, 1], \gamma \geq 0 \quad (13)$$

or

$$\pi(L) = 1 - \frac{\delta(L)}{\beta(L)} \left[1 + \frac{\gamma}{\zeta(1+d)} \sum_{i=0}^{\infty} i^{-1-d} L^i \right], \quad d > 0, \gamma \geq 0, \quad (14)$$

and $\zeta(\cdot)$ is the Riemann zeta function. The index $d > 0$ as usual governs the degree of hyperbolic memory. The case $d > 1$ is ruled out in (13) since it permits negative coefficients. FIGARCH and GARCH correspond to (13) with $\gamma = 1$ and $\gamma = 0$ respectively; IGARCH(1, 1) can be deduced from (13) with $d = 1$ and $\delta(L) = 1$; and $\gamma \geq 1$ in (13) or (14) aligns with non-stationary cases. The tuning parameter γ is key for allowing both hyperbolic decay and covariance stationarity $S < 1$. As long as $d > 0$ then $S < 1$ in both (13) and (14) for any covariance stationary case corresponding to $\gamma < 1$.

Each condition of Davidson's (2004) Theorem 3.1 is satisfied when $\epsilon_t \stackrel{iid}{\sim} (0, 1)$, $\gamma < 1$, and $d \in (0, 1)$ in (13) or $d > 0$ in (14): $\{X_t\}$ is L_2 -bounded and L_1 -NED on $\{\epsilon_t\}$ with size $\lambda = d - \iota$. Invoke Theorem 2.1 and Corollary 3.5 to deduce such HYGARCH processes have well defined tail and non-tail memory properties.

THEOREM 4.4 Let $X_t = \sigma_t \epsilon_t$ satisfy (4) where $\epsilon_t \stackrel{iid}{\sim} (0, 1)$. Specifically, $\pi(L)$ satisfies HYGARCH(p, d, γ, q) (13) with $d \in (0, 1]$ or (14) with $d > 0$, and in general $\gamma \in (0, 1)$. Then $\{I(X_t > b_n e^u), (\ln(X_t/b_n))_+, X_t^s I(X_t \leq b_n)\}$ are L_2 -NED on $\{F_t\}$ with size $d/2 - \iota$.

Remark: Although L_2 -boundedness for X_t is a severe restriction higher moments need not exist, and the tail-trimmed $X_t^s I(X_t \leq b_n)$ is necessarily L_2 -NED. In particular, central limit theory for a tail-trimmed sample variance and covariance exists when X_t^2 and $X_t X_{t-h}$ have an infinite variance (Hill 2009b).

EXAMPLE 6 (HYGARCH): Consider the HYGARCH(1, 2, γ , 1) model

$$X_t = \sigma_t \epsilon_t \quad \text{and} \quad \sigma_t^2 = \pi_0 + \left(1 - \frac{1 - \delta}{1 - \beta} \left[1 + \frac{\gamma}{\zeta(2 + \iota)} \sum_{i=0}^{\infty} i^{-3} L^i \right] \right) X_t^2,$$

where $\beta \in (0, 1)$, $\delta > 0$ and $\gamma \in (0, 1)$. By Theorem 4.4 $\{I(X_t > b_n e^u), (\ln(X_t/b_n))_+, X_t^s I(X_t \leq b_n)\}$ are L_2 -NED with size $1/2$.

4.3 Nonlinear AR-nonlinear GARCH

A large array of nonlinear AR-nonlinear GARCH or SV processes with short-range memory and iid innovations ϵ_t are geometrically ergodic or β -mixing, and therefore geometrically α -mixing. In these cases $\{I(X_t > b_n e^u), (\ln(X_t/b_n))_+, X_t^s I(X_t \leq b_n)\}$ are L_2 -NED with arbitrary size. Examples include Threshold Autoregressions, single layer feed-forward neural networks, random coefficient autoregressions and Multiplicative-, Exponential-, GJR-, and Threshold-GARCH, to name a few. See An and Huang (1996), Ling (1999), Carrasco and Chen (2002), Cline and Pu (2004), and Meitz and Saikkonen (2008).

Compare the following to Examples 1, 2, 4 and 5 where distribution smoothness for the errors is not required and hyperbolic memory cases are covered.

EXAMPLE 7 (A-GARCH): Engle and Ng (1993) propose the following Asymmetric-GARCH(1,1) process $X_t = \sigma_t \epsilon_t$ where

$$\sigma_t^2 = \omega + \alpha (\epsilon_{t-1} - c)^2 \sigma_{t-1}^2 + \beta \sigma_{t-1}^2, \quad \omega > 0, \alpha, \beta \geq 0, c \in \mathbb{R}.$$

Assume $\epsilon_t \stackrel{iid}{\sim} (0, 1)$ has a positive, continuous density with respect to Lebesgue measure on \mathbb{R} , and X_0 is initialized from the invariant distribution. The case $c = 0$ corresponds to GARCH(1,1). If $E[\beta + \alpha(\epsilon_{t-1} - c)^2] < 1$ or $\alpha + \beta < \{E[(\epsilon_{t-1} - c)^2]\}^{-1}$ then $\{X_t\}$ is geometrically β -mixing (Carrasco and Chen 2002: Corollary 6). If the degree of asymmetry is slight $c = .01$, for example, then $\alpha + \beta < .9999$ suffices, and if $c = 1$ then $\alpha + \beta < 1/2$ must hold, exhibiting an asymmetry-memory trade-off.

EXAMPLE 8 (TAR): Consider a Threshold Autoregression of order one:

$$X_t = \phi_1 X_{t-1} I(X_{t-1} \leq c) + \phi_2 X_{t-1} I(X_{t-1} > c) + \epsilon_t = \phi_{t-1} X_{t-1} + \epsilon_t,$$

say, where iid ϵ_t is mean zero with positive density \mathbb{R} -a.e., $E|\epsilon_t| < \infty$, $c \in \mathbb{R}$, and $|\phi_t| \leq \max\{|\phi_1|, |\phi_2|\} \in (0, 1)$. Then $\{X_t\}$ is geometrically ergodic (An and Huang 1996:

Theorem 3.1).

EXAMPLE 9 (STAR): If the STAR process $\{X_t\}$ of Example 3 has innovations ϵ_t that satisfy Example 8 then $\{X_t\}$ is geometrically ergodic (An and Huang 1996: Theorem 3.1).

4.4 Distributed Lags with Random Volatility Errors

Asymptotic theory for NED, E-NED and tail-trimmed NED processes $\{X_t\}$ permits the base $\{\epsilon_t\}$ to satisfy a mixing condition (Davidson 1992, de Jong 1997, Hill 2009a,b, 2010). This is helpful for modeling the memory properties of linear and nonlinear ARMA-GARCH or SV with hyperbolic memory.

By comparison the distributed lag Assumption A imposes regular variation and weak tail orthogonality on the errors and F_{t-i-1} -measurability on $\psi_{t,i}$ to ensure the weaker L_0 -APP property holds. Neither regular variation nor weak tail memory is guaranteed to hold for distributed lags with linear or nonlinear GARCH errors (Chernick et al 1991, Mikosch and Stărică 2000, Borkovec and Klüppelberg 2001, Basrak et al 2002a, Cline 2007), and coefficient measurability fails for random coefficient autoregressions like SETAR and STAR. If we require an NED array like $\{I(X_t > b_n e^u)\}$, $\{(\ln(X_t/b_n)_+)\}$, or $\{X_t^s I(X_t \leq b_n)\}$ to exhibit hyperbolic memory and have a linear or nonlinear GARCH or SV base $\{\epsilon_t\}$, something more than Sections 4.1-4.3 must be developed.

Consider a generalization of (5):

$$X_t = \sum_{i=0}^{\infty} \psi_{t,i} u_{t-i}, \quad |\psi_{t,i}| \leq |\psi_i| \text{ a.s.}, \quad \sum_{i=0}^{\infty} |\psi_i| < \infty, \quad \sup_{t \in \mathbb{N}} \|u_t\|_p < \infty \text{ for } p > 0,$$

assume $\psi_{t,i}$ are measurable, and define the σ -field induced by $\{\psi_{t,i} u_{t-i}\}$:

$$G_t := \sigma(\psi_{\tau,i} u_{\tau-i} : t \leq \tau, i \geq 0).$$

If $\{\psi_{t,i} u_{t-i}\}$ is the base then $|\psi_{t,i}| \leq |\psi_i|$ and uniform L_p -boundedness imply $\{X_t\}$ is L_p -NED.

LEMMA 4.5 $\{X_t\}$ is L_p -NED on $\{G_t\}$ with arbitrary size if $\psi_i = O(\rho^i)$ and size $\lambda = \mu - \iota$ if $\psi_i = O(i^{-\mu})$.

Since L_p -NED with size λ implies L_0 -APP with size $p\lambda$, simply replace F_t with G_t in Theorem 4.2 to deduce Lemma 4.5 implies NED extremes and non-extremes.

THEOREM 4.6 Each $\{I(X_t > b_n e^u), (\ln(X_t/b_n)_+, X_t^s I(X_t \leq b_n))\}$ is L_2 -NED on $\{G_t\}$ with arbitrary size if $\psi_i = O(\rho^i)$ or size $p\mu\kappa/2 - \iota$ if $\psi_i = O(i^{-\mu})$.

Notice we need say nothing about the stochastic nature of $\psi_{t,i}$ nor orthogonality of ϵ_t . The key to a central limit theory for tail and non-tail arrays of $\{X_t\}$ is to demonstrate $\{\psi_{t,i} u_{t-i}\}$ satisfies a mixing property.

EXAMPLE 10 (SETAR-STGARCH): Consider a stationary Self Exciting Threshold-Autoregression with Exponential Smooth Transition GARCH(1,1) errors:

$$X_t = \phi X_{t-1} I(X_{t-1} < 0) + u_t, \quad |\phi| < 1, \quad u_t = \sigma_t \epsilon_t, \quad \epsilon_t \stackrel{iid}{\sim} (0, 1) \quad (15)$$

$$\sigma_t^2 = \omega + \alpha u_{t-1}^2 + \tilde{\alpha} u_{t-1}^2 \exp\{-\gamma u_{t-1}^2\} + \beta \sigma_{t-1}^2$$

where $\gamma \geq 0$, $\omega > 0$, and $\{\alpha, \alpha + \tilde{\alpha}, \beta\} \geq 0$. See Tong and Lim (1980) for SETAR and González-Rivera (1998) for STGARCH, inter alia. Assume $E[(\beta + \max\{\alpha, \alpha + \tilde{\alpha}\} \epsilon_t^2)^r] < 1$. Write $X_t = \phi_{t-1} X_{t-1} + u_t = \sum_{i=0}^{\infty} \psi_{t,i} u_{t-i}$, where $\phi_{t-1} = \phi I(X_{t-1} < 0)$, $\psi_{t,0} = 1 \forall t$, and $\psi_{t,i} = \prod_{j=1}^i \phi_{t-j}$ satisfies $|\psi_{t,i}| \leq |\psi_i| = O(\rho^i)$. Both $\{u_t, x_t\}$ are geometrically ergodic (Meitz and Saikkonen 2008: Proposition 1). Therefore $\psi_{t,i} u_{t-i} = \phi^i \prod_{j=1}^i I(X_{t-j} < 0) \times u_{t-i}$ for $i \geq 1$ is geometrically α -mixing. But this implies $\{X_t\}$ is geometrically L_p -NED for some $p > 0$ on a geometrically α -mixing base by Lemma 4.5, so Theorem 4.6 applies.

5. EXTREMAL AND ROBUST STATISTICS In this section we prove tail index and tail dependence estimators, intermediate order statistics, and a tail-trimmed sum are asymptotically normal for L_0 -APP data. This includes X_t in (1), (4) and (5), and the myriad nonlinear processes discussed above. We also show how the major tail and non-tail results simultaneously deliver new robust estimator asymptotics.

5.1 Order Statistics and Tail Index

Consider a second order Paretian class $P(X_t > x) = cx^{-\kappa}(1 + O(x^{-\varsigma}))$, $c, \varsigma > 0$ (e.g. Hall 1982, Haeusler and Teugels 1985). The class covers linear distributed lags like ARFIMA, and bilinear as long as the errors are iid with power-law tail decay (Cline 1983, Davis and Resnick 1996); and it naturally characterizes GARCH (Mikosch and Stărică 2000, Basrak et al 2002a), linear and nonlinear AR-GARCH (Borkovec and Klüppelberg 2001, Cline 2007), and auction prices near the reserve price (Hill and Schneyerov 2010).

The widely used Hill (1975) estimator of κ^{-1} is $\hat{\kappa}_n^{-1} = 1/k_n \sum_{i=1}^{k_n} \ln(X_{(i)}/X_{(k_n+1)})$. Let $k_n \rightarrow \infty$ and $k_n/n^{2\varsigma/(2\varsigma+\kappa)} \rightarrow 0$. Theorem 2.5 ensures all conditions of Hill's (2010: Theorem 2 and Lemma 3) Gaussian limit theory for $X_{(k_n+1)}$ and $\hat{\kappa}_n^{-1}$ are satisfied. Classic treatments can be found in Galambos (1987) and Embrechts et al (1997). See Hsing (1991) and Hill (2010) for surveys.

THEOREM 5.1 *If $\{X_t\}$ is L_0 -APP with size 1 on α -mixing $\{\epsilon_t\}$ with size 1 then*

$$k_n^{1/2} \ln(X_{(k_n+1)}/b_n) \xrightarrow{d} N(0, w^2) \quad \text{and} \quad k_n^{1/2} (\hat{\kappa}_n^{-1} - \kappa^{-1})/v_n \xrightarrow{d} N(0, 1)$$

where $w^2 = \kappa^{-2} \lim_{n \rightarrow \infty} E(1/k_n^{1/2} \sum_{t=1}^n I_{n,t}(u/k_n^{1/2}))^2 < \infty$ and $v_n^2 := E[(k_n^{1/2}(\hat{\kappa}_n^{-1} - \kappa^{-1}))^2] = O(1)$.

Remark: A similar argument reveals Hill's (2010: Theorem 3) kernel estimator \hat{v}_n^2 is consistent for v_n^2 under L_0 -APP.

Since Theorem 5.1 both encompasses and augments Hill's (2010) results, evidently this is the most general limit theory available for intermediate order statistics and tail index estimators under data dependence and heterogeneity. The size restrictions are irrelevant in the geometric decay case, covering IGARCH, Explosive GARCH, the Example 7 Asymmetric GARCH, Example 8 STAR and Example 9 SETAR-STGARCH models.

Each hyperbolic memory process discussed above is also covered provided memory decay is carefully considered. It is easy to verify the following cases: the Example 1 ARFIMA(1, d , 1) with innovations $\epsilon_t \stackrel{iid}{\sim} (2)$, $\kappa > 0$, and Hurst index $d < 1 - 1/\kappa$ is L_0 -APP with size 1 on iid $\{\epsilon_t\}$ covering long-memory cases when $\kappa > 2$; the Example 6 HYGARCH(p, d, γ, q) with iid innovations $\{\epsilon_t\}$ is L_1 -NED on iid $\{\epsilon_t\}$ with size $\lambda = d - \iota$ (Davidson 2004) and therefore L_0 -APP with size 1 for any $d > 0$.

5.2 Tail Dependence Coefficient (Extremogram)

Consider a bivariate process $\{Z_t\}$, $Z_t = [X_t, Y_t]'$, on $[0, \infty) \times [0, \infty)$ with marginal tails (2) and indices $\kappa_x, \kappa_y > 0$. Denote by $\{b_{x,n}, b_{y,n}\}$ the associated threshold sequences, e.g. $(n/k_n)P(X_t > b_{x,n}) \rightarrow 1$, where $1 \leq k_n < n$, $k_n = o(n)$ and $k_n \rightarrow \infty$, and define

$$\hat{I}_{x,n,t} := I(X_t > X_{(k_n+1)}) \quad \text{and} \quad \hat{I}_{y,n,t} := I(Y_t > Y_{(k_n+1)}).$$

The statistic

$$\hat{r}_n(h) = \frac{1}{k_n} \sum_{t=1}^n \left(\hat{I}_{x,n,t-h} \hat{I}_{y,n,t} - \left(\frac{k_n}{n} \right)^2 \right)$$

non-parametrically estimates the tail event correlation

$$r_n(h) := \frac{P_{h,n} - P_{x,n}P_{y,n}}{(P_{x,n}P_{y,n})^{1/2}} \sim \frac{n}{k_n} (P_{h,n} - P_{x,n}P_{y,n}),$$

where $P_{h,n} := P(X_{t-h} > b_{x,n}, Y_t > b_{y,n})$ and $P_{x,n} := P(X_t > b_{x,n})$. Davis and Mikosch (2009c) call the limit

$$r(h) = \lim_{n \rightarrow \infty} r_n(h) = \lim_{n \rightarrow \infty} \frac{P_{h,n} - P_{x,n}P_{y,n}}{(P_{x,n}P_{y,n})^{1/2}}$$

the "extremogram". Trivially $P_{x,n}P_{y,n}/(P_{x,n}P_{y,n})^{1/2} = (P_{x,n}P_{y,n})^{1/2} = o(1)$ hence the extremogram is identically

$$r(h) = \lim_{n \rightarrow \infty} \frac{P_{h,n}}{(P_{x,n}P_{y,n})^{1/2}}.$$

Define $\tilde{r}(h) := \lim_{n \rightarrow \infty} (n/k_n)r_n(h) = \lim_{n \rightarrow \infty} \{P_{h,n}/(P_{x,n}P_{y,n}) - 1\}$ if the limit exists.

Tail dependence at displacement h is exhibited if $\tilde{r}(h) \neq 0$ covering "local" forms $r(h) = 0$ yet $\tilde{r}(h) \neq 0$; and "distant" forms $r(h) \neq 0$ hence $\tilde{r}(h) = \infty^3$. See Hill (2008, 2011) for details. Note Davis and Mikosch (2009c) only consider distant tail dependence which neglects SV. See Section 6.1 below for details on $r_n(h)$ and its relationship with other tail dependence measures.

Write $r_{n,h} := [r_n(1), \dots, r_n(h)]'$ and $\hat{r}_{n,h} := [\hat{r}_n(1), \dots, \hat{r}_n(h)]'$. If $\{Z_t\}$ is L_0 -APP with size 2 on some possibly vector-valued base $\{\epsilon_t\}$ then it is L_2 -E-NED with size 1 on $\{\epsilon_t\}$ (Corollary 2.7), and therefore L_4 -E-NED with size 1/2 on $\{\epsilon_t\}$ (Theorem 2.1). If ϵ_t is α -mixing with size 1 then $\{X_t, Y_t\}$ satisfies all relevant tail decay and tail memory properties required of Theorem 3.2 of Hill (2008): as long as sufficiently many extremes are used $k_n/n^{2/3} \rightarrow \infty$ then

$$\sqrt{k_n}(\hat{r}_{n,h} - r_{n,h}) \xrightarrow{d} N(0, V),$$

where $V = \lim_{n \rightarrow \infty} k_n E[(\hat{r}_{n,h} - r_{n,h})(\hat{r}_{n,h} - r_{n,h})'] \in \mathbb{R}^{h \times h}$ is positive definite. Hill (2008, 2009a) shows a test of tail independence based on $k_n^{1/2} \hat{r}_{n,h}$ is consistent against local (i.e. $r(h) = 0, \tilde{r}(h) \neq 0$) or distant (i.e. $r(h) \neq 0, \tilde{r}(h) = \infty$) alternatives. See Davis and Mikosch (2009a) for the same estimator under distant dependence, joint regularly variation and α -mixing, a subset of the processes allowed here.

The hyperbolic memory case is complicated by feedback between $\hat{I}_{x,n,s-h} \hat{I}_{y,n,s}$ and $\hat{I}_{x,n,t-h} \hat{I}_{y,n,t}$.

EXAMPLE 11 (Stochastic Volatility): Let $\epsilon_t = [\epsilon_{x,t}, \epsilon_{y,t}]' \in \mathbb{R}^2$ be iid with symmetric marginal tails (2) and indices $\kappa = [\kappa_x, \kappa_y]'$ > 0 . Consider a bivariate random variable $Z_t = [X_t, Y_t]'$ with stochastic volatility

$$X_t = \sigma_{x,t} \epsilon_{x,t} \quad \text{and} \quad Y_t = \sigma_{y,t} \epsilon_{y,t}.$$

Assume $\sigma_t^\kappa = [\sigma_{x,t}^{\kappa_x}, \sigma_{y,t}^{\kappa_y}]' \in \mathbb{R}_+^2$ is independent of ϵ_t and governed by log-ARFIMA(1, d , 1),

$$(I_2 - \Phi L)(I_2 - L)^d \ln \sigma_t^\kappa = \Theta + \zeta_t, \quad \Theta \in \mathbb{R}_+^2, \quad \Phi \in \mathbb{R}_+^{2 \times 2}$$

$$\zeta_t \in \mathbb{R}^2, \quad \zeta_t \stackrel{iid}{\sim} N(0, I_2), \quad d < 1 - 2/\kappa,$$

where the roots of $I_2 - \Phi z$ lie outside the unit circle. By imitating Lemma 4.1 it is easy to show $\{\ln \sigma_t^\kappa\}$ is L_0 -APP on iid $\{\epsilon_t\}$ with size $(1-d)\kappa - \iota \geq 2$. Since $\zeta_t \stackrel{iid}{\sim} N(0, I_2)$ and each $\{E|\epsilon_{x,t}|^{\kappa_x - \iota}, E|\epsilon_{y,t}|^{\kappa_y - \iota}\} < \infty$ it similarly follows $\{Z_t\}$ is L_0 -APP on iid $\{\epsilon_t\}$ with size 2 (Davidson 1994: Theorem 17.22). For example, if $\kappa = 1.5$ then Hurst indices $d < -1/3$ are allowed, and long-memory when $\kappa > 4$. See Hill (2008, 2011) for E-NED characterizations when $d = 0$.

5.3 Tail-Trimmed Sums

Although the use of trimmed or truncated sums for robust estimation has a substantial history⁴, the vast majority of cases cover independent data with rare exceptions including mixing data with finite variance (Hahn et al 1987) and fixed quantile

³Clearly this abuses notation since therefore the limit $\tilde{r}(h) := \lim_{n \rightarrow \infty} (n/k_n)r_n(h)$ does not exist.

⁴See Stigler (1973) for historical details, and Hill (2009b) for a recent survey.

trimmed linear distributed lags (Wu 2005). Theorem 3.3, by comparison, allows for asymptotically negligible trimming for a far larger class of dependent heterogeneous heavy tailed processes.

Assume $X_t := |Y_t|$ for some symmetrically distributed L_p -bounded process $\{Y_t\}$, $p \in (0, 2)$, and let positive real sequences $\{k_n, b_n\}$ satisfy $(n/k_n)P(X_t > b_n) \rightarrow 1$. Recall $\hat{X}_{n,t} := X_t I(X_t \leq b_n) = X_t I_{n,t}$, define $v_n^2 := E(\sum_{t=1}^n \{\hat{X}_{n,t} - E[\hat{X}_{n,t}]\})^2$, and define the self-normalized tail-trimmed level for scalar X_t :

$$\hat{Z}_{n,t}^* := \left(\hat{X}_{n,t}^* - E \left[\hat{X}_{n,t} \right] \right) / v_n \quad \text{where} \quad \hat{X}_{n,t}^* = X_t I \left(X_t \leq X_{(k_{n+1})} \right).$$

Asymmetric trimming is identical in theory with only added notation. The trick is to fuse tail and non-tail asymptotics simultaneously by first showing $\sum_{t=1}^n \{\hat{X}_{n,t}^* - \hat{X}_{n,t}\} = o(v_n)$, which requires the extreme value result $X_{(k_{n+1})}/b_n = 1 + O_p(1/k_n^{1/2})$, and then delivering a central limit theorem for self-normalized trimmed sums $1/v_n \sum_{t=1}^n \{\hat{X}_{n,t} - E[\hat{X}_{n,t}]\}$. The next theorem exploits arguments in Hill (2009b) where only geometric memory is considered.

THEOREM 5.2 *Let $\liminf_{n \rightarrow \infty} \{v_n^2/n\} > 0$, $b_n/n^t \rightarrow \infty$ and $b_n = o(n^{1/2})$. Further, $\{X_t\}$ is L_p -bounded geometrically L_0 -APP on geometrically α -mixing $\{\epsilon_t\}$. Then $\sum_{t=1}^n \hat{Z}_{n,t}^* \xrightarrow{d} N(0, 1)$.*

Remark 1: The regulatory condition $\liminf_{n \rightarrow \infty} \{v_n^2/n\} > 0$ ensures the trimmed sum $\sum_{t=1}^n X_t I(X_t \leq b_n)$ is not degenerate asymptotically.

Remark 2: The trimming threshold b_n must be restricted to ensure sufficiently many tail observations are trimmed for asymptotic normality. The bound $b_n = o(n^{1/2})$ enforces $\max_{1 \leq t \leq n} \{|\hat{X}_{n,t}|\} = o_p(n^{1/2})$, the relative stability property shared by weakly dependent square integrable processes in the maximum domain of attraction of a Type II extreme value distribution (Leadbetter et al 1983, Naveau 2003).

The thresholds b_n are intimately related to the number of trimmed observations k_n through $P(X_t > b_n) \sim k_n/n$. Suppose X_t has tail (2) with index $\kappa \in (0, 2]$, and impose $k_n \sim n^\delta$, $\delta \in (0, 1)$, for simplicity.

EXAMPLE 12 (Paretian Tails and Trimming): Under (2) $b_n = K(n/k_n)^{1/\kappa} \sim K n^{(1-\delta)/\kappa}$. The bound $b_n = o(n^{1/2})$ reduces to $\delta > 1 - \kappa/2$. Heavier tailed data generating processes $\kappa \searrow 0$ result in samples with more extremes on average, hence more observations must be trimmed $k_n \sim n^\delta \nearrow n$ to ensure asymptotic normality. If $\kappa = 1.5$ or $\kappa = .75$ then trimming at least $[n^{.25}]$ or $[n^{.625}]$ observations per sample $\{X_t\}_{t=1}^n$ respectively ensures $\sum_{t=1}^n \hat{Z}_{n,t}^* \xrightarrow{d} N(0, 1)$.

5.4 Tail-Trimmed Method of Moments

The tail-trimmed sum lies at the heart of a new robust Minimum Distance Estimator. Consider estimating the autoregression parameter θ^0 of a stationary AR(1):

$$X_t = \theta^0 X_{t-1} + \epsilon_t, \quad |\theta^0| < 1, \quad \epsilon_t \stackrel{iid}{\sim} (2) \text{ with index } \kappa \in (1, 2], \text{ and } \mathfrak{S}_t := \sigma(X_\tau : \tau \leq t).$$

Assume for simplicity ϵ_t has an \mathbb{R} -a.e. absolutely continuous, positive distribution, symmetric at zero. The $\{X_t\}$ is geometrically α -mixing (An and Huang 1996: Theorem 3.1).

In order to robustify against heavy tails Hill and Renault (2010) propose the Generalized Method of Tail-Trimmed Moments [GMTTM] estimator. Define one estimating equation

$$m_t(\theta) = (X_t - \theta X_{t-1}) X_{t-1}.$$

The parameter of interest θ is identified by $E[m_t(\theta)] = 0$ if and only if $\theta = \theta^0$. Denote by $m_{(j)}^{(a)}(\theta)$ the j^{th} -order statistic of $m_t^{(a)}(\theta) := |m_t(\theta)|$, $m_{(1)}^{(a)}(\theta) \geq m_{(2)}^{(a)}(\theta) \geq \dots$, and assume there exists a sequence of deterministic functions $\{c_n(\theta)\}$ that satisfies, uniformly on compact $\Theta \subset (-1, 1)$, $c_n(\theta) \rightarrow \infty$ and $(n/k_n)P(|m_t(\theta)| > c_n(\theta)) \rightarrow 1$, where $k_n \in \mathbb{N}$, $1 \leq k_n < n$, $k_n \rightarrow \infty$, $k_n/n \rightarrow 0$.

Now construct deterministically and stochastically trimmed equations $m_{n,t}(\theta) = m_t(\theta) \times I(|m_t(\theta)| \leq c_n(\theta))$ and $\hat{m}_{n,t}(\theta) = m_t(\theta) \times I(|m_t(\theta)| \leq m_{(k_n+1)}^{(a)}(\theta))$. The criterion is

$$\hat{Q}_n(\theta) = \left(\frac{1}{n} \sum_{t=1}^n \hat{m}_{n,t}(\theta) \right)^2$$

and the GMTTME is $\hat{\theta}_n = \arg \min_{\theta \in \Theta} \{\hat{Q}_n(\theta)\}$. Intuitively $\hat{\theta}_n$ is that θ that renders the average of non-tail equations closest to zero. Since $E[m_t(\theta^0)] = 0$ by integrability, the negligibly trimmed equations satisfy $E[m_{n,t}(\theta^0)] \rightarrow 0$ by Lebesgue's dominated convergence, so $\hat{\theta}_n \xrightarrow{P} \theta^0$ by standard arguments (Hill and Renault 2010: Theorem 2.1).

Define

$$v_n^2 = nE[m_{n,t}^2(\theta^0)].$$

Symmetric trimming for a symmetric DGP ensures $E[m_{n,t}(\theta)] = 0$ if and only if $\theta = \theta^0$ for any threshold sequences $\{c_n(\theta^0)\}$. Further, absolute continuity and the linear data generating process imply $\hat{Q}_n(\theta)$ is almost surely twice differentiable at $\hat{\theta}_n$: $(\partial/\partial\theta)\hat{Q}_n(\theta)|_{\hat{\theta}_n} = 0$ a.s. (Čížek 2008: Lemma 1). Indeed, differentiability of $\hat{Q}_n(\theta)$ along with negligibility of trimming and distribution smoothness imply asymptotic linearity

$$n^{1/2} \frac{\hat{J}_n}{(E[m_{n,t}^2(\theta^0)])^{1/2}} \times (\hat{\theta}_n - \theta^0) = \frac{1}{v_n} \sum_{t=1}^n \hat{m}_{n,t}(\theta^0)$$

where $\hat{J}_n := 1/n \sum_{t=1}^n X_{t-1}^2 I(|m_t(\theta^0)| \leq m_{(k_n+1)}^{(a)}(\theta^0))$. Under the stated assumptions $\hat{J}_n = E[X_{t-1}^2 I(|m_t(\theta^0)| \leq c_n(\theta^0))] \times (1 + o_p(1))$, and $E[X_{t-1}^2 I(|m_t(\theta^0)| \leq c_n(\theta^0))]/(E[m_{n,t}^2(\theta^0)])^{1/2} \rightarrow \infty$ if $\kappa < 2$. In the latter infinite variance case super- \sqrt{n} -consistency is achieved. See Hill and Renault (2010).

Clearly if $\{\hat{m}_{n,t}(\theta^0)\}$ satisfies a central limit theorem $1/v_n \sum_{t=1}^n \hat{m}_{n,t}(\theta^0) \xrightarrow{d} N(0, 1)$ then the GMTTME $\hat{\theta}_n$ is asymptotically normal. Theorem 5.2 contains the required limit theory. As long as $\{m_t(\theta^0)\}$ is geometrically L_0 -APP on a geometrically α -mixing base, $\liminf_{n \rightarrow \infty} \{v_n^2/n\} > 0$, and sufficiently many equations $m_t(\theta)$

are trimmed $c_n(\theta^0) = o(n^{1/2})$ then $1/v_n \sum_{t=1}^n \hat{m}_{n,t}(\theta^0) \xrightarrow{d} N(0, 1)$.

EXAMPLE 13 (L₀-Approximable Equations): The product convolution $m_t(\theta^0) = \epsilon_t X_{t-1}$ of independent random variables, one iid and one geometrically α -mixing, is also geometrically α -mixing. Simply define the base as $\epsilon_t X_{t-1}$ and define the base σ -field $F_t = \sigma(\epsilon_\tau X_{\tau-1} : \tau \leq t)$. The sequence $\{m_t(\theta^0)\}$ is trivially L₀-APP on $\{F_t\}$ with arbitrary size on a geometrically α -mixing base.

EXAMPLE 14 (Tail-Trimmed QML): Since QML can be couched in GMM, we may similarly couch a robust version of QML in GMTTM simply by re-defining the estimating equations. See Hill and Renault (2010) for details and comparisons with existing trimmed M-estimators, and examples concerning GARCH and AR-GARCH estimation.

6. MEASURES OF TAIL and NON-TAIL MEMORY We complete this paper with an expanded discussion on extant tail memory properties, and a comparison with the concepts and usages considered here. Two final unifying examples concerning SV and GARCH are given in Sections 6.2 and 6.3, with details summarized in Tables 1 and 2.

6.1 Tail Memory

Extremal Index Leadbetter (1974, 1983) shows the maximum of many weakly dependence processes $\{X_t\}$ satisfies

$$\lim_{n \rightarrow \infty} P \left(\frac{1}{u_n} \max_{1 \leq t \leq n} |X_t| \leq z \right) = e^{-\theta z^{-\kappa}}, \quad z \geq 0, \theta \in [0, 1]$$

if and only if $nP(|X_t| > u_n) \rightarrow 1$. Cf. Loynes (1965), O'Brien (1974). Recall $1/\theta$ approximates the mean number of high threshold exceedances, $\theta = 1$ implies independence, $\theta \in (0, 1)$ short-range dependence, and $\theta = 0$ long-range dependence. A stationary AR(1) $X_t = \phi X_{t-1} + \epsilon_t$ with $\phi \in (0, 1)$ and iid Cauchy ϵ_t satisfies $X_t \sim (2)$ with tail index $\kappa = 1$ and extremal index $\theta = 1 - \phi$. Greater geometric memory ($\phi \nearrow 1$) aligns with larger extremal clusters ($1/\theta \nearrow \infty$), irrespective of tail thickness κ . The result generalizes to linear distributed lags $X_t = \sum_{i=0}^{\infty} \psi_i \epsilon_{t-i}$ with $\epsilon_t \stackrel{iid}{\sim} (2)$. See Chernick et al (1991).

The maximum of a GARCH sample $\{X_t\}_{t=1}^n$ has this property and exhibits power-law tail decay (2) with index $\kappa > 0$ (de Haan et al 1989, Mikosch and Stărică 2000, and Davis and Mikosch 2009a). Mikosch and Stărică (2000: Theorem 4.1) characterize θ for GARCH(1,1), shown in Table 1 below. The formula reveals a tight moment-memory relationship: no GARCH affects ($\alpha_1 = \beta_1 = 0$) are associated with small average threshold exceedance (mean threshold exceedance $1/\kappa \searrow 0$) and no extremal clustering (mean extremal cluster size $1/\theta = 1$); and large α_1 and β_1 are associated with heavier tails ($1/\kappa \nearrow \infty$) and stronger geometric tail memory or larger extreme cluster size ($1/\theta \nearrow \infty$).

Although θ has been characterized for Markov chains, linear distributed lags, GARCH and SV, alone θ does not contain enough information to support limit the-

ory for tail or tail-trimmed arrays due to insufficient details on memory decay⁵. Asymptotics for estimators of θ require more information, where a mixing condition is a popular route (e.g. Smith and Weissman 1994). See also de Haan et al (1989), Chernick et al (1991), Smith (1992), Hsing (1993), and Davis and Mikosch (2009b).

Tail Mixing Mixing properties have been used to analyze extremes at least since Loynes (1965). Leadbetter (1974, 1983) and Leadbetter et al (1983) extend the mixing concept to probability tails of weakly dependent, stationary sequences. The so-called D-mixing property has been used to deliver limit theory for sample maxima and point processes of stationary sequences (e.g. Leadbetter 1974; de Haan, Resnick, Rootzén and de Vries 1989; Hsing et al 1989; Hsing 1993; Stărică 1999; Leadbetter et al 2001).

Measurable functions of D-mixing random variables are not necessarily D-mixing, and few attempts to characterize stochastic processes as D-mixing exist. See, e.g., Chernick et al (1991) for stationary moving averages and autoregressions.

Hsing (1991: p. 1555) improves the D-mixing construction so that functions of tail mixing random variables are tail mixing, and Hill (2010) generalizes Hsing's (1991) property to F-mixing to cover arbitrary triangular array functions of $\{X_t\}$, including extremes events $I(X_t > b_n)$ and values $X_t I(X_t > b_n)$, but also non-extreme events $I(X_t \leq b_n)$ and values $X_t I(X_t \leq b_n)$. Neither property requires stationarity; both are implied by α -mixing hence GARCH class (1) is F-mixing; and as with other mixing properties they are difficult to verify and have not been verified for hyperbolic memory processes (e.g. FIGARCH, HYGARCH).

Tail Event Correlation and Extremogram The coefficient $r_n(h) := (P_{h,n} - P_{x,n}P_{y,n}) / (P_{x,n}P_{y,n})^{1/2} \sim (n/k_n) \times (P_{h,n} - P_{x,n}P_{y,n})$ defined in Section 5.2 reveals a broad range of tail dependence properties including arbitrarily small deviations from tail independence, accommodates tail memory decay, and is easy to estimate. The extremogram $r(h) = \lim_{n \rightarrow \infty} r_n(h)$ characterizes large or "distant" forms of tail dependence, and has been characterized for ARMA and SV processes (Hill 2008, 2011, Davis and Mikosch 2009c), and bounded for GARCH (Davis and Mikosch 2009c). Hill (2008, 2011) shows a variety of bivariate tails, including those of SV, exhibit small or "local" tail dependence $\tilde{r}(h) := \lim_{n \rightarrow \infty} (n/k_n)r_n(h) = \lim_{n \rightarrow \infty} \{P_{h,n} / (P_{x,n}P_{y,n}) - 1\} \rightarrow 0$ as $h \rightarrow \infty$. See Section 6.2.

Like extant tail dependence estimators we must say something more to deliver limit theory for the non-parametric estimator $\hat{r}_n(h) = 1/k_n \sum_{t=1}^n (\hat{I}_{x,n,t-h} \hat{I}_{y,n,t} - (k_n/n)^2)$. Either E-NED or E-APP without tail restrictions suffice by Section 5.2 (Hill 2008, 2009a), or more restrictive α -mixing with multivariate regular variation (Davis and Mikosch 2009c).

⁵It is important to distinguish between domains of attraction (Ibragimov and Linnik 1971, Leadbetter et al 1983, Resnick 1987). The index θ itself is a limiting distribution characteristic for weakly dependent, stationary sequences $\{X_t\}$ in the *maximum* domain of attraction: $\lim_{n \rightarrow \infty} P(u_n^{-1} \max_{1 \leq t \leq n} |X_t| \leq z) = e^{-\theta z^{-\kappa}}$ (Leadbetter 1983). But knowing θ does not provide sufficient information to conclude, e.g., the tail array $\{I(X_t > b_{k_n} e^u)\}$ or non-tail array $\{X_t I(X_t \leq b_{k_n} e^u)\}$ belong to the domain of attraction of a normal law. Further, the long-range dependence case $\theta = 0$ clearly provides insufficient knowledge since it says nothing about long memory decay.

Hsing (1993: Theorem 2.1) estimates the extremal index θ and imposes two short-range dependence properties for proving consistency of his estimator $\hat{\theta}$, including an intermediate order generalization of Leadbetter's (1983) extreme order D' -mixing property, $\sum_{h=1}^{\infty} |r(h)| < \infty$. Hsing imposes *finite* dependence to prove asymptotic normality of $\hat{\theta}$ due to substantial technical challenges (Hsing 1993: p. 2049-2050).

Bivariate Tail Index Ledford and Tawn (1997, 2003) propose a power-law bivariate tail decay model for tail dependence to improve upon methods dating at least to Loynes (1965). In a simple setting there exists $\eta_h \in [0, 1]$ for a joint process $\{X_t, Y_t\}$ with unit Fréchet marginals such that

$$P(X_t > z, Y_{t-h} > z) = z^{-\eta_h} L_h(z) \quad \text{as } z \rightarrow \infty,$$

for some slowly varying L_h . All values $\eta_h < 1$ imply $P(X_t > z | Y_{t-h} > z) \rightarrow 0$ as $z \rightarrow \infty$ hence "asymptotic independence" (cf. Loynes 1965), and if $\eta_h = 1$ then $P(X_t > z | Y_{t-h} > z) \rightarrow 0$ hence "asymptotic dependence". The above model is argued to be useful for detailing pre-asymptotic dependence since $\eta_h < 1/2$, $\eta_h = 1/2$ and $\eta_h \in (1/2, 1)$ respectively imply large values are negatively associated, independent, and positively associated (Ledford and Tawn 1997). It is, however, easy to prove values of η_h need not logically coordinate with tail dependence. For example $\eta_h = 1/2$ can align with tail dependence even though it means doubly "asymptotically independent" and "independence of large values" (Ledford and Tawn 1997), and negative tail dependence can align with $\eta_h > 1/2$. See Hill (2008, 2011) for examples, and see Section 6.2, below. Further, bivariate tail dependence decay as $h \rightarrow \infty$ for time series data is rarely modeled by bivariate power-law decay (Ledford and Tawn 2003, Ramos and Ledford 2009) or theory is ignored (Ledford and Tawn 2003), or decay is ignored (Stărică (1999)).

Apparently a characterization of η_h and its decay do not exist for parametric classes of processes like ARFIMA and GARCH. But, tautologically, if Ledford and Tawn's (1997, 2003) model is valid for GARCH then $\eta_h = 1$ with $\lim_{s \rightarrow \infty} L_h(z)$ decaying at a geometric rate in h . This aligns with "asymptotic dependence" a la Loynes (1965) and Ledford and Tawn (1997). See Section 6.3.

Further, knowledge of η_h has never been shown to suffice for tail arrays of dependent heterogeneous $\{X_t, Y_t\}$ to belong to any domain of attraction. Ledford and Tawn (2003: p. 534) estimate η_h for stationary sequences but do not characterize a limit distribution, hence a nonparametric block bootstrap is applied for inference. Although they never state memory assumptions nor prove the bootstrap works in the above environment, at least strong mixing is required given the literature cited.

Tail Copula Copula functions have risen dramatically as a way to characterize dependence in stationary sequences (Joe 1997). Let $F_x(x)$ and $F_y(y)$ denote the marginal distribution functions of $\{X_t, Y_t\}$, and write the survival probability $\bar{F}_z(z) := 1 - F_z(z)$ with generalized inverse $\bar{F}_z^{-1}(u) := \inf\{|z| \in \mathbb{R} | \bar{F}_z(z) \geq u\}$. Then there exists a unique mapping, the copula, $C : [0, 1]^2 \rightarrow [0, 1]$ satisfying $C[F_x(x), F_y(y)] = P(X_t \leq x, Y_t \leq y)$, cf. Sklar (1959). The survival copula is $\bar{C}[u, v] = P(X_t \geq \bar{F}_x^{-1}(u), Y_t \geq \bar{F}_y^{-1}(v))$, the joint probability X_t and Y_t exceed their u^{th} and v^{th} marginal quantiles.

In its simplest form the right tail copula $\Lambda_{x,y} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is defined by

$$\Lambda_{x,y} := \lim_{u \rightarrow 0} \frac{1}{u} \bar{C}[u, u].$$

Since independent random variables satisfy $\bar{C}[u, u]/u = u^2/u = u \rightarrow 0$ as $u \rightarrow 0$, $\Lambda_{x,y}$ near zero implies smaller degrees of tail dependence and $\Lambda_{x,y} = 0$ is interpreted as tail-independence.

A generalized version of $\Lambda_{x,y}$ permits displacement and arbitrary thresholds. Let $\{b_{x,n}, b_{y,n}\}$ satisfy (6) respectively for $\{X_t, Y_t\}$ with common fractile sequence $\{k_n\}$. Then the right-tailed tail copula $\Lambda_{x,y,h} : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ over displacement h is (e.g. Schmidt and Stadtmüller 2006, Klüppelberg et al 2008)

$$\Lambda_{x,y,h} := \lim_{n \rightarrow \infty} \frac{n}{k_n} \times P(X_{t-h} > b_{x,n}, Y_t > b_{y,n})$$

and since $(n/k_n)P(X_{t-h} > b_{x,n})P(Y_t > b_{y,n}) \rightarrow 0$ trivially

$$\Lambda_{x,y,h} = \lim_{n \rightarrow \infty} r_n(h) = r(h).$$

Tail independence is assumed to be captured by $\Lambda_{x,y,h} = 0$, displacement is ignored in this literature ($h = 0$), estimators are only offered for iid marginals X_t and Y_t , and only stationary joint distributions are considered. See Hill (2008, 2009a).

Since $\Lambda_{x,y,h} = r(h)$ the tail copula identically depicts Davis and Mikosch's (2009c) extremogram, and Hsing's (1993) short range dependence property reduces to tail copula summability $\sum_{h=1}^{\infty} |r(h)| = \sum_{h=1}^{\infty} |\Lambda_{x,y,h}| < \infty$. Otherwise explicit usage of $\Lambda_{x,y,h}$ appears to be restricted to applied contexts.

The tail copula $\Lambda_{x,y,h}$ is nearly universally used to measure contemporaneous tail dependence $h = 0$ and is easily estimable non-parametrically (e.g. Schmidt and Stadtmüller 2006, Hill 2008). In a rare instance the decay of $\Lambda_{x,y,h}$ is used to prove consistency of a short-range tail dependence estimator for stationary sequences (Hsing 1993). E-NED and E-APP, however, are abstractions used to characterize minimal and verifiable tail dependence and heterogeneity properties required to support limit theory for a wide array of tail and non-tail estimators like order statistics, tail indices, tail dependence and tail-trimmed objects.

The differences in usage can be explained by a direct comparison. Assume E-NED size is 1/2 with displacement $l_n \rightarrow \infty$ as $n \rightarrow \infty$ and argument $u = 0$, and use Remark 1 of the definition of E-NED and iterated expectations to deduce

$$\text{E-NED} : \lim_{n \rightarrow \infty} l_n \sup_{1 \leq t \leq n} \frac{n}{k_n} \left\{ P(X_t > b_n) - E \left[P \left(X_t > b_n | F_{t-l_n}^{t+l_n} \right)^2 \right] \right\} = 0$$

$$\text{Tail Copula} : \Lambda_{x,x,h} = \lim_{n \rightarrow \infty} \frac{n}{k_n} \left\{ P(X_{t-h} > b_n, X_t > b_n) - P(X_{t-h} > b_n)^2 \right\}.$$

E-NED is a prediction error statement of *conditional* tail memory decay: as information amasses $l_n \rightarrow \infty$ the minimum mean-squared-error predictor $P(X_t > b_n e^u | F_{t-l_n}^{t+l_n})$

convergences to $I(X_t > b_n e^u)$ faster than the rate at which $I(X_t > b_n e^u)$ degenerates in L_2 -norm (i.e. k_n/n). The tail copula, however, measures the asymptotic discrepancy between *joint* and *marginal* tail probabilities at a fixed displacement h .

We now give two examples demonstrating the information content of various measures of tail dependence for two random volatility models.

6.2 Tail Dependence in Log-Autoregressive Stochastic Volatility

Consider a univariate Log-Autoregressive Stochastic Volatility [LASV] model

$$y_t = \sigma_t \epsilon_t \quad \text{where} \quad \ln \sigma_t^\kappa = \theta + \phi \ln \sigma_{t-1}^\kappa + \zeta_t, \quad \text{and} \quad |\phi| < 1,$$

where $\epsilon_t \stackrel{iid}{\sim} (2)$ with index κ and $\zeta_t \stackrel{iid}{\sim} N(0, 1)$, as in Example 11. Let X_t be $|y_t|$, $-y_t I(y_t < 0)$, or $y_t I(y_t > 0)$. It is easy to show $X_t \sim (2)$ with index κ (Hill 2008, 2011, Davis and Mikosch 2009b,c).

Hill (2011) shows the bivariate tail index $\eta_h = 1/2$, which Ledford and Tawn (1997) interpret as independence asymptotically and for large values. Similarly the extremogram $r(h) = 0$ (Davis and Mikosch 2009b,c) and copula $\Lambda_{x,x,h} = 0$ (Hill 2008, 2011) for all $h \geq 1$, and extremal index $\theta = 1$ (Davis and Mikosch 2009b) each reflect a lack of tail dependence.

Yet X_t exhibits local tail dependence at a geometric rate (Hill 2011):

$$\tilde{r}(h) = \lim_{n \rightarrow \infty} \left(\frac{n}{k_n} \right) r_n(h) = \lim_{n \rightarrow \infty} \frac{P(X_{t-h} > b_{x,n}, X_t > b_{x,n})}{P(X_t > b_{x,n}) P(X_{t-h} > b_{x,n})} - 1 = K \phi^h.$$

Further, $\{X_t\}$ is geometrically β -mixing (Carrasco and Chen 2002) and L_2 -E-NED on $\{\epsilon_t, \zeta_t\}$ with coefficients that depend on ϕ^h (Hill 2011). Describing tail dependence by $\tilde{r}(h)$ and E-NED better captures the tail memory characteristics of LASV data. The use of η_h , $r(h)$, $\Lambda_{x,x,h}$ or θ may be misleading, and typically does not provide enough information to push through limit theory for tail or non-tail estimators. That $\theta = 1$, $\eta_h = 1/2$, and $\Lambda_{x,x,h} = r(h) = 0$ need not tell us anything about whether estimators of κ , θ , η_h , $\Lambda_{x,x,h}$ or $r_n(h)$ are consistent or asymptotically normal, while the E-NED property supports asymptotics at least for Hill's (1975) $\hat{\kappa}_n$ and the non-parametric $\hat{r}_n(h)$ (Hill 2008, 2009a, 2010). Since estimators of θ are popularly based on intermediate order statistics as in Hsing (1993) and Smith and Weissman (1994), it is likely they too are asymptotically normal under E-NED, cf. Theorem 5.1. See Table 1 for a summary of known LASV tail dependence properties.

TABLE 1: Tail Dependence for LASV

Property	Coeff.	Value	Interpretation
Extremal Index	θ	1	no extremal clustering
Bivariate Tail Index	η_h	1/2	asymptotic independence
Extremogram	$r(h)$	0	uncorrelated tail events
Tail Dep. Coefficient	$\tilde{r}(h)$	$O(\rho^h)$	locally correlated tail events
Tail Copula	$\Lambda_{x,x,h}$	0	uncorrelated tail events
E-NED	ψ_{l_n}	$O(\rho^{l_n})$	nonlinearly dependent tail events

6.3 Tail Dependence in GARCH(1,1)

Consider a strong-GARCH(1,1):

$$y_t = \sigma_t \epsilon_t, \quad \epsilon_t \stackrel{iid}{\sim} (0, 1), \quad \sigma_t^2 = \omega + \alpha y_{t-1}^2 + \beta \sigma_{t-1}^2, \quad \omega > 0, \quad \alpha, \beta \in [0, 1).$$

Then $\sigma_t^2 = \pi_0 + \sum_{i=1}^{\infty} \alpha \beta^{i-1} y_{t-i}^2$ satisfies $0 \leq \alpha \beta^{i-1} \leq C \rho^i$ where $\rho \in (0, 1)$ and $C \in [0, 1/\rho)$, covering covariance stationary, integrated and explosive cases. Let X_t be $|y_t|$, $-y_t I(y_t < 0)$, or $y_t I(y_t > 0)$. Tail dependence properties are detailed in Table 2.

In summary, κ and θ provide ample tail dependence details but alone do not dictate if tail or non-tail estimators converge in any sense. The extremogram $r(h)$ and tail copula $\Lambda_{x,x,h}$ reflect distant geometric tail dependence hence Hsing's (1993) estimator is consistent $\hat{\theta} \xrightarrow{P} \theta$ under additional regularity conditions, but apparently no other asymptotic theory has been deduced from summability of $r(h)$ or $\Lambda_{x,x,h}$.

The extremogram can be used to deduce a bound on the bivariate tail index, if it exists, a la Ledford and Tawn (1997, 2003). Davis and Mikosch (2009c) show the above GARCH process satisfies for $\rho \in (0, 1)$

$$r(h) = \lim_{n \rightarrow n} \frac{P(x_{t-h} > b_{x,n}, y_t > b_{y,n}) - P(x_t > b_{x,n})P(y_t > b_{y,n})}{[P(x_t > b_{x,n})P(y_t > b_{y,n})]^{1/2}} = O(\rho^h)$$

hence $\lim_{n \rightarrow n} P(x_{t-h} > b_{x,n}, y_t > b_{y,n})/[P(x_t > b_{x,n})P(y_t > b_{y,n})]^{1/2} = O(\rho^h)$. Let $\{x_t^*, y_t^*\}$ be unit Fréchet transforms of $\{x_t, y_t\}$. Assuming Ledford and Tawn's (1997, 2003) model is valid, use the fact that x_t^* is a monotonic transform and $P(x_t^* > z) = 1 - \exp\{-1/z\} = z^{-1} \times (1 + o(1))$ to deduce

$$r(h) = \lim_{n \rightarrow n} \frac{P(x_{t-h}^* > z, x_t^* > z)}{P(x_{t-h}^* > z)P(x_t^* > z)} = \lim_{n \rightarrow n} z^{2-1/\eta_h} L_h(z) \times (1 + o(1))$$

if and only if $\eta_h = 1$ and $\lim_{z \rightarrow \infty} L_h(z) = r(h) = O(\rho^h)$ since $L_h(z)$ is slowly varying (Resnick 1987). Therefore GARCH is asymptotically dependent (Loynes 1965, Ledford and Tawn 1997).

Finally, geometric α -mixing or E-NED ensure Hill's (1975) $\hat{\kappa}_n$ and the nonparametric $\hat{r}_n(h)$ are consistent and asymptotically normal (Hill 2008, 2010, Davis and Mikosch 2009c). Use the identity $\Lambda_{x,x,h} = \lim_{n \rightarrow \infty} r_n(h)$ to deduce a nonparametric copula estimator $\hat{\Lambda}_{x,x,h} = \hat{r}_n(h)$ is also covered (e.g. Schmidt and Stadtmüller 2006, Hill 2008, 2009a).

TABLE 2: Tail Dependence for GARCH(1,1)

Property	Coeff.	Value	Interpretation
Tail Index	κ	$E \left[(\alpha \epsilon_t^2 + \beta)^{\kappa/2} \right] = 1$	$\kappa \uparrow$ implies tail memory \uparrow
Extremal Index	θ	$\theta(\alpha, \beta)^\dagger$	extremal clustering
Bivariate Tail Index	η_h	1	asymptotic dependence
Extremogram	$r(h)$	$O(\rho^h), \rho \in (0, 1)$	correlated tail events
Tail Dep. Coefficient	$\tilde{r}(h)$	∞	globally correlated tail events
Tail Copula	$\Lambda_{x,x,h}$	$O(\rho^h)$	correlated tail events
E-NED	ψ_{l_n}	$O(\rho^{l_n}), l_n \rightarrow \infty$ as $n \rightarrow \infty$.	nonlinearly dependent tail events

$$\dagger \theta(\alpha, \beta) := \frac{1}{E|\epsilon_1|^\kappa} \lim_{l \rightarrow \infty} E \left(|\epsilon_1|^\kappa - \max_{2 \leq j \leq l+1} \left\{ \epsilon_j^2 \prod_{i=2}^j (\alpha \epsilon_i^2 + \beta) \right\}^{\kappa/2} \right)_+.$$

6.3 SUMMARY

Tail memory properties are predominantly aimed at applied researchers: the goal here is to *characterize* tail dependence (e.g. θ , η_h , $\Lambda_{x,y,h}$, $r(h)$ and $\tilde{r}(h)$). Very few properties are consequently applicable for asymptotic theory associated with extreme value estimators since they have not been, or cannot be, shown to reveal enough information for general central and weak limit theory. Further, in general there do not exist non-tail memory notions for either empirical characterizations or asymptotic theory.

By comparison, Near Epoch Dependence and L_0 -Approximability can be straightforwardly applied solely to tails or non-tails. The extensions are not vacuous since very heavy-tailed random volatility data may have NED extremes and non-extremes even if population NED is unknown. These tail and non-tail memory properties cover nonstationary and hyperbolic memory data; they link X_t to some "base" ϵ_t within a prediction premise, so multivariate dependence is allowed; and the notion of "size" permits easy characterizations of tail and non-tail memory decay. Further, since memory decay can be concisely depicted our notions of dependence suffice for extreme value and robust central limit theory.

Extremal-NED and APP provide substantial generality beyond conventional mixing conditions since little is known about mixing in processes with hyperbolic memory, and ensuring mixing in short memory data typically requires restrictions on error distribution smoothness. The results of this paper permit a series of useful limit theory extensions to extremal and non-extremal processes based on parametric classes (e.g. nonlinear ARFIMA and Explosive GARCH), and a new path for delivering robust asymptotic theory with applications to new robust estimators.

APPENDIX 1: PROOFS

We list for future reference several primitive results concerning moment and probability bounds. We only prove non-trivial assertions. In the following $\{U_t, X_t, Y_t, Z_t\}$

are arbitrary random variables on some measure space; $\{A_{n,t}, B_{n,t}, C_{n,t}\}$ are positive, deterministic triangular arrays; and R is an arbitrary subset of \mathbb{R} with positive Lebesgue measure. Define

$$I_{x,t}(A_{n,t}) := I(X_t \leq A_{n,t}).$$

LEMMA A.1 *For any X_t in $L_2(\Omega, F, P)$, any σ -field $\mathfrak{S} \subseteq F$ and all \mathfrak{S} -measurable random variables Y_t ,*

$$E(X_t - E[X_t|\mathfrak{S}])^2 \leq E(X_t - Y_t)^2. \quad (16)$$

LEMMA A.2

i. If U_t is uniformly almost surely bounded then, for any $q > 0$,

$$E|U_t|^q \leq E[|U_t|^q I(Z_t \in R)] + K \times P(Z_t \notin R). \quad (17)$$

ii. If U_t is $L_{r \times q}$ -bounded for $r > 1$ and $q > 0$ then

$$E|U_t|^q \leq E[|U_t|^q I(Z_t \in R)] + \|U_t\|_{r \times q}^q \times P(Z_t \notin R)^{(r-1)/r}. \quad (17')$$

LEMMA A.3 *For any $q > 0$*

$$\begin{aligned} E[|I_{x,t}(A_{n,t}) - I_{y,t}(A_{n,t})|^q \times I(|X_t - Y_t| \leq B_{n,t})] \\ \leq 2P(A_{n,t} - B_{n,t} \leq X_t \leq A_{n,t} + B_{n,t}). \end{aligned} \quad (18)$$

LEMMA A.4 *Let $\{X_t\}$ be L_p -bounded, $p > 0$, and define $\bar{F}_t(x) := P(X_t > x)$. For all arrays $\{A_{n,t}, C_{n,t}\}$ satisfying $\inf_{t \in \mathbb{Z}} \{A_{n,t}\} \rightarrow \infty$ and $\limsup_{n \rightarrow \infty} \sup_{1 \leq t \leq n} \{C_{n,t}\} \in (0, 1)$, there exists a triangular array $\{r_{n,t}\}$ satisfying $\liminf_{n \rightarrow \infty} \inf_{1 \leq t \leq n} \{r_{n,t}\} \geq 0$ and $r_{n,t} = o(1)$ for each $1 \leq t \leq n$, such that*

$$\bar{F}_t(A_{n,t}) = A_{n,t}^{-p} \times r_{n,t} \quad \text{and} \quad \bar{F}_t(A_{n,t}(1 \pm C_{n,t})) = K \times A_{n,t}^{-p} \times C_{n,t} \times r_{n,t}. \quad (19)$$

LEMMA A.5 *For any $\{B_{n,t}\}$ almost surely*

$$I(|I_{x,t}(A_{n,t}) - I_{y,t}(A_{n,t})| > B_{n,t}) \leq |I_{x,t}(A_{n,t}) - I_{y,t}(A_{n,t})|. \quad (20)$$

LEMMA A.6 *If $\limsup_{n \rightarrow \infty} \sup_{1 \leq t \leq n} \{B_{n,t}\} \in (0, 1)$ then*

$$E[|I_{x,t}(A_{n,t}) - I_{y,t}(A_{n,t})|^q I(|I_{x,t}(A_{n,t}) - I_{y,t}(A_{n,t})| \leq B_{n,t})] = 0. \quad (21)$$

Proof of Lemma A.2. Since $E|U_t|^q = E[|U_t|^q I(Z_t \in R)] + E[|U_t|^q I(Z_t \notin R)]$ claim (i) follows from boundedness $E[|U_t|^q I(Z_t \notin R)] \leq K \times E[I(Z_t \notin R)] = K \times P(Z_t \notin R)$, and (ii) from Hölders inequality. ■

Proof of Lemma A.3. Note $|I_{x,t}(A_{n,t}) - I_{y,t}(A_{n,t})|^q \times I(|X_t - Y_t| \leq B_{n,t}) = 1$ if $-B_{n,t} \leq X_t - Y_t \leq B_{n,t}$, and $X_t \leq A_{n,t} < Y_t$ or $Y_t \leq A_{n,t} < X_t$; and $|I_{x,t}(A_{n,t}) - I_{y,t}(A_{n,t})|^q \times I(|X_t - Y_t| \leq B_{n,t}) = 0$ otherwise. Now use $A_{n,t}, B_{n,t} > 0$ to deduce

$$\begin{aligned}
& E [|I_{x,t}(A_{n,t}) - I_{y,t}(A_{n,t})|^q \times I(|X_t - Y_t| \leq B_{n,t})] \\
&= P(-B_{n,t} \leq X_t - Y_t \leq B_{n,t} \cap X_t \leq A_{n,t} \cap A_{n,t} < Y_t) \\
&\quad + P(-B_{n,t} \leq X_t - Y_t \leq B_{n,t} \cap Y_t \leq A_{n,t} \cap A_{n,t} < X_t) \\
&\leq P(A_{n,t} < Y_t \cap Y_t - B_{n,t} \leq X_t \leq A_{n,t}) \\
&\quad + P(A_{n,t} < X_t \leq B_{n,t} + Y_t \cap Y_t \leq A_{n,t}) \\
&\leq P(A_{n,t} - B_{n,t} \leq X_t \leq A_{n,t} + B_{n,t}) \\
&\quad + P(A_{n,t} - B_{n,t} < X_t \leq A_{n,t} + B_{n,t}).
\end{aligned}$$

■

Proof of Lemma A.4. Apply the L_p -boundedness implications (9), $\lim_{x \rightarrow \infty} x^p \bar{F}_t(x) = 0 \forall p < \kappa$, and $\inf_{t \in \mathbb{Z}} \{A_{n,t}\} \rightarrow \infty$ to deduce $\bar{F}_t(A_{n,t}) = A_{n,t}^{-p} \times r_{n,t}$ where $r_{n,t} = o(1)$ for each $1 \leq t \leq n$. Since $C_{n,t} > 0$ and $\limsup_{n \rightarrow \infty} \sup_{1 \leq t \leq n} \{C_{n,t}\} < 1$ clearly $A_{n,t}(1 \pm C_{n,t}) \rightarrow \infty$. Now apply the first claim and the mean-value-theorem to obtain $\bar{F}_t(A_{n,t}(1 \pm C_{n,t})) = A_{n,t}^{-p}(1 \pm C_{n,t})^{-p} r_{n,t} = K A_{n,t}^{-p} C_{n,t} r_{n,t}$. ■

Proof of Lemma A.6. Since uniformly $B_{n,t} \in (0, 1)$ it follows $|I_{x,t}(A_{n,t}) - I_{y,t}(A_{n,t})| \leq B_{n,t}$ only if $I(X_t \leq A_{n,t}) = I(Y_t \leq A_{n,t}) = 0$ or 1, in which cases $|I_{x,t}(A_{n,t}) - I_{y,t}(A_{n,t})| = 0$ a.s. ■

Proof of Theorem 2.1.

Claim (i): Recall $\bar{I}_{n,t}(u) := I(X_t > b_n e^u)$ and define $\eta_n := b_n e^u \vartheta_{l_n}^{1/2}$. For any $q \geq 2$ and some uniformly positive triangular array $\{r_{n,t}\}$

$$\begin{aligned}
& E \left| \bar{I}_{n,t}(u) - E \left[\bar{I}_{n,t}(u) | \mathcal{F}_{t-l_n}^{t+l_n} \right] \right|^q \leq E \left(\bar{I}_{n,t}(u) - E \left[\bar{I}_{n,t}(u) | \mathcal{F}_{t-l_n}^{t+l_n} \right] \right)^2 \\
&\leq 2 \times P(b_n e^u - \eta_n \leq X_t \leq b_n e^u + \eta_n) + K \times P \left(\left| X_t - E \left[X_t | \mathcal{F}_{t-l_n}^{t+l_n} \right] \right| > \eta_n \right) \\
&\leq K r_{n,t} b_n^{-p} e^{-pu} \vartheta_{l_n}^{1/2} + K \times E \left(\left| X_t - E \left[X_t | \mathcal{F}_{t-l_n}^{t+l_n} \right] \right|^p \right) \times \eta_n^{-p} \\
&\leq K r_{n,t} b_n^{-p} e^{-pu} \vartheta_{l_n}^{1/2} + K d_t^p \vartheta_{l_n}^p \times \left\{ e^{-pu} b_n^{-p} \vartheta_{l_n}^{-p/2} \right\} \\
&\leq K (1 + r_{n,t}) b_n^{-p} e^{-pu} \max_{1 \leq t \leq n} \{d_t^p\} \vartheta_{l_n}^{\min\{p,1\}/2}.
\end{aligned}$$

The first inequality follows from $\bar{I}_{n,t}(u) - E[\bar{I}_{n,t}(u) | \mathcal{F}_{t-l_n}^{t+l_n}] \in [-1, 1]$ and $q \geq 2$; the second from (16), (17) and (18); the third from (19) given $\sup_{l \geq 1} \vartheta_l \in [0, 1)$, and Markov's inequality, where $r_{n,t} = o(1)$ for each $1 \leq t \leq n$; and the fourth is L_p -NED. Therefore $\|\bar{I}_{n,t}(u) - E[\bar{I}_{n,t}(u) | \mathcal{F}_{t-l_n}^{t+l_n}]\|_q$ is bounded by

$$\begin{aligned}
& \left\{ \left(\frac{k_n}{n} \right)^{1/q} e^{-up/q} \right\} \times \left[K (1 + r_{n,t}) \left(\frac{n}{k_n} \right)^{1/q} l_n^{-1} b_n^{-p/q} \max_{1 \leq t \leq n} \left\{ d_t^{p/q} \right\} \right] l_n^u \vartheta_{l_n}^{\min\{p,1\}/(2q)} \\
&= d_{n,t}(u) \times \psi_{l_n},
\end{aligned}$$

say, for arbitrarily tiny $\iota > 0$. Clearly $\sup_{1 \leq t \leq n} \{d_{n,t}(u)\} = K(k_n/n)^{1/q} e^{-up/q}$ is Lebesgue integrable on \mathbb{R}_+ and $\sup_{1 \leq t \leq n} \sup_{u \geq 0} \{d_{n,t}(u)\} = O((k_n/n)^{1/q})$. Finally, we can always choose $l_n \rightarrow \infty$ sufficiently fast such that $(K + r_{n,t})(n/k_n)^{1/q} l_n^{-\iota} b_n^{-p/q} \max_{1 \leq t \leq n} \{d_t^{p/q}\} = O(1)$, and for $\iota > 0$ sufficiently small $l_n^\iota \vartheta_{l_n}^{\min\{p,1\}/(2q)} = o(l_n^{-\lambda \min\{p,1\}/(2q)})$ by continuity and $\vartheta_{l_n} = o(l_n^{-\lambda})$.

Claim (ii): The assertion follows from boundedness $|\bar{I}_{n,t}(u) - E[\bar{I}_{n,t}(u)|F_{t-l_n}^{t+l_n}]| \leq 1$ and Lyapunov's inequality. ■

Proof of Lemma 2.3. Let $\{\eta_n\}$ be any positive sequence that satisfies $\eta_n \geq l_n^{\lambda+\iota}$, where $\lambda > 0$ is the L_0 -APP size. For some sequence $\{f_t\}$, $\inf_{t \in \mathbb{Z}} f_t = f > 0$, and constant $\delta > 0$ let $\{f_{n,t}\}$ be any triangular array and $\{\delta_n\}$ any sequence that satisfy

$$f_{n,t} \delta_n = b_n e^u \eta_n l_n^{-\lambda-\iota} f_t \delta \geq b_n e^u f_t \delta \geq e^u f_t \delta \geq f_t \delta > 0.$$

Define $h_t^{(l_n)}(u) := I(g_t^{(l_n)} > b_n e^u)$. It follows for tiny $\iota > 0$

$$\begin{aligned} & P\left(\left|I(X_t > b_n e^u) - h_t^{(l_n)}(u)\right| > f_{n,t} \delta_n\right) \\ & \leq E\left[I\left(\left|I(X_t > b_n e^u) - h_t^{(l_n)}(u)\right| > b_n e^u l_n^{-\lambda-\iota} f \delta\right) \times I\left(\left|X_t - g_t^{(l_n)}\right| \leq b_n e^u l_n^{-\lambda-\iota} f \delta\right)\right] \\ & \quad + \left\|I\left(\left|I(X_t > b_n e^u) - h_t^{(l_n)}(u)\right| > e^u f \delta\right)\right\|_{1/\iota} \times P\left(\left|X_t - g_t^{(l_n)}\right| > f_t \delta\right)^{1-\iota} \\ & \leq E\left[I\left(\left|I(X_t > b_n e^u) - h_t^{(l_n)}(u)\right| > b_n e^u l_n^{-\lambda-\iota} f \delta\right) \times I\left(\left|X_t - g_t^{(l_n)}\right| \leq b_n e^u l_n^{-\lambda-\iota} f \delta\right)\right] \\ & \quad + K e^{-\iota u} \times P\left(\left|X_t - g_t^{(l_n)}\right| > f_t \delta\right)^{1-\iota} \\ & \leq K \times P\left(b_n e^u - b_n e^u l_n^{-\lambda-\iota} f \delta < X_t < b_n e^u + b_n e^u l_n^{-\lambda-\iota} f \delta\right) + K e^{-\iota u} \times v_{l_n}^{1-\iota} \\ & \leq K b_n^{-p} e^{-pu} l_n^{-\lambda-\iota} + K e^{-\iota u} v_{l_n}^{1-\iota} \\ & = K \left\{\frac{k_n}{n} e^{-\iota u} \times (b_n^{-p} + 1) \times \frac{n}{k_n} l_n^{-\iota}\right\} \times \left\{l_n^\iota \times o\left(l_n^{-\lambda}\right)\right\} \\ & \leq K \times e_{n,t}(u) \times \varphi_{l_n}. \end{aligned}$$

The first inequality follows from $f_{n,t} \delta_n \geq b_n e^u l_n^{-\lambda-\iota} f \delta$ and $f_{n,t} \delta_n \geq e^u f_t \delta \geq e^u f \delta$, and property (17'). The second exploits $\{f, \delta\} > 0$ and Markov's inequality. The third follows from (20), then (18), and L_0 -APP. The fourth follows from (19). Simply choose $l_n \rightarrow \infty$ sufficiently fast to ensure $e_{n,t}(u) = (k_n/n) e^{-\iota u} \in [0, 1]$ and $\iota > 0$ sufficiently small such that $\varphi_{l_n} = o(l_n^{-\lambda})$ by a continuity argument. ■

Proof of Lemma 2.4. Define $\bar{I}_{n,t}(u) := I(X_t > b_n e^u)$.

Claim 1: Let $h_t^{(l_n)}(u) := I(g_t^{(l_n)} > b_n e^u)$ for some $F_{t-l_n}^{t+l_n}$ -measurable random variable $g_t^{(l_n)}$. Use (16), (17), (21) and the L_0 -E-APP property to deduce for any η

$\in (0, 1)$

$$\begin{aligned} E \left(\bar{I}_{n,t}(u) - E \left[\bar{I}_{n,t}(u) | \mathcal{F}_{t-l_n}^{t+l_n} \right] \right)^2 &\leq E \left(\bar{I}_{n,t}(u) - h_t^{(l_n)}(u) \right)^2 I \left(\left| \bar{I}_{n,t}(u) - h_t^{(l_n)}(u) \right| \leq \eta \right) \\ &\quad + K \times P \left(\left| \bar{I}_{n,t}(u) - h_t^{(l_n)}(u) \right| > \eta \right) \\ &\leq 0 + K \times e_{n,t}(u) \times \varphi_{l_n}. \end{aligned}$$

The E-NED claim follows from the stated L_0 -E-APP properties of $e_{n,t}(u)$ and φ_{l_n} .

Claim 2: Let $h_t^{(l_n)}(u) := P(X_t > b_n e^u | \mathcal{F}_{t-l_n}^{t+l_n})$ and invoke the L_0 -E-APP property with constants $e_{n,t}(u) = (k_n/n)e^{-u}$ for each $n \geq 1$. For any positive sequences $\{\eta_n, \delta_n\}$, $\eta_n \in (0, 1)$ and $\delta_n < \eta_n$, and $K > 0$

$$\begin{aligned} E \left(\bar{I}_{n,t}(u) - h_t^{(l_n)}(u) \right)^2 I \left(\left| \bar{I}_{n,t}(u) - h_t^{(l_n)}(u) \right| \leq \eta_n \right) &\quad (22) \\ &\leq \delta_n + \int_{\delta_n}^{\eta_n} P \left(\left| \bar{I}_{n,t}(u) - h_t^{(l_n)}(u) \right| > x^{1/2} \right) dx \\ &\leq K \delta_n + K e_{n,t}(u) \varphi_{l_n}. \end{aligned}$$

Similarly, $E[(\bar{I}_{n,t}(u) - h_t^{(l_n)}(u))^2 I(|\bar{I}_{n,t}(u) - h_t^{(l_n)}(u)| > \eta_n)]$ is bounded by

$$\int_{\eta_n}^1 P \left(\left| \bar{I}_{n,t}(u) - h_t^{(l_n)}(u) \right| > x^{1/2} \right) dx \leq K e_{n,t}(u) \varphi_{l_n}. \quad (23)$$

Together (22) with $\delta_n = e_{n,t}(u) \varphi_{l_n}$ and (23) imply L_2 -E-NED. ■

Proof of Lemma 3.2. Under the stated suppositions $\{X_t\}$ is L_0 -APP with coefficients $v_{l_n} = o(l_n^{-\lambda})$ and approximator $\{g_t^{(l_n)}\}$, and L_0 -E-APP with coefficients $\varphi_{l_n} = o(l_n^{-\lambda})$ and approximator $\{I(g_t^{(l_n)} > b_n e^u)\}$ by Lemma 2.3. By construction this implies $I_{n,t}(u)$ is L_0 -APP with coefficients φ_{l_n} and approximator $\{I(g_t^{(l_n)} \leq b_n e^u)\}$.

We need only demonstrate the conditions of Theorem 17.22 of Davidson (1994) are satisfied to prove $X_t^s I_{n,t}(u)$ and $X_t^s \bar{I}_{n,t}(u)$ are also L_0 -APP with coefficients $v_{l_n} + \varphi_{l_n}$ of size $\min\{\lambda, \lambda\} = \lambda$ and approximators $\{(g_t^{(l_n)})^s I(g_t^{(l_n)} \leq b_n e^u)\}$ and $\{(g_t^{(l_n)})^s I(g_t^{(l_n)} > b_n e^u)\}$, respectively.

Fix $s = 1$ and consider $X_t I_{n,t}(u)$, the proofs for arbitrary $s > 0$ and $X_t^s \bar{I}_{n,t}(u)$ are identical. Define vectors $a = [a_1, a_2]' \in \mathbb{R}^2$ and $b = [b_1, b_2]' \in \mathbb{R}^2$, and define a mapping $B : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ by $B(a, b) = |a_1| + |b_2|$. Thus $B(a, b)$ sums the absolute values of the first element of a and the second element of b . Let $\{X_t^{(1)} I_{n,t}^{(1)}(u), X_t^{(2)} I_{n,t}^{(2)}(u)\}$ be two copies of $X_t I_{n,t}(u)$. Then

$$\begin{aligned} &\left| X_t^{(1)} I_{n,t}^{(1)}(u) - X_t^{(2)} I_{n,t}^{(2)}(u) \right| \\ &\leq \left\{ \left| X_t^{(1)} \right| + \left| I_{n,t}^{(2)}(u) \right| \right\} \times \left\{ \left| X_t^{(1)} - X_t^{(2)} \right| + \left| I_{n,t}^{(1)}(u) - I_{n,t}^{(2)}(u) \right| \right\} \\ &= B \left(\left\{ X_t^{(1)}, I_{n,t}^{(1)}(u) \right\}, \left\{ X_t^{(2)}, I_{n,t}^{(2)}(u) \right\} \right) \times \left\{ \left| X_t^{(1)} - X_t^{(2)} \right| + \left| I_{n,t}^{(1)}(u) - I_{n,t}^{(2)}(u) \right| \right\}. \end{aligned}$$

But since X_t and $I_{n,t}(u)$ are L_p -bounded, clearly

$$\begin{aligned} \left\| B \left(\{X_t, I_{n,t}(u)\}, \left\{ g_t^{(l_n)}, I \left(g_t^{(l_n)} \leq b_n e^u \right) \right\} \right) \right\|_p &= \left\| |X_t| + \left| I \left(g_t^{(l_n)} \leq b_n e^u \right) \right| \right\|_p \\ &\leq \|X_t\|_p + 1 \leq K \text{ if } p \geq 1 \\ &\leq K (E |X_t|^p + 1)^{1/p} \leq K \text{ if } p \in (0, 1), \end{aligned}$$

by Minkowski's and Loève's inequalities, respectively. This verifies the conditions of Davidson's (1994) Theorem 17.22. ■

Proof of Theorem 3.3. We prove the claim for $b_n^{-1} X_t I(|X_t| \leq b_n)$ since the argument for $b_n^{-s} X_t^s I(|X_t| \leq b_n)$ and arbitrary $s > 0$ is identical.

Write $\hat{Z}_{n,t}^* := b_n^{-1} \hat{X}_{n,t} = b_n^{-1} X_t I(|X_t| \leq b_n)$ and $\hat{g}_{n,t}^{(l_n)} := g_t^{(l_n)} I(g_t^{(l_n)} \leq b_n)$. Since $|\hat{Z}_{n,t}^*| \leq 1$ a.s. uniformly in n and t , clearly

$$\limsup_{n \rightarrow \infty} \sup_{1 \leq t \leq n} \left\{ \left| \hat{Z}_{n,t}^* - E \left[\hat{Z}_{n,t}^* | F_{t-l_n}^{t+l_n} \right] \right| \right\} \leq K < \infty.$$

Further, $\{\hat{X}_{n,t}\}$ is L_0 -APP on $\{F_t\}$ with coefficients $w_{l_n} \in [0, 1)$ of size λ and approximator $\{\hat{g}_{n,t}^{(l_n)}\}$ by Lemma 3.2. Now use (16), boundedness $b_n^{-1} |\hat{X}_{n,t} - \hat{g}_{n,t}^{(l_n)}| \leq 1$ a.s. with (17), and the L_0 -APP property to deduce for any sequence of uniformly positive numbers $\{\eta_n\}$ and any $p \geq 2$

(24)

$$\begin{aligned} E \left| \hat{Z}_{n,t}^* - E \left[\hat{Z}_{n,t}^* | F_{t-l_n}^{t+l_n} \right] \right|^p &\leq E \left[\left(\hat{Z}_{n,t}^* - E \left[\hat{Z}_{n,t}^* | F_{t-l_n}^{t+l_n} \right] \right)^2 \right] \\ &\leq b_n^{-1} E \left[\left(\hat{X}_{n,t} - \hat{g}_{n,t}^{(l_n)} \right)^2 \times I \left(\left| \hat{X}_{n,t} - \hat{g}_{n,t}^{(l_n)} \right| \leq \eta_n \right) \right] \\ &\quad + K \times P \left(\left| \hat{X}_{n,t} - \hat{g}_{n,t}^{(l_n)} \right| > \eta_n \right) \\ &\leq b_n^{-1} E \left[\left(\hat{X}_{n,t} - \hat{g}_{n,t}^{(l_n)} \right)^2 \times I \left(\left| \hat{X}_{n,t} - \hat{g}_{n,t}^{(l_n)} \right| \leq \eta_n \right) \right] \\ &\quad + K \times w_{l_n}. \end{aligned}$$

Similar to (22) use L_0 -APP to deduce for any sequence $\{\delta_n\}$, $0 < \delta_n < \eta_n$,

$$\begin{aligned} E \left[\left(\hat{X}_{n,t} - \hat{g}_{n,t}^{(l_n)} \right)^2 \times I \left(\left| \hat{X}_{n,t} - \hat{g}_{n,t}^{(l_n)} \right| \leq \eta_n \right) \right] & \quad (25) \\ &= \int_0^{\delta_n} P \left(\left| \hat{X}_{n,t} - \hat{g}_{n,t}^{(l_n)} \right| > v^{1/2} \right) dv + \int_{\delta_n}^{\eta_n} P \left(\left| \hat{X}_{n,t} - \hat{g}_{n,t}^{(l_n)} \right| > v^{1/2} \right) dv \\ &\leq \delta_n + (\eta_n - \delta_n) \times w_{l_n}. \end{aligned}$$

Together, (24) and (25) imply

$$E \left| \hat{Z}_{n,t}^* - E \left[\hat{Z}_{n,t}^* | F_{t-l_n}^{t+l_n} \right] \right|^p \leq b_n^{-1} [\delta_n + (\eta_n - \delta_n) \times w_{l_n}] + w_{l_n}.$$

Choose $\delta_n = w_{l_n} \in [0, 1]$ and $\eta_n \in (1, 2]$ to get

$$E \left| \hat{Z}_{n,t}^* - E \left[\hat{Z}_{n,t}^* | \mathcal{F}_{t-l_n}^{t+l_n} \right] \right|^p \leq K (1 + b_n) w_{l_n} \leq K w_{l_n}.$$

hence $\|\hat{Z}_{n,t}^* - E[\hat{Z}_{n,t}^* | \mathcal{F}_{t-l_n}^{t+l_n}]\|_p \leq K w_{l_n}^{1/p} = o(l_n^{-\lambda/p})$.

Now apply Lyapunov's inequality to deduce for any $p \in (0, 2)$, $\|\hat{Z}_{n,t}^* - E[\hat{Z}_{n,t}^* | \mathcal{F}_{t-l_n}^{t+l_n}]\|_p \leq \|\hat{Z}_{n,t}^* - E[\hat{Z}_{n,t}^* | \mathcal{F}_{t-l_n}^{t+l_n}]\|_2 \leq o(l_n^{-\lambda/2})$. This completes the proof. ■

Proof of Theorem 3.4. Apply Theorem 3.3 to deduce for any $s > 0$ and $p > 0$, $\|\hat{X}_{n,t}^s - E[\hat{X}_{n,t}^s | \mathcal{F}_{t-l_n}^{t+l_n}]\|_p \leq b_n^s l_n^{-\iota} o(l_n^{-\lambda/\max\{p,2\}})$. Choose $\iota > 0$ sufficiently tiny and $l_n \rightarrow \infty$ sufficiently fast to complete the proof. ■

Proof of Lemma 4.5. If $p \geq 1$ the proof follows from Minkowski's inequality (e.g. Davidson 1994: p. 263). If $p \in (0, 1)$ apply Loève's inequality, $\sup_{t \in \mathbb{N}} \|u_t\|_p$, and $|\psi_{t,i}| \leq |\psi_i|$ to deduce $E|X_t - E[X_t | \mathcal{G}_{t-l}^{t+l}]|^p \leq K (\sum_{i=l+1}^{\infty} |\psi_i|)^{1/p}$. Exploit $\psi_i = O(\rho^i)$ or $\psi_i = O(i^{-\mu})$ to finish the proof. ■

Proof of Theorem 5.1. Under the stated conditions and Theorem 2.5 $\{X_t\}$ is L_2 -E-NED with size $1/2$, constants $d_{n,t}(u) = K(k_n/n)^{1/2} e^{-u/2}$, and α -mixing base with size 1. Coupled with the tail property $P(X_t > x) = c x^{-\kappa} (1 + O(x^{-\zeta}))$ and fractile bound $k_n = O(n^{2\zeta/(2\zeta+\kappa)})$ all conditions of Lemma 3 and Theorem 2 of Hill (2010) are satisfied delivering both limits. ■

Proof of Theorem 5.2. By supposition $\{X_t\}$ is L_p -bounded, $p > 0$ and $E[\hat{X}_{n,t}] = 0$ for each $n \geq 1$ and $1 \leq t \leq n$. We need only verify Assumptions A-C of Hill (2009b) to invoke his Theorem 3.1 central limit theorem: $\sum_{t=1}^n (\hat{X}_{n,t}^* - E[\hat{X}_{n,t}^*]) / v_n \xrightarrow{d} N(0, 1)$. The assumptions are listed here for reference using our notation.

(A) $\liminf_{n \rightarrow \infty} \{v_n^2/n\} > 0$; $b_n = O(n^{1/2-\iota})$; and $k_n/n^\iota \rightarrow \infty$ for tiny $\iota > 0$.
(B) $\{\hat{X}_{n,t}\}$ is geometrically L_2 -NED on geometrically α -mixing $\{\epsilon_t\}$. (C) $\{I(|X_t| > c_n e^u)\}$ is geometrically L_2 -NED on geometrically α -mixing $\{\epsilon_t\}$ with constants $e_{n,t}(u)$ Lebesgue integrable on \mathbb{R}_+ and $\sup_{1 \leq t \leq n} \sup_{u \geq 0} \{e_{n,t}(u)\} \leq K(n/k_n)^{1/2}$, and coefficients with size $1/2$.

Assumption A holds by supposition. Since α -mixing implies NED and NED implies L_0 -APP, Assumptions B and C hold by Theorems 3.3 and 2.5 respectively. ■

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