Robust Estimation and Inference for Extremal Dependence in Time Series

Appendix C: Omitted Proofs and Supporting Lemmata

Jonathan B. Hill*
Dept. of Economics
University of North Carolina - Chapel Hill

January 24, 2009

This appendix contains the omitted proof of Lemma A.3 (Section C.1), and supporting Lemmas B1-B4 (Section C.2).

Recall $x_t$ and $y_t$ take values on $[0, \infty)$ with probability one, $z_t$ denotes either $x_t$ or $y_t$ and

$$P(z_t > \epsilon) = \epsilon^{-\alpha} L_{z}(\epsilon) \text{ as } \epsilon \to \infty, \text{ slowly varying } L_{z}, \quad (C.1)$$

and

$$\frac{n}{m_n} P(z_t > b_{z,n}) \to 1. \quad (C.2)$$

C.1 LEMMA A.3 We present an expanded version of Lemma A.3. The proofs of Lemmas A.2 and A.6 use Lemma A.3.iii,iv which is proved by Lemma A.3.i,ii.

Recall

$$\tilde{I}_{z,n,t}(u) := I(z_t > b_{z,n} \epsilon^u) - P(z_t > b_{z,n} \epsilon^u), \ u \in \mathbb{R}_+.$$

LEMMA A.3 Let Assumption A hold.

i. $\{I_{z,n,t}(u)/m_n^{1/2}\}$ and $\{T_{z,n,t}/m_n^{1/2}\}$ are $L_4$-NED on $\{F_t\}$ with constants

$$\left\{ \frac{d_{z,n,t}(u)}{m_n}, \int d_{z,n,t}(u) du \right\} = O\left( m_n^{-1/2} (m_n/n)^{1/r} \right) \text{ uniformly in } t \geq 1$$

for arbitrary $r \geq 2$, and common coefficients $\psi_{z,n,t} = o((m_n/n)^{1/2-1/r} m_n^{-1/2})$.

ii. For any $i \geq 1$, $\{I_{x,n,t-i}(u)/m_n^{1/2}\}$ and $\{T_{y,n,t-i}/m_n^{1/2}\}$ are $L_2$-NED on $\{F_t\}$ with constants

$$\left\{ D_{n,t}^{(i)}, \tilde{D}_{n,t}^{(i)} \right\} = O\left( m_n^{-1/2} (m_n/n)^{1/r} \right) \text{ uniformly in } t \geq 1$$

*Dept. of Economics, University of North Carolina-Chapel Hill, www.unc.edu/~jbhill, jbhill@email.unc.edu.
and common coefficients \( \Psi_{t_n} = o((m_n/n)^{1/2-1/2\ell_n^{-1/2}}) \).

iii. For any \( \pi \in \mathbb{R}^h \), \( \pi' = 1 \), \( \{ \sum_{i=1}^h \pi_i \bar{Y}_{x,n,t-i} + (u) \bar{Y}_{y,n,t}(u)/m_n^{1/2} \} \) and \( \{ \sum_{i=1}^h \pi_i T_{x,n,t-i} T_{y,n,t}/m_n^{1/2} \} \) are \( L_2\)-NED on \( \{ F_t \} \) with constants

\[
\left\{ c_{n,t}^{(h)}(u), f_{n,t}^{(h)} \right\} = \left\{ \max_{1 \leq i \leq h} \left\{ D_{n,t}^{(i)}(u) \right\}, \max_{1 \leq i \leq h} \left\{ \tilde{D}_{n,t}^{(i)} \right\} \right\}
\]

\[
= \left( m_n^{-1/2} (m_n/n)^{1/2} \right) \text{ uniformly in } t \geq 1
\]

and common coefficients \( \Psi_{t_n}^{(h)} = o((m_n/n)^{1/2-1/2\ell_n^{-1/2}}) \).

iv. Under the conditions of (iii), \( \{ \sum_{i=1}^h \pi_i \bar{Y}_{x,n,t-i} + (u) \bar{Y}_{y,n,t}(u)/m_n^{1/2}, F_t \} \) and \( \{ \sum_{i=1}^h \pi_i T_{x,n,t-i} T_{y,n,t}/m_n^{1/2}, F_t \} \) form \( L_2 \)-mixingale sequences with size 1/2 and constants \( \{ c_{n,t}, c_{n,t}^{*} \} = O(n^{-1/2}) \) uniformly in \( t \).

Proof.

Claim i. We tackle \( \bar{Y}_{x,n,t}(u) \) and \( T_{x,n,t} \) in two steps.

Step 1 (\( \bar{I}_{z,n,t}(u) \)): By the definition of E-NED and the statement of Assumption A, it follows for some \( d_{z,n,t}(u) = O((m_n/n)^{1/2}) \) and \( \psi_{z,l_n} = o((m_n/n)^{1/2-1/2\ell_n^{-1/2}}) \), where \( r \geq 1 \),

\[
\left\| I_{z,n,t}(u)/m_n^{1/2} - E \left[ I_{z,n,t}(u)/m_n^{1/2} \left| F_{t-l_n}^{t+l_n} \right. \right] \right\|_4 \leq m_n^{-1/2} d_{z,n,t}(u) \times \psi_{z,l_n}
\]

where

\[
d_{z,n,t}(u) = d_{z,n,t}(u)/m_n^{1/2} = O \left( m_n^{-1/2} (m_n/n)^{1/2} \right)
\]

is Lebesgue integrable by Assumption A. Therefore \( \{ I_{z,n,t}(u)/m_n^{1/2} \} \) is \( L_4 \)-NED on \( \{ F_t \} \) with constants \( d_{z,n,t} \) and coefficients \( \psi_{z,l_n} \) with size 1/2.

Step 2 (\( T_{x,n,t} \)): Define

\[
g_{n,t}(u) := I \left( z_t > b_{z,n} \right) - P \left( z_t > b_{z,n} \right) F_{t-l_n}^{t+l_n}
\]
and note
\[
\| T_{z,n,t} - E[T_{z,n,t}|F_{t-l_n}^{t+l_n}] \|_4 \\
= \| (\ln (zt/b_{z,n}))+ - E (\ln (zt/b_{z,n}))+ |F_{t-l_n}^{t+l_n}) \|_4 \\
= \left[ E \left( \int_0^\infty \left[ I (zt > b_{z,n}e^u) - P (zt > b_{z,n}e^u|F_{t-l_n}^{t+l_n}) \right] du \right) \right]^{1/4} \\
= \left[ E \left( \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \prod_{i=1}^4 g_{n,t}(u_i)du_i \right) \right]^{1/4} \\
= \left[ \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \prod_{i=1}^4 g_{n,t}(u_i)du_1du_2du_3du_4 \right]^{1/4} \\
\leq \left( \int_0^\infty d_{z,n,t}(u)du \right) \times \psi_{z,n,t}.
\]

The second equality follows from \((\ln(z_t/b_{z,n}))+ = \int_0^\infty I(z_t > b_{z,n}e^u)du\) and the Fubini-Tonelli Theorem: \(E[(\ln(z_t/b_{z,n}))+|F_{t-l_n}^{t+l_n}] = E[\int_0^\infty I(z_t > b_{z,n}e^u)du|F_{t-l_n}^{t+l_n}] = \int_0^\infty P(z_t > b_{z,n}e^u|F_{t-l_n}^{t+l_n})\). The fourth equality follows from the Fubini-Tonelli Theorem. The first inequality is Cauchy-Schwarz’s. The last inequality follows from Assumption A. This proves \(T_{z,n,t}\) is L_4-NED on \(F_t\) with constants \(\int_0^\infty d_{z,n,t}(u)du = O((m/n)\|\_{1/4})\) uniformly in \(t \geq 1\) under Assumption A.

Finally, the proof that \(T_{z,n,t}/m^{1/2}\) is L_4-NED on \(F_t\) with constants \(\int_0^\infty d_{z,n,t}(u)du = O(m^{-1/2}(m/n)\|\_{1/4})\) uniformly in \(t \geq 1\) and coefficients \(\psi_{z,n,}\) follows similarly.

Claim ii. Use the L_4-NED properties of \(\tilde{I}_{x,n,t}(u)\) and \(\tilde{I}_{y,n,t}(u/m^{1/2})\) by Claim (i) to deduce the convolution \(\tilde{I}_{n,t,i}(u) := \tilde{I}_{x,n,t-i}(u_1)\tilde{I}_{y,n,t}(u_2)/m^{1/2}\) satisfies (see Davidson 1994: Theorem 17.9)

\[
\left\| \tilde{I}_{n,t,i}(u) - E[\tilde{I}_{n,t,i}(u)|F_{t-l_n}^{t+l_n}] \right\|_2 \\
\leq 3\left\| I_{x,n,t-i}(u_1) \|_4 \times d_{y,n,t}(u_2) \times \psi_{y,n} + \left\| I_{y,n,t}(u_2)/m^{1/2} \right\|_4 \times d_{x,n,t-i}(u_1) \times \psi_{x,n} \right\|_2 \\
= 3\left\| I_{x,n,t-i}(u_1) \|_4 \times d_{y,n,t}(u_2) \times \psi_{y,n} + \left\| I_{y,n,t}(u_2)/m^{1/2} \right\|_4 \times d_{x,n,t-i}(u_1) \times \psi_{x,n} \right\|_2.
\]

Now use Lemma B.1 to deduce

\[
\left\| I_{n,t,i}(u) - E[I_{n,t,i}(u)|F_{t-l_n}^{t+l_n}] \right\|_2 \\
\leq K(m_n/n)^{1/4} \times \max \left\{ d_{x,n,t-i}(u_1), d_{y,n,t}(u_2) \right\} \times \max \left\{ \psi_{x,n}, \psi_{y,n} \right\} \\
= D_{n,t}^{(i)}(u) \times \Psi_{n}.
\]
say, where by Claim (i)
\[ D_{n,t}^{(i)}(u) = (m_n/n)^{1/4} \times \max \left\{ d_x, n, t - i(u_1), d_y, n, t(u_2) \right\} \]
\[ = O \left( (m_n/n)^{1/4} m_n^{1/r - 1/2} n^{-1/r} \right) \]
\[ = O \left( m_n^{-1/2} (m_n/n)^{1/r} \right) \text{ uniformly in } t \geq 1 \]
and
\[ \Psi_n = \max \left\{ \psi_{x, n}, \psi_{y, n} \right\} = o \left( (m_n/n)^{1/2 - 1/r} t_n^{-1/2} \right). \]

Therefore \( \{ I_{x, n, t - i}(u_1)I_{y, n, t}(u_2)/m_n^{1/2} \} \) is \( L_2 \)-NED on \( \{ F_t \} \) with constants \( D_{n,t}^{(i)}(u) = O(m_n^{-1/2} (m_n/n)^{1/r}) \) and coefficients \( \Psi_n = o((m_n/n)^{1/2 - 1/r} t_n^{-1/2}). \)

The proof that \( \{ T_{x, n, t - i}T_{y, n, t}/m_n^{1/2} \} \) is \( L_2 \)-NED follows from an argument similar to Step 2 of Claim (i), where the constants are
\[ \tilde{D}_{n,t}^{(i)} = (m_n/n)^{1/4} \times \max \left\{ \int_0^\infty d_x, n, t - i(u)du, \int_0^\infty d_y, n, t(u)du \right\} \]
\[ = O \left( m_n^{-1/2} (m_n/n)^{1/r} \right) \text{ uniformly in } t \geq 1 \]
and the coefficients are also \( \Psi_n \).

**Claim iii.** Use Minkowski’s inequality to deduce
\[ \left\| \sum_{i=1}^h \pi_i I_{x, n, t - i}(u)I_{y, n, t}(u)/m_n^{1/2} - E \left[ \sum_{i=1}^h \pi_i I_{x, n, t - i}(u)I_{y, n, t}(u)/m_n^{1/2} I_{t - t_n} \right] \right\|_2 \]
\[ \leq \sum_{i=1}^h D_{n,t}^{(i)}(u) \times \Psi_n \leq \max_{1 \leq i \leq h} \left\{ D_{n,t}^{(i)}(u) \right\} \times (h \times \Psi_n) \]
\[ = e_{n,t}(u) \times \Psi_n^{(h)}, \]
say, where \( e_{n,t}^{(h)}(u) = O(m_n^{-1/2} (m_n/n)^{1/r}) \) uniformly in \( t \geq 1 \) and \( \Psi_n^{(h)} = o((m_n/n)^{1/2 - 1/r} t_n^{-1/2}) \) under Claim (ii). An identical argument applies to \( \sum_{i=1}^h \pi_i T_{x, n, t - i}T_{y, n, t}/m_n^{1/2} \).

**Claim iv.** The final claim follows from Claim (iii) and arguments in Davidson (1994: p. 264-265). Consider \( \Pi_{n,t}(\pi, u) := \sum_{i=1}^h \pi_i I_{x, n, t - i}(u)I_{y, n, t}(u)/m_n^{1/2} \), the proof for \( \sum_{i=1}^h \pi_i T_{x, n, t - i}T_{y, n, t}/m_n^{1/2} \) being similar.

If the base \( \{ \epsilon_i \} \) is strong mixing with coefficients \( \zeta_n \) then (see Davidson 1994: eq. 17.18)
\[ \left\| E \left[ \Pi_{n,t}(\pi, u) | F_{t - t_n} \right] \right\|_2 \leq \max \left\{ \left\| \Pi_{n,t}(\pi, u) \right\|_r, e_{n,t}(u) \right\} \times \left( \zeta_n^{1/2 - 1/r} + \Psi_n^{(h)} \right). \]

The Minkowski and the Cauchy-Schwarz inequalities, \( \pi = 1 \), and \( \left\| I_{x, n, t}(u) \right\|_r = O((m_n/n)^{1/r}) \) uniformly in \( t \geq 1 \) by Lemma B.1, below, imply for any \( r \geq 2 \)
\[ \left\| \Pi_{n,t}(\pi, u) \right\|_r \leq K \times m_n^{-1/2} \left\| \Pi_{x, n, t} \right\|_{2r} \times m_n \left\| \Pi_{y, n, t} \right\|_{2r} = O \left( m_n^{-1/2} (m_n/n)^{1/r} \right). \]

Since \( e_{n,t}(u) = O(m_n^{-1/2} (m_n/n)^{1/r}) \) uniformly in \( t \geq 1 \) and \( \Psi_n^{(h)} = o((m_n/n)^{1/2 - 1/r} t_n^{-1/2}) \) by Claim (iii), and \( \zeta_n^{1/2 - 1/r} = o((m_n/n)^{1/2 - 1/r} \times t_n^{1/2 - 1/r} \times t_n^{1/2 - 1/r}) = o((m_n/n)^{1/2 - 1/r}) \).
\times I_n^{-1/2} \text{ by Assumption A, it follows}
\begin{align*}
\|E[ \bar{I}_{n,t}(\pi,u) | F_{t-I_n}] \|_2 \\
&\le Km_n^{-1/2} (m_n/n)^{1/r} \left( \zeta_{I_n}^{-1/2-1/r} + \Psi_{I_n}^{(b)} \right) \\
&= Km_n^{-1/2} (m_n/n)^{1/r} \left( o((m_n/n)^{1/2-1/r} I_n^{-1/2}) + o((m_n/n)^{1/2-1/r} I_n^{-1/2}) \right) \\
&= Km_n^{-1/2} (m_n/n)^{1/r-1/2} \left( o((m_n/n)^{1/2-1/r} I_n^{-1/2}) + o((m_n/n)^{1/2-1/r} I_n^{-1/2}) \right) \\
&= Km_n^{-1/2} \times o(I_n^{-1/2}) = c_{0,t} \times o(I_n^{-1/2}),
\end{align*}
say. A similar argument applies to the remaining mixingale inequality \(|I_{n,t}(\pi,u) - E[I_{n,t}(\pi,u) | F_{t+I_n}]| \|_2 \le c_{0,t} \times o(I_n^{-1/2})\), and in the uniform mixing case, cf. Davidson (1994: eqs. (17.19)-(17.20)).

C.2 SUPPORTING LEMMAS B.1-B.4 Throughout \(k_{n,s,t} = k((s-t)/\gamma_n)\).

**Lemma B.1** Under (C.1)-(C.2), \(|T_{z,n,t}| = O((m_n/n)^{1/r})\) and \(|I_{z,n,t}| = O((m_n/n)^{1/r})\) uniformly in \(t \geq 1\) for any \(r \geq 1\).

**Lemma B.2** Under Assumptions A, B and E, for all \(l_1, l_2 \in \mathbb{N}\)
\[
\frac{1}{m_n} \sum_{s,t=1}^{n} |k_{n,s,t}| \times \hat{S}_{s, t, m_n}(l_1, l_2) = o_p(1).
\]

**Lemma B.3** Under Assumptions A, B and E:
\[
i. \frac{1}{m_n} \sum_{s,t=1}^{n} k_{n,s,t} \left( \hat{T}_{x,n,t-l} \hat{T}_{y,n,t} - T_{x,n,t-l} T_{y,n,t} \right) = o_p(n/m_n^{1/2});
\]
\[
ii. \frac{1}{m_n} \sum_{s,t=1}^{n} k_{n,s,t} \left( T_{x,n,t-l} T_{y,n,t} - E[T_{x,n,t-l} T_{y,n,t}] \right) = o_p(n/m_n^{1/2}).
\]

**Lemma B.4** Under Assumptions A, B and E
\[
\frac{1}{n} \sum_{s,t=1}^{n} k_{n,s,t} \times \left( \frac{n}{m_n} \left( \frac{T_{x,n,t-l} T_{y,n,t}}{2\alpha_{x,m_n} \alpha_{y,m_n}} - \hat{q}_{m_n}(l) \right) \right) = o_p \left( n^2/m_n^{3/2} \right).
\]

**Proof of Lemma B.1.** Use Minkowski’s and Liapunov’s inequalities and arguments in Hsing (1991: p. 1548), cf. (C.1)-(C.2), to deduce for all \(r \geq 1\)
\[
\|T_{z,n,t}\|_r \leq 2 \left( E \left[ \left( \ln \left( \frac{z_t}{b_{z,n}} \right) \right)^{r} \right] \right)^{1/r} \sim 2 \left( (m_n/n) \times r! \times \alpha_{z}^{-1} \right)^{1/r} = O((m_n/n)^{1/r}),
\]
which does not depend on \(t\). Similarly, the construction of \(b_{z,m_n}\) implies for any \(r \geq 1\)
\[
\|I_{z,n,t}\|_r \leq P \left( z_t > b_{z,n} \right)^{1/r} + P \left( z_t > b_{z,n} \right) \sim (m_n/n)^{1/r} + (m_n/n) = O((m_n/n)^{1/r}).
\]
Proof of Lemma B.2. Write $\Delta \hat{T}_{z,n,t,m_n} := \hat{T}_{z,n,t} - T_{z,n,t}$ and

\begin{equation}
\delta_{s,t,m_n}(l_1,l_2) = \hat{T}_{z,n,s-l_1} T_{y,n,s} T_{x,n,t-l_2} \hat{T}_{y,n,t} - T_{z,n,s-l_1} T_{y,n,s} T_{x,n,t-l_2} T_{y,n,t} \\
= T_{x,n,s-l_1} \Delta \hat{T}_{y,n,s} T_{x,n,t-l_2} \Delta \hat{T}_{y,n,t} \\
+ \Delta \hat{T}_{x,n,s-l_1} T_{y,n,s} \Delta \hat{T}_{y,n,t} - T_{x,n,t-l_2} \Delta \hat{T}_{y,n,t} \\
+ \Delta \hat{T}_{x,n,s-l_1} T_{y,n,s} \Delta \hat{T}_{y,n,t} - T_{x,n,t-l_2} \Delta \hat{T}_{y,n,t} \\
+ \cdots \\
+ \Delta \hat{T}_{x,n,s-l_1} T_{y,n,s} \Delta \hat{T}_{y,n,t} - T_{x,n,t-l_2} \Delta \hat{T}_{y,n,t}.
\end{equation}

(C.3)

We will prove

$$
\frac{1}{m_n} \sum_{s,t=1}^{n} |k_{n,s,t}| \times \left| T_{x,n,s-l_1} T_{y,n,s} T_{x,n,t-l_2} \Delta \hat{T}_{y,n,t} \right| = o_p(1).
$$

Nearly identical arguments suffice to show kernel weighted sum of each remaining term in (C.3) is $o_p(1)$.

Exploiting $\hat{\alpha}_{z,m_n} = \alpha_z^{-1} + O_p(1/m_n^{1/2})$ by Theorem 5.1 of Hill (2009b) and Assumption B we may write (cf. Hsing 1991: p. 1554)

$$
\hat{T}_{z,n,s} - T_{z,n,s} = (\ln(z_t/z_{(m_n+1)}))_+ - (\ln(z_t/b_{z,n}))_+ \\
- (m_n/n) \{\alpha_z^{-1} - (n/m_n)E(\ln(z_t/b_{z,n}))_+\} \\
- (m_n/n) \{\hat{\alpha}_{z,m_n} - \alpha_z^{-1}\} \\
= (\ln(z_t/z_{(m_n+1)}))_+ - (\ln(z_t/b_{z,n}))_+ + o(m_n^{1/2}/n).
$$

By cases it is straightforward to show

$$
\left| (\ln(z_t/z_{(m_n+1)}))_+ - (\ln(z_t/b_{z,n}))_+ \right| \leq \left| \ln(z_{(m_n+1)}/b_{z,n}) \right|.
$$

Note

$$
\frac{1}{m_n} \sum_{s,t=1}^{n} k_{n,s,t} T_{x,n,s-l_1} T_{y,n,s} T_{x,n,t-l_2} \left( \hat{T}_{y,n,t} - T_{y,n,t} \right) \\
= \frac{1}{m_n} \sum_{s,t=1}^{n} k_{n,s,t} T_{x,n,s-l_1} T_{y,n,s} T_{x,n,t-l_2} \\
\times \left\{ (\ln(y_t/y_{(m_n+1)}))_+ - (\ln(y_t/b_{y,n}))_+ \right\} \\
+ \frac{1}{m_n} \sum_{s,t=1}^{n} k_{n,s,t} T_{x,n,s-l_1} T_{y,n,s} T_{x,n,t-l_2} \times o((m_n/n)^{1/2}) \times n^{-1/2} \\
= \frac{1}{m_n} \sum_{s,t=1}^{n} k_{n,s,t} T_{x,n,s-l_1} T_{y,n,s} T_{x,n,t-l_2} \\
\times \left\{ (\ln(y_t/y_{(m_n+1)}))_+ - (\ln(y_t/b_{y,n}))_+ \right\} + o_p(1).
$$
Similarly, because

\[ \frac{1}{m_n} \sum_{s,t=1}^n |k_{n,s,t}| = o(n^{1/2}) \]

under Assumption E to get

\[
\left\| \frac{1}{m_n} \sum_{s,t=1}^n k_{n,s,t}T_{x,n,s-l_1}T_{y,n,s}T_{x,n,t-l_2} \right\|_1 \leq \frac{1}{m_n} \sum_{s,t=1}^n |k_{n,s,t}| \times \|T_{y,n,s}\|_2 \times \|T_{x,n,t}\|_2
\]

\[
= o \left( n^{1/2} \left( \frac{m_n}{n} \right)^{1/2} \left( \frac{n}{m_n} \right)^{1/4} \right)^2 = o \left( \frac{m_n}{n^{1/2}} \right)
\]

and \( o(m^{1/2}/(m_n/n)^{1/2}) = o((m_n/n)^{1/2}) = o(1) \).

Therefore

\[
\left\| \frac{1}{m_n} \sum_{s,t=1}^n k_{n,s,t}T_{x,n,s-l_1}T_{y,n,s}T_{x,n,t-l_2} \left( (\ln (y_{(m_n+1)})_+ - (\ln (y_{(m_n+1)/b_{x,n}})_+) \right) \right\|_1
\]

\[
\leq \frac{1}{m_n} \sum_{s,t=1}^n |k_{n,s,t}| \times \|T_{x,n,s-l_1}T_{y,n,s}T_{x,n,t-l_2}\|_2 \times \left\| \ln \left( (y_{(m_n+1)})_+ - (\ln (y_{(m_n+1)/b_{x,n}})_+) \right) \right\|_2
\]

\[
+ \frac{1}{n^{1/2}} \frac{1}{m_n} \sum_{s,t=1}^n |k_{n,s,t}| \times \|T_{x,n,s}\|_8 \times \|T_{y,n,s}\|_8 \|T_{x,n,t}\|_4
\]

\[
+ o((m_n/n)^{1/2})
\]

\[
= O(1) \times \frac{1}{n^{1/2}} \frac{1}{m_n} \sum_{s,t=1}^n |k_{n,s,t}| + o((m_n/n)^{1/2}) = o(1).
\]

Similarly, because \( \ln \left( z_{(m_n+1)/b_{x,z_n}} \right) = O_p(1/m_n^{1/2}) \) by Theorem 5.1 Hill (2009b),

\[
m_n^{-1} \sum_{s,t=1}^n k_{n,s,t} \left( T_{x,n,s-l_1} - T_{x,n,s-l_1} \right) \left( T_{y,n,s} - T_{y,n,s} \right)
\]

\[
\times \left( T_{x,n,t-l_2} - T_{x,n,t-l_2} \right) \left( T_{y,n,t} - T_{y,n,t} \right)
\]

\[
\leq \left| \ln \left( x_{(m_n+1)/b_{x,n}} \right) \right|^2 \times \left| \ln y_{(m_n+1)/b_{y,n}} \right|^2 \times \frac{1}{m_n} \sum_{s,t=1}^n |k_{n,s,t}|
\]

\[
= O_p(1/m_n) \times \frac{1}{m_n} \sum_{s,t=1}^n |k_{n,s,t}|
\]

\[
= O_p(n^{1/2}/m_n) \times \frac{1}{n^{1/2}} \times \frac{1}{m_n} \sum_{s,t=1}^n |k_{n,s,t}| = o_p(1).
\]

- Proof of Lemma B.3.

Claim 1: Since the kernel \( k_{n,s,t} \) symmetrically and negligibly trims over \( s,t = 1...n \), and \( 1/n \sum_{s=1}^n |k_{n,s,t}| = o(m_n^{1/2}/n^{3/4}) = o(1) \) under Assumption E and \( (m_n/n) = o(1) \),
the claim follows by an argument identical to the proof of Lemma A.1.

Claim ii: Simply augment the argument used to prove Lemma A.2. Since $1/n \sum_{s=1}^{n} |k_{n,s,t}| = o(m_n^{1/2}/n^{3/4}) = o(1)$ it is easy to prove under Assumptions A and B

$$
\left\{ \frac{n^{3/4}}{m_n^{1/2}} \times \frac{1}{n} \sum_{s=1}^{n} k_{n,s,t} T_{x,n,t-l} T_{y,n,t} \right\}
$$

is $L_2$-NED on $\{F_t\}$ with constants $f_{n,t}^{(h)} = O(m_n^{1-r/2}n^{-r/2})$ uniformly in $t \geq 1$ and coefficients $\Psi_{\nu}^{(h)} = o((m_n/n)^{1/2-1/r}n^{1/2})$, cf. Lemma A.3. Now apply Corollary 3.3 of Hill (2009b) to deduce

$$
\frac{1}{m_n} \sum_{s.t=1}^{n} \left( \frac{n^{3/4}}{m_n^{1/2}} \times \frac{1}{n} \sum_{s=1}^{n} k_{n,s,t} T_{x,n,t-l} T_{y,n,t} - E \left[ \frac{n^{3/4}}{m_n^{1/2}} \times \frac{1}{n} \sum_{s=1}^{n} k_{n,s,t} T_{x,n,t-l} T_{y,n,t} \right] \right) = O_p(1/m_n^{1/2}).
$$

Multiply both sides by $n$ and $m_n^{1/2}/n^{3/4}$ and compress the summation to conclude

$$
\frac{1}{m_n} \sum_{s.t=1}^{n} (k_{n,s,t} T_{x,n,t-l} T_{y,n,t} - E[k_{n,s,t} T_{x,n,t-l} T_{y,n,t}]) = O_p \left( \frac{n^{1/2} m_n^{1/2}}{m_n^{1/2} n^{3/4}} \right) = o_p(1/m_n^{1/2})
$$
given $m_n/n = o(1)$.

Proof of Lemma B.4. Write $B_\alpha := (1/2) \alpha_x \alpha_y$ and $\hat{B}_\alpha := (1/2) \hat{\alpha}_x \hat{\alpha}_y$. We first show that everywhere $\hat{B}_\alpha$ and $\hat{T}_{x,n,t-l} \hat{y}_{n,t}$ can be replaced with $B_\alpha$ and $T_{x,n,t-l} T_{y,n,t}$

$$
\frac{1}{n} \sum_{s.t=1}^{n} k_{n,s,t} \left( \frac{1}{m_n} \hat{B}_\alpha \hat{T}_{x,n,t-l} \hat{y}_{n,t} - \hat{q}_{m_n}(l) \right)
$$

$$
= \frac{1}{n} \sum_{s.t=1}^{n} k_{n,s,t} \left( \frac{n}{m_n} B_\alpha T_{x,n,t-l} T_{y,n,t} - q_{m_n}(l) \right) + o_p(1).
$$

and then prove

$$
\frac{1}{n} \sum_{s.t=1}^{n} k_{n,s,t} \left( \frac{n}{m_n} B_\alpha T_{x,n,t-l} T_{y,n,t} - q_{m_n}(l) \right) = o_p \left( n^2 / m_n^{3/2} \right).
$$

Step 1: The arguments in the proof of Lemma A.1 carry over entirely if we replace $\hat{T}_{x,n,t-l} \hat{y}_{n,t}$ with $k_{n,s,t} T_{x,n,t-l} T_{y,n,t}$. In particular it is straightforward to show

$$
\frac{1}{n} \sum_{s=1}^{n} \left( \frac{1}{m_n} \sum_{t=1}^{n} k_{n,s,t} \hat{B}_\alpha \hat{T}_{x,n,t-l} \hat{y}_{n,t} \right)
$$

$$
= \frac{1}{n} \sum_{s=1}^{n} \left( \frac{n}{m_n} \sum_{s,t=1}^{n} k_{n,s,t} \hat{B}_\alpha T_{x,n,t-l} T_{y,n,t} \right) + o_p \left( 1/m_n^{1/2} \right).
$$
Since $\hat{B}_\alpha = B_\alpha + O_p(1/m_n^{1/2})$ is implied by Theorem 5.1 of Hill (2009b) it follows
\[
\frac{1}{n} \sum_{s=1}^{n} \left( \frac{1}{m_n} \sum_{t=1}^{n} k_{n,s,t} \tilde{B}_\alpha \tilde{T}_{x,n,t-l \tilde{T}_{y,n,t}} \right) \\
= \frac{1}{n} \sum_{s=1}^{n} \left( \frac{1}{m_n} \sum_{t=1}^{n} k_{n,s,t} B_\alpha T_{x,n,t-l T_{y,n,t}} \right) + o_p \left( 1/m_n^{1/2} \right) \\
+ O_p \left( 1/m_n^{1/2} \right) \times \frac{1}{n} \sum_{s=1}^{n} \left( \frac{1}{m_n} \sum_{t=1}^{n} k_{n,s,t} B_\alpha T_{x,n,t-l T_{y,n,t}} \right).
\]

Since $\|T_{x,n,t}\|_2 = O((m_n/n)^{1/2})$ uniformly in $t$ by Lemma B.1, use Assumption E and the Cauchy-Schwartz inequality to deduce
\[
\left\| \frac{1}{n} \sum_{s=1}^{n} \left( \frac{1}{m_n} \sum_{t=1}^{n} k_{n,s,t} B_\alpha T_{x,n,t-l T_{y,n,t}} \right) \right\|_1 \leq K \frac{1}{m_n} \sum_{s,t=1}^{n} |k_{n,s,t}| \times \frac{m_n}{n} \\
= o\left( m_n/n^{1/2} \right).
\]

Therefore
\[
\frac{1}{n} \sum_{s=1}^{n} \left( \frac{1}{m_n} \sum_{t=1}^{n} k_{n,s,t} B_\alpha T_{x,n,t-l T_{y,n,t}} \right) \\
= \frac{1}{n} \sum_{s=1}^{n} \left( \frac{1}{m_n} \sum_{t=1}^{n} k_{n,s,t} B_\alpha T_{x,n,t-l T_{y,n,t}} \right) + o_p \left( 1/m_n^{1/2} \right) + O_p \left( (m_n/n)^{1/2} \right) \\
= \frac{1}{n} \sum_{s=1}^{n} \left( \frac{1}{m_n} \sum_{t=1}^{n} k_{n,s,t} B_\alpha T_{x,n,t-l T_{y,n,t}} \right) + o_p (1).
\]

Now observe Lemma A.1, Theorem 3.1 and $\hat{B}_\alpha = B_\alpha + O_p(1/m_n^{1/2})$ imply
\[
\hat{q}_{m_n}(l) = \frac{1}{m_n} \sum_{t=1}^{n} \hat{B}_\alpha \hat{T}_{x,n,t-l \hat{T}_{y,n,t}} \\
= q_{m_n}(l) + B_\alpha \times \frac{1}{m_n} \sum_{t=1}^{n} \left( \hat{T}_{x,n,t-l \hat{T}_{y,n,t}} - T_{x,n,t-l T_{y,n,t}} \right) \\
+ \left( \hat{B}_\alpha - B_\alpha \right) \frac{1}{m_n} \sum_{t=1}^{n} \hat{T}_{x,n,t-l \hat{T}_{y,n,t}} \\
= q_{m_n}(l) + o_p \left( 1/m_n^{1/2} \right),
\]
and $1/m_n \sum_{s,t=1}^{n} |k_{n,s,t}| = o(n^{1/2})$ under Assumption E, hence
\[
\frac{1}{n} \sum_{s,t=1}^{n} k_{n,s,t} \left( \frac{n}{m_n} \hat{B}_\alpha \hat{T}_{x,n,t-l \hat{T}_{y,n,t}} - \hat{q}_{m_n}(l) \right) \\
= \frac{1}{n} \sum_{s,t=1}^{n} k_{n,s,t} \left( \frac{n}{m_n} B_\alpha T_{x,n,t-l T_{y,n,t}} - q_{m_n}(l) \right) + \frac{1}{n} \sum_{s,t=1}^{n} k_{n,s,t} \times o_p \left( 1/m_n^{1/2} \right) \\
= \frac{1}{n} \sum_{s,t=1}^{n} k_{n,s,t} \left( \frac{n}{m_n} B_\alpha T_{x,n,t-l T_{y,n,t}} - q_{m_n}(l) \right) + o_p \left( (m_n/n)^{1/2} \right).
\]
**Step 2:** By construction

\[
\frac{1}{n} \sum_{s,t=1}^{n} k_{n,s,t} \left( \frac{n}{m_n} B \alpha T_{x,n,t-i} T_{y,n,t} - q_{m_n}(t) \right)
\]

\[
= B \sum_{s=1}^{n} \frac{1}{m_n} \sum_{t=1}^{n} k_{n,s,t} (T_{x,n,t-i} T_{y,n,t} - E[T_{x,n,t-i} T_{y,n,t}])
\]

so we need only prove

\[
\frac{1}{n} \sum_{s=1}^{n} \frac{1}{m_n} \sum_{t=1}^{n} k_{n,s,t} (T_{x,n,t-i} T_{y,n,t} - E[T_{x,n,t-i} T_{y,n,t}]) = O_p \left(1/m_n^{1/2}\right),
\]

since \(1/m_n^{1/2} = o(n/m_n^{3/2})\) follows trivially from \(m_n/n = o(1)\). The rate follows instantly from Lemma B.3 and \(m_n/n = o(1)\). ■