Technical Appendix for "Gaussian Tests of Extremal White Noise for Dependent, Heterogeneous Processes with an Application"

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February 29, 2008

In this appendix we study the bounds and decay properties of the co-relation, and present omitted lemmas. Section A details co-relation bounds, and Section B characterizes co-relation decay in linear and nonlinear distributed lags (Lemmas B.1-B.4). Section C contains the supporting Lemma C.1 used in the main paper. The proofs of Lemmas B.1-B.4.

Write $(z)_- = \min\{z, 0\}$ and $(z)_+ = \max\{z, 0\}$. Recall for all t as $\varepsilon \to \infty$ there exist indices $\alpha_i > 0$ and scales $c_1(x) \geq 0$ and $c_2(x) > 0$ such that

$$P\left(X_t < -\varepsilon\right) = c_1(x)\varepsilon^{-\alpha_1}(1 + O(\varepsilon^{-\theta_1})) \quad \text{(B.1)}$$
$$P\left(X_t > \varepsilon\right) = c_2(x)\varepsilon^{-\alpha_2}(1 + O(\varepsilon^{-\theta_2})),$$

and

$$\alpha_0 := \min\{\alpha_1, \alpha_2\} \text{ and } \theta_0 := \min\{\theta_1, \theta_2\}.$$

A. CO-RELATION BOUNDS Write $\Im_t \equiv \sigma(X_\tau : \tau \leq t)$. Suppose there exists a Borel-measurable function $g : \mathbb{R} \to \mathbb{R}$ such that for all $t \in \mathbb{Z}$

$$X_t = g(X_{t-h}). \quad \text{(B.2)}$$

Then $\sigma(X_t) \subseteq \sigma(X_{t-h})$ for all t hence

$$\Im_t = \sigma\left(\cup_{\tau \leq t}\sigma(X_\tau)\right) \subseteq \Im_{t-h} = \sigma\left(\cup_{\tau \leq t-h}\sigma(X_\tau)\right).$$

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But $\Im_{t-h} \subseteq \Im_t$ by assumption, therefore $\Im_{t-h} = \Im_t$. This implies g must be a Borel-measurable isomorphism (1-1 onto, where g and g^1 are Borel measurable), and therefore monotonic. See Davidson (1994) and Doob (1994).

In this section we will prove almost everywhere continuity and $|g(\varepsilon)| \to \infty$ as $\varepsilon \to \infty$ imply g must be a power function in order for $g(X_t) \sim$ (B.1). Further, (B.1) and (B.2) simultaneously hold *if and only if* g is asymptotically affine. Co-relation bounds will then easily follow.

Although we do not claim the following discourse is exhaustive, it does reveal from a reasonably general perspective that the co-relation measures the degree to which extremes are linearly related. We do *not* prove that if a bound is reached it must be one characterized below *nor* that the extremes must be governed by a linear data generating process.

LEMMA B.1 *Let* $X_t \sim$ (B.1). *Let* $g : \mathbb{R} \to \mathbb{R}$ *be continuous* $\mathbb{R}$- *a.e., and let* $|g(\varepsilon)| \to \infty$ *as* $|\varepsilon| \to \infty$. *Then*

$$P(g(X_t) < -\varepsilon) = c_1(g)\varepsilon^{-\alpha_1(g)}\ (1 + O(\varepsilon^{-\theta_1(g)}))$$
$$P(g(X_t) < \varepsilon) = c_2(g)\varepsilon^{-\alpha_2(g)}(1 + O(\varepsilon^{-\theta_2(g)}))$$

with finite $c_i(g) > 0$ *and* $\alpha_2(g) > 0$ if and only if $\lim_{\varepsilon\to\infty} g(-\varepsilon)/\varepsilon^{\gamma_1} = b_1$ *and* $\lim_{\varepsilon\to\infty} g(\varepsilon)/\varepsilon^{\gamma_2} = b_2$ *for some* $\gamma_1, \gamma_2 > 0$ *and finite* $b_1, b_2 \in \mathbb{R}$.

If $X_t \sim$ (B.1) *and* (B.2) holds then g must be a positive or negative identity function asymptotically .

LEMMA B.2 *Let* $X_t \sim$ (B.1).

i. (B.2) *holds if and only if* g *is* asymptotically affine *in the sense* $g(\pm\varepsilon)/\varepsilon \to 1$ or $g(\pm\varepsilon)/\varepsilon \to 1$ *as* $\varepsilon \to \infty$.

ii. If $g(\pm\varepsilon)/\varepsilon \to 1$ *then*

$$\rho^{(0)}(X_t, X_{t-h}) = 1 \ \text{ and } \ \rho^{(i)}(X_t, X_{t-h}) = 2^{\alpha_i} - 1 \text{ for } i = 1, 2.$$

If $g(\pm\varepsilon)/\varepsilon \to -1$ *then*

$$\rho^{(0)}(X_t, X_{t-h}) = 1 - 2^{\alpha_0 - 1} \ \text{ and } \ \rho^{(i)}(X_t, X_{t-h}) = -1 \text{ for } i = 1, 2.$$

Together,

$$\min\{1 - 2^{\alpha_0-1}, 0\} \le \rho^{(0)}(h) \le 1$$
$$-1 \le \rho^{(i)}(h) \le \max\{2^{\alpha_i - 1} - 1, 0\}\}, \ i = 1, 2.$$

B. NONLINEAR DISTRIBUTED LAGS We characterize the tail shape of extremal threshold and simple bilinear processes.

B.1 Extremal Threshold and Random Coefficient Autoregressions

Consider first order extremal threshold processes

$$X_{n,t} = \phi X_{n,t-1} I(|u_{t-1}| \leq v_n) + u_t$$
$$W_{n,t} = \phi W_{n,t-1} I(|u_{t-1}| > v_n) + u_t$$

for some real sequence $v_n \to \infty$ as $n \to \infty$, where $|\phi| < 1$ and $u_t \overset{iid}{\sim}$ (B.1) has symmetric tails $c_1(u) = c_2(u) = 1$, $\alpha_1 = \alpha_2 = \alpha$ and $\theta_1 = \theta_2 = \theta$. Clearly if $u_{t-1} > v_n$ then $X_{n,t}$ is independent noise and $W_{n,t}$ is AR(1). We again exploit independence, this time to simplify arguments due to the nonlinear $I(|u_{t-1}| \leq v_n)$.

Notice $X_{n,t}$ and $W_{n,t}$ trivially have Random Coefficient Autoregression representations:

$$X_{n,t} = \phi_{n,t-1} X_{n,t-1} + u_t, \;\; \phi_{n,t} = \phi I(|u_{t-1}| \leq v_n)$$
$$W_{n,t} = \phi_{n,t-1} W_{n,t-1} + u_t, \;\; \phi_{n,t} = \phi I(|u_{t-1}| > v_n).$$

Write $\rho_n^{(0)}(h)$ to denote the possibility that the co-relation depends on n through v_n.

LEMMA B.3

i. Each $\{X_{n,t}, X_{n,t} \pm X_{n,t-h}\}$ *and* $\{W_{n,t}, W_t \pm W_{t-h}\}$ *satisfies (B.1) with indices* α_1 *and* α_2.

ii. $\{X_{n,t}\}$ *is extremal white noise:*

$$\limsup_{n\to\infty} \sup_{h\in\mathbb{N}} \left|\rho_n^{(0)}(X_{n,t}, X_{n,t-h})\right| = 0.$$

$\{W_{n,t}\}$ *is first-order extremal dependent:*

$$\lim_{n\to\infty} \rho_n^{(0)}(W_{n,t}, W_{n,t-h}) \;\; = \;\; (1-(1-\phi)^{\alpha})/2 \;\; \text{for} \;\; h = 1,$$

and $\lim_{n\to\infty} \rho_n^{(0)}(W_{n,t}, W_{n,t-h}) = 0$ *for all* $h \geq 2$.

B.2 Bilinear

Consider a simple bilinear process

$$X_t = \beta X_{t-1} u_{t-1} + u_t, \;\; u_t \overset{iid}{\sim} \text{(B.1)}, \;\; P(u_t < 0) = 0.$$

Let $\beta \in (0, 1)$ satisfy $\beta^{\alpha/2} E[u_t^{\alpha/2}] < 1$.

LEMMA B.4

i. Each $Z_t \in \{X_t, X_t \pm X_{t-h}\}$ *has tails (B.1) with tail index* $\alpha/2$.

ii. The co-relations are identically represented by (B.4) with $\psi_i = \beta^i$.

C. LEMMA C.1

The following lemma is invoke in the proofs of Theorem 6.2 and Lemma A.5.

LEMMA C.1 *Recall S_m is the subset of*

$$\left\{\{m_n\} : 1 \leq m_n \leq n,\ m_n \sim n^{\delta},\ 1/2 < \delta < \min_{0 \leq 1 \leq 2}\left\{\frac{2\theta_i}{2\theta_i + \alpha_i}\right\}\right\}.$$

such that $m_n/\tilde{m}_n = 1 + o(1)\ \forall\{m_n, \tilde{m}_n\} \in S_m$. Under Assumptions A, B.1 and C $\hat{\alpha}_{i,m_n} = \hat{\alpha}_{i,\tilde{m}_n} + o_p(1/\sqrt{\tilde{m}_n})\ \forall m,_n \tilde{m}_n \in S_m$.

Proof of Lemma C.1. Consider $\hat{\alpha}_{0,m_n}^{-1}$: an identical argument proofs holds for any $\hat{\alpha}_{i,m_n}^{-1}$. For notational convenience drop the tail signifier "i", simply write $\hat{\alpha}_{m_n}^{-1}$, and write X_t for the associated process with support on $[0, \infty)$ and tail

$$P\left(X_t > x\right) = cx^{-\alpha}(1 + o(x^{-\theta})), \quad c > 0.$$

Denote by b_{m_n} the associated threshold sequence: $(n/m_n)P(X_t > b_m) \to 1$.

Define for any $v \in \mathbb{R}$

$$\{U_{m_n,t}\} \equiv \left\{(\ln X_t - \ln b_{m_n})_+ - E[(\ln X_t - \ln b_{m_n})_+]\right\}$$
$$\{U_{m_n,t}^*(v/\sqrt{m_n})\} \equiv \left\{I\left(X_t > b_m e^{v/\sqrt{m_n}}\right) - E[I\left(X_t > b_m e^{v/\sqrt{m_n}}\right)]\right\}.$$

Lemma C.1.1, below, implies

$$\frac{1}{m_n^{1/2}}\sum_{t=1}^{n}\left(U_{m_n,t} - \alpha^{-1}U_{m_n,t}^*(v/\sqrt{m_n})\right)$$
$$- (m_n/\tilde{m}_n)^{1/2} \times \frac{1}{\tilde{m}_n^{1/2}}\sum_{t=1}^{n}\left(U_{\tilde{m}_n,t} - \alpha^{-1}U_{\tilde{m}_n,t}^*(v/\sqrt{\tilde{m}_n})\right) = o_p(1),$$

where $(m_n/\tilde{m}_n)^{1/2} = 1 + o(1)$ by assumption, hence

$$\frac{1}{m_n^{1/2}}\sum_{t=1}^{n}\left(U_{m_n,t} - \alpha^{-1}U_{m_n,t}^*(v/\sqrt{m_n})\right)$$
$$- \frac{1}{\tilde{m}_n^{1/2}}\sum_{t=1}^{n}\left(U_{\tilde{m}_n,t} - \alpha^{-1}U_{\tilde{m}_n,t}^*(v/\sqrt{\tilde{m}_n})\right) = o_p(1).$$

Now use Lemma A.1 to deduce

$$\sqrt{m_n}\left(\hat{\alpha}_{m_n}^{-1} - \alpha^{-1}\right) - \sqrt{\tilde{m}_n}\left(\hat{\alpha}_{\tilde{m}_n}^{-1} - \alpha^{-1}\right)$$
$$= \frac{1}{m_n^{1/2}}\sum_{t=1}^{n}\left(U_{m_n,t} - \alpha^{-1}U_{m_n,t}^*(v/\sqrt{m_n})\right)$$
$$- \frac{1}{\tilde{m}_n^{1/2}}\sum_{t=1}^{n}\left(U_{\tilde{m}_n,t} - \alpha^{-1}U_{\tilde{m}_n,t}^*(v/\sqrt{\tilde{m}_n})\right)$$
$$= o_p(1).$$

Therefore

$$\begin{aligned}
&\sqrt{m_n}\left(\hat{\alpha}_{m_n}^{-1}-\alpha^{-1}\right)-\sqrt{\tilde{m}_n}\left(\hat{\alpha}_{\tilde{m}_n}^{-1}-\alpha^{-1}\right)\\
&\quad=\sqrt{m_n}\left(\hat{\alpha}_{m_n}^{-1}-\alpha^{-1}\right)\left(1-\sqrt{\tilde{m}_n}/\sqrt{m_n}\right)+\sqrt{\tilde{m}_n}\left(\hat{\alpha}_{m_n}^{-1}-\hat{\alpha}_{\tilde{m}_n}^{-1}\right)\\
&\quad=o_p(1).
\end{aligned}$$

Theorem 5 of Hill (2005) implies $\hat{\alpha}_{m_n}^{-1}-\alpha^{-1}=O(1/m_n^{1/2})$, and by assumption $1-\tilde{m}_n^{1/2}/m_n^{1/2}=o(1)$. Therefore $\tilde{m}_n^{1/2}(\hat{\alpha}_{m_n}^{-1}-\hat{\alpha}_{\tilde{m}_n}^{-1})=o_p(1)$. ■

LEMMA C.1.1 *Under Assumptions A, B, and C*

$$\frac{1}{\sqrt{m_n}}\sum_{t=1}^{n}U_{m_n,t}-\frac{1}{\sqrt{\tilde{m}_n}}\sum_{t=1}^{n}U_{\tilde{m}_n,t}=o_p(1)$$

$$\frac{1}{\sqrt{m_n}}\sum_{t=1}^{n}U_{m_n,t}^{*}(v/\sqrt{m_n})-\frac{1}{\sqrt{\tilde{m}_n}}\sum_{t=1}^{n}U_{\tilde{m},t}^{*}(v/\sqrt{\tilde{m}_n})=o_p(1).$$

Proof. For notational convenience assume $\tilde{m}_n\leq m_n$ for each n, and write

$$\frac{1}{\sqrt{m_n}}\sum_{t=1}^{n}U_{m_n,t}-\frac{1}{\sqrt{\tilde{m}_n}}\sum_{t=1}^{n}U_{\tilde{m}_n,t}=\frac{1}{\sqrt{m_n}}\sum_{t=1}^{n}\left(U_{m_n,t}-(m_n/\tilde{m}_n)^{1/2}U_{\tilde{m}_n,t}\right). \tag{B.4}$$

Clearly

$$\begin{aligned}
&U_{m_n,t}-(m_n/\tilde{m}_n)^{1/2}U_{\tilde{m}_n,t}\\
&\quad=(\ln X_t/b_{m_n})_+-(m_n/\tilde{m})_n^{1/2}(\ln X_t/b_{\tilde{m}_n})_+\\
&\qquad+(m_n/\tilde{m})_n^{1/2}E(\ln X_t/b_{\tilde{m}_n})_+-E(\ln X_t/b_{m_n})_+.
\end{aligned}$$

By Lemma C.1.2, $(m_n/\tilde{m}_n)^{1/2}E(\ln X_t/b_{\tilde{m}_n})_+-E(\ln X_t/b_{m_n})_+=o(m_n^{1/2}/n)$.

It is easy to show $b_{m_n}\leq b_{\tilde{m}_n}$ *if and only if* $\tilde{m}_n\leq m_n$ as $n\rightarrow\infty$. Assume without loss of generality that n is large enough such that $b_{m_n}\leq b_{\tilde{m}_n}$. Define

$$A_{n,1}=\{t:b_{\tilde{m}_n}\geq X_t>b_{m_n}\},\quad A_{n,2}=\{t:X_t\geq b_{\tilde{m}_n}\}.$$

Identity (B.4) reduces to

$$\begin{aligned}
&\frac{1}{\sqrt{m_n}}\sum_{t=1}^{n}\left(U_{m_n,t}-(m_n/\tilde{m}_n)^{1/2}U_{\tilde{m}_n,t}\right)\\
&\quad=\frac{1}{\sqrt{m_n}}\sum_{t=1}^{n}\left((\ln X_t/b_{m_n})_+-(m_n/\tilde{m}_n)^{1/2}(\ln X_t/b_{\tilde{m}_n})_+\right)+o(1)\\
&\quad=\frac{1}{\sqrt{m_n}}\sum_{t\in A_{n,1}}\ln X_t/b_{m_n}+\frac{1}{\sqrt{m_n}}\sum_{t\in A_{n,2}}\left(\ln X_t/b_{m_n}-(m_n/\tilde{m}_n)^{1/2}\ln X_t/b_{\tilde{m}_n}\right)+o(1)\\
&\quad\leq\frac{1}{\sqrt{m_n}}\sum_{t\in A_{n,1}}\ln X_t/b_{m_n}+\ln(b_{\tilde{m}_n}/b_{m_n})\times\frac{1}{\sqrt{m_n}}\sum_{t=1}^{n}I(X_t\geq b_{\tilde{m}_n})+o(1).
\end{aligned}$$

We will show each term on the right-hand-side is $o_p(1)$. First, by Assumptions A and B, Minkowski's and the Cauchy-Schwartz inequalities,

$$\begin{aligned}
&\left\| \frac{1}{\sqrt{m_n}} \sum_{t \in A_{n,1}} \ln X_t/b_{m_n} \right\|_1 \\
&\quad = \left\| \frac{1}{\sqrt{m_n}} \sum_{t=1}^{n} I(b_{\tilde{m}_n} \geq X_t \geq b_{m_n}) \left(\ln X_t/b_{m_n}\right)_+ \right\|_1 \\
&\quad \leq (n/m_n^{12}) \left[P(X_t \geq b_{\tilde{m}_n}) - P(X_t \geq b_{m_n})\right]^{1/2} \left\| (\ln X_t/b_{m_n})_+ \right\|_2 \\
&\quad \sim (n/m_n^{12}) \left[\tilde{m}/_n n - m_n/n\right]^{1/2} \left\| (\ln X_t/b_{m_n})_+ \right\|_2 \\
&\quad = O\left(1\right) \times \left[\tilde{m}/_n m_n - 1\right]^{1/2} \times O((m_n/n)^{1/2}) \\
&\quad = o(1) \times O((m_n/n)^{1/2}) = o((m_n/n)^{1/2}).
\end{aligned}$$

The first equality follows Assumptions A and B and arguments in Hsing (1991: p. 1548).

For the second term, Assumptions A and B and Lemma C.1.3 imply

$$\begin{aligned}
&(\ln b_{\tilde{m}_n}/b_{m_n}) \times \left\| \frac{1}{\sqrt{m_n}} \sum_{t=1}^{n} I\left(X_t \geq b_{\tilde{m}_n}\right) \right\|_1 \\
&\quad \leq o(m_n^{-1/2}) \times (n/\sqrt{m_n}) P\left(X_t \geq b_{\tilde{m}_n}\right) = o(1) \times O(1) = o(1).
\end{aligned}$$

■

Lemma C.1.2 *Under Assumptions A, B, and C* $(m_n/\tilde{m}_n)^{1/2} E\left(\ln X_t/b_{\tilde{m}_n}\right)_+ - E\left(\ln X_t/b_{m_n}\right)_+ = o(m_n^{1/2}/n)$.

Proof. Using Assumption A and $(\ln X_t/b_{m_n})_+ \geq 0$,

$$\begin{aligned}
&(m_n/\tilde{m}_n)^{1/2} E\left(\ln X_t/b_{\tilde{m}_n}\right)_+ - E\left(\ln X_t/b_{m_n}\right)_+ \\
&\quad = (m_n/\tilde{m}_n)^{1/2} \int_0^\infty \bar{F}\left(b_{\tilde{m}_n} e^v\right) du - \int_0^\infty \bar{F}\left(b_{m_n} e^v\right) dudu \\
&\quad = (m_n/\tilde{m}_n)^{1/2} \bar{F}\left(b_{\tilde{m}_n}\right) \int_0^\infty \frac{\bar{F}\left(b_{\tilde{m}_n} e^v\right)}{\bar{F}\left(b_{\tilde{m}_n}\right)} du - \bar{F}\left(b_{m_n}\right) \int_0^\infty \frac{\bar{F}\left(b_{m_n} e^v\right)}{\bar{F}\left(b_{m_n}\right)} du \\
&\quad \sim (m/_n \tilde{m}_n)^{1/2} \bar{F}\left(b_{\tilde{m}_n}\right) \int_0^\infty e^{-\alpha u} du - \bar{F}\left(b_{m_n}\right) \int_0^\infty e^{-\alpha u} du \\
&\quad = \alpha^{-1} \times \bar{F}\left(b_{m_n}\right) \left[\left(\frac{m_n}{\tilde{m}_n}\right)^{1/2} \frac{\bar{F}\left(b_{\tilde{m}_n}\right)}{\bar{F}\left(b_{m_n}\right)} - 1 \right].
\end{aligned}$$

Assumption A and the construction of b_{m_n} imply $\bar{F}\left(b_{m_n}\right) = O(m_n/n)$. The claim now follows from Lemma C.1.3. ■

Lemma C.1.3 *Under Assumptions A, B, and C,* $\forall m_n, \tilde{m}_n \in S_m$

$$\left(\frac{m_n}{\tilde{m}_n}\right)^{1/2} \frac{\bar{F}(b_{\tilde{m}_n})}{\bar{F}(b_{m_n})} = 1 + o(m_n^{-1/2}) \quad \text{and} \quad b_{\tilde{m}_n}/b_{m_n} = 1 + o(1/m_n^{1/2}).$$

Proof. Under Assumptions A and B Goldie and Smith's (1987) condition (SR1) is satisfied (see Haeusler and Teugels 1985), hence

$$(n/m_n)\bar{F}(b_{m_n}) = (n/m_n)b_{m_n}^{-\alpha}L(b_{m_n}) = 1 + o(1/m_n^{1/2})$$

and

$$\left(\frac{m_n}{\tilde{m}_n}\right)\frac{\bar{F}(b_{m_n})}{\bar{F}(b_{m_n})} \sim \frac{(n/\tilde{m}_n)b_{\tilde{m}_n}^{-\alpha}}{(n/m_n)b_{m_n}^{-\alpha}} \sim \frac{1+o(1/\tilde{m}_n^{1/2})}{1+o(1/m_n^{1/2})} = 1 + o(1/m_n^{1/2}), \quad \text{(B.5)}$$

given, $\forall m_n, \tilde{m}_n \in S_m$,

$$\begin{aligned} m_n^{1/2}\left(\frac{o(\tilde{m}_n^{-1/2}) - o(m_n^{-1/2})}{1+o(\tilde{m}_n^{-1/2})}\right) &= \left(\frac{o(m_n^{1/2}/\tilde{m}_n^{1/2}) - o(1)}{1+o(\tilde{m}_n^{-1/2})}\right) \\ &= \left(\frac{(1+o(1))\times o(1) - o(1)}{1+o(\tilde{m}_n^{-1/2})}\right) = o(1). \end{aligned}$$

But (B.5) and $m/_n\tilde{m}_n = 1 + o(1)$ imply $b_{\tilde{m}n}/b_{m_n} = 1 + o(1/m_n^{1/2})$, hence $\bar{F}(b_{m_n})/\bar{F}(b_{m_n}) = 1 + o(1/m_n^{1/2})$ and

$$\left(\frac{m_n}{\tilde{m}_n}\right)^{1/2}\frac{\bar{F}(b_{m_n})}{\bar{F}(b_{m_n})} = 1 + o(1/m_n^{1/2}).$$

■

Appendix 1: Proofs of Lemmas B.1-B.7

Proof of Lemma B.1. For notational simplicity assume $\alpha_1 = \alpha_2 = \alpha$. All arguments extend to the general case $\alpha_1 \lesseqgtr \alpha_2$. Note $|X_t|$ has a right-tail (B.1) with scale $c_1(x) + c_2(x)$ and index α.

Step 1 (only if): Let $\lim_{\varepsilon\to\infty} g(-\varepsilon)/\varepsilon^{\gamma_1} = b_1$ and $\lim_{\varepsilon\to\infty} g(\varepsilon)/\varepsilon^{\gamma_2} = b_2$. By cases, assume $b_1 < 0$ and $b_2 > 0$. For some finite $c_2(g) > 0$ property (B.1) implies

$$\begin{aligned} c_2(g) &= \lim_{\varepsilon\to\infty} \varepsilon^{\alpha/\gamma_2} P(g(X_t) > \varepsilon) = \lim_{\varepsilon\to\infty} \varepsilon^{\alpha/\gamma_2} b_2^{\alpha/\gamma_2} P\left(X_t > \varepsilon^{1/\gamma_2}\right) \\ &= b_2^{\alpha/\gamma_2} \times c_2(x). \end{aligned}$$

An identical argument implies $c_1(g) = b_1^{\alpha/\gamma_1} \times c_1(x)$.

If $b_1 > 0$ and $b_2 < 0$ then

$$c_i(g) = b_j^{\alpha/\gamma_j} \times c_j(x), \ \ i \neq j.$$

If $b_1 > 0$ and $b_2 > 0$ then $c_1(g) = 0$ and

$$\begin{aligned} c_2(g) &= \lim_{\varepsilon\to\infty} \varepsilon^{\alpha/\max\{\gamma_1,\gamma_2\}} P\left(g(X_t) > \varepsilon\right) \\ &= \lim_{\varepsilon\to\infty} \varepsilon^{\alpha/\max\{\gamma_1,\gamma_2\}} P\left(X_t > b_2^{-1/\gamma_2}\varepsilon^{1/\gamma_2}\right) \\ &\quad + \lim_{\varepsilon\to\infty} \varepsilon^{\alpha/\max\{\gamma_1,\gamma_2\}} P\left(X_t < b_1^{-1/\gamma_1}\varepsilon^{1/\gamma_1}\right) \\ &= b_2^{\alpha/\gamma_2} \lim_{\varepsilon\to\infty} \varepsilon^{\alpha/\max\{\gamma_1,\gamma_2\}-\alpha/\gamma_2}\varepsilon^{\alpha/\gamma_2} P\left(X_t > \varepsilon^{1/\gamma_2}\right) \\ &\quad + b_1^{\alpha/\gamma_1} \lim_{\varepsilon\to\infty} \varepsilon^{\alpha/\max\{\gamma_1,\gamma_2\}-\alpha/\gamma_1}\varepsilon^{\alpha/\gamma_1} P\left(X_t < \varepsilon^{1/\gamma_1}\right) \\ &= b_1^{\alpha/\gamma_1} c_1(x) \times I\left(\gamma_1 \leq \gamma_2\right) + b_2^{\alpha/\gamma_2} c_2(x) \times I\left(\gamma_2 \leq \gamma_1\right), \end{aligned}$$

etc. Therefore $g(X_t) \sim$ (B.1) with indices α/γ_i.

Step 2 (if): Now assume

$$P\left(g(X_t) < -\varepsilon\right) = c_1(g)\varepsilon^{-\alpha_1(g)} \left(1 + O(\varepsilon^{-\theta_1(g)})\right)$$

$$P\left(g(X_t) > \varepsilon\right) = c_2(g)\varepsilon^{-\alpha_2(g)}(1 + O(\varepsilon^{-\theta_2(g)})),$$

for some $\alpha_i(g), \theta_i(g) > 0$, where $0 \leq c_1(g), c_2(g) < \infty$ with at least one $c_i(g) > 0$. Let $c_2(g) > 0$. Because $|g(\varepsilon)| \to \infty$ as $|\varepsilon| \to \infty$ and g has at most countably many discontinuity points, there exists functions $\tilde{g}_1, \tilde{g}_2 : \mathbb{R} \to \mathbb{R}_+$ satisfying $\tilde{g}_i(\varepsilon) \to \infty$ as $\varepsilon \to \infty$ and[1]

$$\begin{aligned} c_2(g) &= \lim_{\varepsilon\to\infty} \varepsilon^{\alpha_2(g)} P(g(X_t) > \varepsilon) \\ &= \lim_{\varepsilon\to\infty} \varepsilon^{\alpha_2(g)} \left[P(X_t < -\tilde{g}_1(\varepsilon)) + P(X_t > \tilde{g}_2(\varepsilon))\right] \\ &= \lim_{\varepsilon\to\infty} \varepsilon^{\alpha_2(g)} c_1(x)\tilde{g}_1(\varepsilon)^{-\alpha_1} + \lim_{\varepsilon\to\infty} \varepsilon^{\alpha_2(g)} c_2(x)\tilde{g}_2(\varepsilon)^{-\alpha_2}. \end{aligned}$$

Therefore both limits exist and at most one $\lim_{\varepsilon\to\infty} \varepsilon^{\alpha_2(g)} c_i(x)\tilde{g}_i(\varepsilon)^{-\alpha_i} = 0$. Consider the case $\lim_{\varepsilon\to\infty} \varepsilon^{\alpha_2(g)} c_1(x)\tilde{g}_1(\varepsilon)^{-\alpha_1} = 0$:

$$\lim_{\varepsilon\to\infty} \varepsilon^{\alpha_2(g)/\alpha_2}/\tilde{g}_2(\varepsilon) = \left[\frac{c_2(g)}{c_2(x)}\right]^{1/\alpha_1},$$

hence as $\varepsilon \to \infty$

$$\begin{aligned} P(g(X_t) > \varepsilon) &= P\left(X_t > \tilde{g}_2(\varepsilon)\right) \\ &= P\left(X_t > \varepsilon^{\alpha_2(g)/\alpha_2}\left[\frac{c_2(x)}{c_2(g)}\right]^{1/\alpha_1}\right) \\ &= \varepsilon^{-\alpha_2(g)} \times \left[\frac{c_2(x)}{c_2(g)}\right]^{\alpha_2/\alpha_1} \times \left(1 + O(\varepsilon^{-\theta_2})\right) \end{aligned}$$

[1] Notice $\tilde{g}_i(\varepsilon) = \infty$ is possible. If $g(\varepsilon)/\varepsilon \to 1$ as $\varepsilon \to \pm\infty$ then $\tilde{g}_1(\varepsilon) = \infty$ for all ε. If $g(\varepsilon)/\varepsilon^2 \to 1$ as $\varepsilon \to \pm\infty$ then $\tilde{g}_i(\varepsilon) = |\varepsilon|^{1/2}$.

But this implies

$$\frac{g(\varepsilon)}{\varepsilon^{\alpha_2/\alpha_2(g)}} \to \left[\frac{c_2(x)}{c_2(g)}\right]^{1/\alpha_1} \quad \text{as } \varepsilon \to \infty.$$

Similar arguments apply to $P(g(X_t) < -\varepsilon)$ and to all other cases concerning $\lim_{\varepsilon\to\infty} \varepsilon^{\alpha_2(g)} c_i(x) \tilde{g}_i(\varepsilon)^{-\alpha_i} \in [0, \infty)$. ■

Proof of Lemma B.2. Assume $X_t \sim$ (B.1) for all t.

Claim (i):

Step 1: Let $X_t = g(X_{t-h})$. Then $\{X_t, X_{t-h}, g(X_{t-h})\}$ all have an identical tail shape (B.1) for all t:

$$P\left(X_t > \varepsilon\right) = P\left(g(X_t) > \varepsilon\right) \text{ and } P\left(X_t < -\varepsilon\right) = P\left(g(X_t) < -\varepsilon\right).$$

Moreover, g must be a Borel measurable isomorphism, and therefore monotonic (see the discussion proceeding the statement of the lemma).

Suppose g is monotonically *increasing*. Then there exists a function $\tilde{g}_2 : \mathbb{R} \to \mathbb{R}_+$ satisfying $\tilde{g}_2(\varepsilon) \to \infty$ as $\varepsilon \to \infty$, and $x_1 \equiv g(\varepsilon_1) \leq x_2 \equiv g(\varepsilon_2) \to \infty$ *if and only if* $\tilde{g}_2(x_2) \geq \tilde{g}_2(x_1) \to \infty$ as $\varepsilon_1 \to \infty$. Therefore, as $\varepsilon \to \infty$

$$\begin{aligned} P\left(X_t > \varepsilon\right) &= c_2(x)\varepsilon^{-\alpha_2}(1 + o(1)) = P\left(g(X_{t-h}) > \varepsilon\right) \\ &= P\left(X_{t-h} > \tilde{g}_2(\varepsilon)\right) = c_2(x)\tilde{g}_2(\varepsilon)^{-\alpha_2}(1 + o(1)), \end{aligned}$$

hence $\tilde{g}_2(\varepsilon)/\varepsilon \to 1$ as $\varepsilon \to \infty$, and

$$g(\varepsilon) = a + \varepsilon(1 + o(1)) \text{ for any } \in \mathbb{R}.$$

An identical argument holds for the left tail $P(X_t < -\varepsilon)$.

For $\tilde{Y}_{t,h} \equiv X_t - X_{t-h}$ we deduce as $\varepsilon \to \infty$

$$P\left(\tilde{Y}_{t,h} > \varepsilon\right) = P\left(g(X_{t-h}) - X_{t-h} > \varepsilon\right) = P\left(X_{t-h} - X_{t-h} > \varepsilon\right) = 0$$

$$P(\tilde{Y}_{t,h} < -\varepsilon) = P\left(g(X_{t-h}) - X_{t-h} < -\varepsilon\right) = P\left(X_{t-h} - X_{t-h} < -\varepsilon\right) = 0,$$

hence

$$c_1(\tilde{y}_h) = c_2(\tilde{y}_h) = 0.$$

Similarly

$$P(Y_{t,h} > \varepsilon) = P\left(g(X_{t-h}) + X_{t-h} > \varepsilon\right) = P\left(2X_{t-h} > \varepsilon\right) = 2^{\alpha_2} P\left(X_{t-h} > \varepsilon\right)$$

$$P(Y_{t,h} < -\varepsilon) = P\left(g(X_{t-h}) + X_{t-h} < -\varepsilon\right) = P\left(2X_{t-h} < -\varepsilon\right) = 2^{\alpha_1} P\left(X_{t-h} < -\varepsilon\right),$$

hence

$$c_i(y_h) = 2^{\alpha_i} \times c_i(x), \;\; i = 1, 2.$$

Together the above implies

$$\rho^{(0)}(h) = 1 - \frac{c_0(\tilde{y}_h)}{2c_0(x)} = 1 - \frac{c_1(\tilde{y}_h) + c_2(\tilde{y}_h)}{2c_0(x)} = 1$$
$$\rho^{(i)}(h) = \frac{c_i(y_h)}{2c_i(x)} - 1 = \frac{2^{\alpha_i} c_i(x)}{2c_i(x)} - 1 = 2^{\alpha_i - 1} - 1.$$

Now suppose g is monotonically *decreasing*. There exists a function $\tilde{g}_1 : \mathbb{R} \to \mathbb{R}_+$ satisfying $\tilde{g}_1(\varepsilon) \to \infty$ as $\varepsilon \to \infty$, and $x_2 \equiv g(\varepsilon_2) \leq x_1 \equiv g(\varepsilon_1) \to -\infty$ *if and only if* $\tilde{g}_1(x_2) \geq \tilde{g}_1(x_1) \to -\infty$ as $\varepsilon_1 \to \infty$. Thus as $\varepsilon \to \infty$

$$\begin{aligned} P(X_t > \varepsilon) &= P(g(X_{t-h}) > \varepsilon) = P(X_{t-h} < \tilde{g}_1(\varepsilon)) \\ &= P(g(X_{t-2h}) < \tilde{g}_1(\varepsilon)) = P(X_{t-2h} > \tilde{g}_1(\tilde{g}_1(\varepsilon))) \end{aligned}$$

hence

$$\lim_{\varepsilon \to \infty} g(\tilde{g}_1(\tilde{g}_1(\varepsilon)))/g(\varepsilon) = 1 \implies g(\tilde{g}_1(\tilde{g}_1(\varepsilon))) = g(\varepsilon)(1 + o(1)).$$

This implies the slope between the two points $\{\varepsilon, g(\varepsilon)\}$ and $\{g(\tilde{g}_1(\tilde{g}_1(\varepsilon))), \varepsilon\}$ satisfies

$$\frac{g(\varepsilon) - \varepsilon}{\varepsilon - g(\tilde{g}_1(\tilde{g}_1(\varepsilon)))} = \frac{g(\varepsilon) - \varepsilon}{\varepsilon - g(\varepsilon)[1 + o(1)]} = -1 + o(1).$$

From $g(\varepsilon) \to -\infty$ and $g(\tilde{g}_1(\tilde{g}_1(\varepsilon))) \to -\infty$ as $\varepsilon \to \infty$, it must be the case that $g(\varepsilon)/\varepsilon \to -1$.

In this case X_t must have symmetric tails $\alpha_1 = \alpha_2 = \alpha_0$: as $\varepsilon \to \infty$

$$P(X_t > \varepsilon) = P(g(X_{t-h}) > \varepsilon) = P(X_{t-h} < -\varepsilon),$$

and we assume X_t has for each t the same tail shape.

For $Y_{t,h}$ we deduce as $\varepsilon \to \infty$

$$\begin{aligned} P(Y_{t,h} > \varepsilon) &= P(g(X_{t-h}) + X_{t-h} > \varepsilon) \\ &= P(-X_{t-h} + X_{t-h} > \varepsilon) = 0 \end{aligned}$$

$$\begin{aligned} P(Y_{t,h} < -\varepsilon) &= P(g(X_{t-h}) + X_{t-h} < -\varepsilon) \\ &= P(-X_{t-h} + X_{t-h} < -\varepsilon) = 0 \end{aligned}$$

hence

$$c_1(y_h) = c_2(y_h) = 0.$$

Similarly

$$\begin{aligned} P(\tilde{Y}_{t,h} > \varepsilon) &= P(g(X_{t-h}) - X_{t-h} > \varepsilon) \\ &= P(-X_{t-h} - X_{t-h} > \varepsilon) = 2^{\alpha_1} P(X_{t-h} < -\varepsilon) \\ P(\tilde{Y}_{t,h} < -\varepsilon) &= P(g(X_{t-h}) - X_{t-h} < -\varepsilon) \\ &= P(-X_{t-h} - X_{t-h} < -\varepsilon) = 2^{\alpha_2} P(X_{t-h} > \varepsilon) \end{aligned}$$

hence

$$c_i(\tilde{y}_h) = 2^{\alpha_j} \times c_j(x), \;\; i \neq j.$$

In this case

$$\begin{aligned}\rho^{(0)}(h) &= 1 - \frac{c_1(\tilde{y}_h) + c_2(\tilde{y}_h)}{2c_0(x)} = 1 - \frac{2^{\alpha_0 - 1}c_0(x) + 2^{\alpha_0 - 1}c_0(x)}{c_0(x)} = 1 - 2^{\alpha_0 - 1} \\ \rho^{(i)}(h) &= c_i(y_h)/2c_i(x) - 1 = -1.\end{aligned}$$

Step 2: If $g(\varepsilon)/\varepsilon = 1 + o(1)$ then $g(X_{t-h}) = X_{t-h}(1 + o(1))$ hence

$$\begin{aligned}&P(g(X_{t-h}) > \varepsilon) = P(X_{t-h}(1 + o(1)) > \varepsilon) \\ &\qquad\qquad \sim (1 + o(1))^{-\alpha} \times P(X_{t-h} > \varepsilon) \sim P(X_{t-h} > \varepsilon).\end{aligned}$$

The case $g(\varepsilon)/\varepsilon = -1 + o(1)$ is identical.

We deduce $Y_{t,h} \equiv X_t + X_{t-h}$ satisfies as $\varepsilon \to \infty$

$$\begin{aligned}P\left(Y_{t,h} > \varepsilon\right) &= P\left(g(X_{t-h}) + X_{t-h} > \varepsilon\right) = P\left(2X_{t-h} > \varepsilon\right) \\ &= 2^{\alpha_2} c_2(x)\varepsilon^{-\alpha_2}(1 + o(1)).\end{aligned}$$

In general $c_i(y_h) = 2^{\alpha_i} c_i(x)$.

Step 3: Recall $\rho^{(\cdot)}(h) = 0$ if X_t is serially independent. Thus, the above bounds must be amended to captured this case:

$$\begin{aligned}&\min\{1 - 2^{\alpha_0 - 1}, 0\} \leq \rho^{(0)}(h) \leq 1 \\ &-1 \leq \rho^{(i)}(h) \leq \max\{2^{\alpha_i - 1} - 1, 0\}\}, \; i = 1, 2.\end{aligned}$$

Claim (ii): Immediate from Steps 1 and 2. ■

Proof of Lemma B.3. Decompose

$$\begin{aligned}X_{n,t} &= u_t + \sum_{i=1}^{\infty} \phi^i u_{t-i} \prod_{j=1}^{i} I_{n,t-j}, \; a.s. \text{ where } I_{n,t-j} \equiv I\left(|u_{t-j}| \leq v_n\right) \\ W_{n,t} &= u_t + \sum_{i=1}^{\infty} \phi^i u_{t-i} \prod_{j=1}^{i} \bar{I}_{n,t-j}, \; a.s. \text{ where } \bar{I}_{n,t-j} \equiv I\left(|u_{t-j}| > v_n\right),\end{aligned}$$

and define

$$P_n \equiv P(|u_{t-j}| \leq v_n) \;\; \text{and} \;\; \bar{P}_n \equiv P(|u_{t-j}| > v_n).$$

Let $\phi \in [0, 1)$ for brevity. Derivations in the general case $|\phi| < 1$ are similar but tedious.

Claim (i):

Step 1 ($X_{n,t}, W_{n,t} \sim$ (B.1)): Consider $X_{n,t}$, and note u_t is independent of $\sum_{i=1}^{\infty} \phi^i u_{t-i} \prod_{j=1}^{i} I_{n,t-j}$. Moreover, the event

$$\sum\nolimits_{i=1}^{\infty} \phi^i u_{t-i} \prod\nolimits_{j=1}^{i} I_{n,t-j} > \varepsilon > 0$$

occurs if $\phi u_{t-1} > \varepsilon$, $v_n \geq u_{t-1}$ and $u_{t-2} > v_n$; or $\sum_{i=1}^{2} \phi^i u_{t-i} > \varepsilon$, $\{v_n \geq u_{t-i}\}_{i=1}^{2}$, and $\{u_{t-3} > v_n\}$; or $\sum_{i=1}^{3} \phi^i u_{t-i} > \varepsilon$, $\{v_n \geq u_{t-i}\}_{i=1}^{3}$, and $\{u_{t-4} > v_n\}$; etc. Therefore,

$$\begin{aligned}
&P\left(\sum_{i=1}^{\infty} \phi^i u_{t-i} \prod_{j=1}^{i} I_{n,t-j} > \varepsilon\right) \\
&\quad = \sum_{i=1}^{\infty} P\left(\sum_{j=1}^{i} \phi^j u_{t-i} > \varepsilon \cap \{u_{t-j} \leq v_n\}_{j=1}^{i}\right) \times \bar{P}_n \\
&\quad \sim \sum_{i=1}^{\infty} \sum_{j=1}^{i} \phi^{j\alpha} P\left(v_n \geq u_{t-i} > \varepsilon\right) \times P_n^{i-1} \times \bar{P}_n \\
&\quad = \bar{P}_n \sum_{i=1}^{\infty} P_n^{i-1} \sum_{j=1}^{i} \phi^{j\alpha} \times \varepsilon^{-\alpha} \left[1 - (\varepsilon/v_n)^{\alpha} + O\left((\varepsilon/v_n)^{\alpha}\right)\right]_+ \\
&\quad = \varepsilon^{-\alpha} c_n(\phi),
\end{aligned}$$

say. This implies $\sum_{i=1}^{\infty} \phi^i u_{t-i} \prod_{j=1}^{i} I_{n,t-j}$ has tail (B.1), with index α and scale $c_n(\phi)$.

Clearly for each $n \geq 1$ there exists a $\varepsilon_n \geq 1$ such that $[1 - (\varepsilon/v_n)^{\alpha} + O((\varepsilon/v_n)^{\alpha})]_+ = 0 \; \forall \varepsilon > \varepsilon_n$, and

$$\bar{P}_n \sum_{i=1}^{\infty} P_n^{i-1} \sum_{j=1}^{i} \phi^{j\alpha} = \frac{\phi^{\alpha}}{1 - P_n \phi^{\alpha}},$$

hence convergence is guaranteed:

$$\begin{aligned}
\lim_{\varepsilon \to \infty} \varepsilon^{\alpha} P\left(X_{n,t} > \varepsilon\right) &= \lim_{\varepsilon \to \infty} \varepsilon^{\alpha} P\left(u_t > \varepsilon\right) + \lim_{\varepsilon \to \infty} \varepsilon^{\alpha} P\left(\sum_{i=1}^{\infty} \phi^i u_{t-i} \prod_{j=1}^{i} I_{n,t-j} > \varepsilon\right) \\
&= \lim_{\varepsilon \to \infty} \varepsilon^{\alpha} P\left(u_t > \varepsilon\right) = 1.
\end{aligned}$$

Similarly $\lim_{\varepsilon \to \infty} \varepsilon^{\alpha} P(X_{n,t} < -\varepsilon) = 1$ implies

$$\lim_{\varepsilon \to \infty} \varepsilon^{\alpha} P\left(|X_{n,t}| > \varepsilon\right) = 2.$$

Consider $W_{n,t}$. Using the same logic as above

$$
\begin{aligned}
&P\left(\sum_{i=1}^{\infty}\phi^i u_{t-i}\prod_{j=1}^{i}\bar{I}_{n,t-j} > \varepsilon\right) \\
&\quad = \sum_{i=1}^{\infty} P\left(\sum_{j=1}^{i}\phi^j u_{t-j} > \varepsilon \cap \{u_{t-j} > v_n\}_{j=1}^{i}\right) \times P_n \\
&\quad \sim \sum_{i=1}^{\infty}\sum_{j=1}^{i}\phi^{j\alpha} P\left(u_{t-i} > \max\{\varepsilon, v_n\}\right) \times \bar{P}_n^{i-1} \times P_n \\
&\quad = \max\{\varepsilon, v_n\}^{-\alpha}\left(1 + O(\max\{\varepsilon, v_n\}^{-\theta})\right) \times \phi^{\alpha} P_n \sum_{i=0}^{\infty}\bar{P}_n^i\left(\frac{1-\phi^{i\alpha}}{1-\phi^{\alpha}}\right) \\
&\quad = \max\{\varepsilon, v_n\}^{-\alpha}\left(1 + O(\max\{\varepsilon, v_n\}^{-\theta})\right) \times \frac{\bar{P}_n\phi^{\alpha}}{1-\bar{P}_n\phi^{\alpha}}
\end{aligned}
$$

Therefore

$$\lim_{\varepsilon\to\infty}\varepsilon^{\alpha} P\left(W_{n,t} > \varepsilon\right) = 1 + \frac{\bar{P}_n\phi^{\alpha}}{1-\bar{P}_n\phi^{\alpha}} = \frac{1}{1-\bar{P}_n\phi^{\alpha}}$$

and

$$\lim_{\varepsilon\to\infty}\varepsilon^{\alpha} P\left(|W_{n,t}| > \varepsilon\right) = \frac{2}{1-\bar{P}_n\phi^{\alpha}},$$

hence $W_{n,t} \sim$ (B.1).

Step 2 ($X_{n,t} \pm X_{n,t-h}$ (B.1)): It is straightforward to show

$$
\begin{aligned}
&X_{n,t} + X_{n,t-h} \\
&\quad = \sum_{i=0}^{\infty}\phi^i u_{t-i}\prod_{j=1}^{i} I(|u_{t-j}| \leq v_n) + \sum_{i=0}^{\infty}\phi^i u_{t-h-i}\prod_{j=1}^{i} I(|u_{t-h-j}| \leq v_n) \\
&\quad = u_t + u_{t-h} + \sum_{i=1}^{h-1}\phi^i u_{t-i}\prod_{j=1}^{i} I(|u_{t-j}| \leq v_n) \\
&\qquad + \sum_{i=h}^{\infty}\phi^i u_{t-i}\prod_{j=1}^{i} I(|u_{t-j}| \leq v_n) + \sum_{i=1}^{\infty}\phi^i u_{t-h-i}\prod_{j=1}^{i} I(|u_{t-h-j}| \leq v_n)
\end{aligned}
$$

$$
\begin{aligned}
&= u_t + u_{t-h}\left(1 + \phi^h \prod_{j=1}^{h} I(|u_{t-j}| \le v_n)\right) + \sum_{i=1}^{h-1} \phi^i u_{t-i} \prod_{j=1}^{i} I\left(|u_{t-j}| \le v_n\right) \\
&\quad + \sum_{i=1}^{\infty} \phi^{i+h} u_{t-h-i} \prod_{j=1}^{i+h} I(|u_{t-j}| \le v_n) + \sum_{i=1}^{\infty} \phi^i u_{t-h-i} \prod_{j=1}^{i} I\left(|u_{t-h-j}| \le v_n\right) \\
&= u_t + u_{t-h}\left(1 + \phi^h \prod_{j=1}^{h} I(|u_{t-j}| \le v_n)\right) + \sum_{i=1}^{h-1} \phi^i u_{t-i} \prod\nolimits_{j=1}^{i} I(|u_{t-j}| \le v_n) \\
&\quad + \sum_{i=1}^{\infty} \phi^i u_{t-h-i} \prod_{j=1}^{i} I(|u_{t-h-j}| \le v_n) \left(1 + \phi^h \prod_{j=1}^{h} I(|u_{t-j}| \le v_n)\right)
\end{aligned}
$$

Arguments similar to Step 1 imply

$$
\begin{aligned}
&\varepsilon^\alpha P\left(u_{t-h}\left[1 + \phi^h \prod\nolimits_{j=1}^{h} I(|u_{t-j}| \le v_n)\right] > \varepsilon\right) \\
&\qquad = \varepsilon^\alpha P\left(v_n \ge u_{t-h} > \varepsilon/(1+\phi^h)\right) \times P_n^{h-1} \\
&\qquad\qquad + \varepsilon^\alpha P\left(u_{t-h} > \max\{\varepsilon, v_n\}\right) \\
&\qquad \to 1 \quad \text{as} \quad \varepsilon \to \infty, \quad \forall n \ge 1,
\end{aligned}
$$

and

$$
\begin{aligned}
&P\left(\sum_{i=1}^{h-1} \phi^i u_{t-i} \prod_{j=1}^{i} I_{n,t-j} > \varepsilon\right) \\
&\qquad = \sum_{i=1}^{h-1} P\left(\sum_{j=1}^{i} \phi^j u_{t-i} > \varepsilon \cap \left\{u_{t-j} \le v_n\right\}_{j=1}^{i}\right) \times \bar{P}_n \\
&\qquad \sim \sum_{i=1}^{h-1} \sum_{j=1}^{i} \phi^{j\alpha} P\left(v_n \ge u_{t-i} > \varepsilon\right) \times P_n^{i-1} \times \bar{P}_n \\
&\qquad = \varepsilon^{-\alpha} \bar{P}_n \sum_{i=1}^{h-1} P_n^{i-1} \sum_{j=1}^{i} \phi^{j\alpha} \left(1 - (\varepsilon/v_n)^\alpha + O\left((\varepsilon/v_n)^\alpha\right)\right)_+ \\
&\qquad \to 0 \text{ as } \; n \to \infty,
\end{aligned}
$$

and

$$\varepsilon^{\alpha} P\left(\sum_{i=1}^{\infty}\phi^{i}u_{t-h-i}\prod_{j=1}^{i}I(|u_{t-h-j}|\leq v_n)\left[1+\phi^{h}\prod_{j=1}^{h}I(|u_{t-j}|\leq v_n)\right]>\varepsilon\right)$$

$$= \varepsilon^{\alpha}\sum_{i=1}^{\infty}P\left(\sum_{j=1}^{i}\phi^{j}u_{t-h-j}\left[1\pm\phi^{h}\right]>\varepsilon\cap\{u_{t-h-j}\leq v_n\}_{j=1}^{i}\right)\times P_n^{h-1}\times\bar{P}_n$$

$$+\varepsilon^{\alpha}\sum_{i=1}^{\infty}P\left(\sum_{j=1}^{i}\phi^{j}u_{t-h-j}>\varepsilon\cap\{u_{t-h-j}\leq v_n\}_{j=1}^{i}\right)\times\left(1-P_n^{h-1}\right)\times\bar{P}_n$$

$$\sim \left[1\pm\phi^{h}\right]^{\alpha}P_n^{h-1}\bar{P}_n\sum_{i=1}^{\infty}P_n^{i}\sum_{j=1}^{i}\phi^{j\alpha}\varepsilon^{\alpha}P\left(v_n\geq u_{t-h-j}>\varepsilon\right)$$

$$+\left(1-P_n^{h-1}\right)\bar{P}_n\sum_{i=1}^{\infty}P_n^{i}\sum_{j=1}^{i}\phi^{j\alpha}\varepsilon^{\alpha}P\left(v_n\geq u_{t-h-j}>\varepsilon\right)$$

$$\to 0 \quad \text{as} \quad \varepsilon\to\infty,$$

where we use

$$\varepsilon^{\alpha}P\left(v_n\geq u_{t-h-j}>\varepsilon\right)\to 0 \quad \text{as} \quad \varepsilon\to\infty$$

and

$$P_n^{h-1}\bar{P}_n\sum_{i=1}^{\infty}P_n^{i}\sum_{j=1}^{i}\phi^{j\alpha} = \frac{\phi^{\alpha}}{1-\phi^{\alpha}}P_n^{h}\bar{P}_n\left(\frac{1}{\bar{P}_n}-\frac{\phi^{\alpha}}{1-P_n\phi^{i\alpha}}\right)$$

$$= \frac{\phi^{\alpha}P_n^{h}}{1-P_n\phi^{\alpha}}\to\frac{\phi^{\alpha}}{1-\phi^{\alpha}}<\infty.$$

Therefore

$$\varepsilon^{\alpha}P\left(X_{n,t}+X_{n,t-h}>\varepsilon\right)\to 2$$

An identical argument can be used to show

$$\lim_{\varepsilon\to\infty}\varepsilon^{\alpha}P(X_{n,t}\pm X_{n,t-h}>\varepsilon)=\lim_{\varepsilon\to\infty}\varepsilon^{\alpha}P(X_{n,t}\pm X_{n,t-h}<-\varepsilon)=2,$$

hence

$$\lim_{\varepsilon\to\infty}\varepsilon^{\alpha}P(|X_{n,t}\pm X_{n,t-h}|>\varepsilon)=4.$$

Step 3 ($W_{n,t}\pm W_{n,t-h}\sim$ (B.1)): Similarly

$$
\begin{aligned}
&W_{n,t} \pm W_{n,t-h} \\
&\quad = u_t \pm u_{t-h} \left(1 \pm \phi^h \prod_{j=1}^{h} I(|u_{t-j}| > v_n) \right) + \sum_{i=1}^{h-1} \phi^i u_{t-i} \prod_{j=1}^{i} I(|u_{t-j}| > v_n) \\
&\quad \pm \sum_{i=1}^{\infty} \phi^i u_{t-h-i} \prod_{j=1}^{i} I(|u_{t-h-j}| > v_n) \left(1 \pm \phi^h \prod_{j=1}^{h} I(|u_{t-j}| > v_n) \right),
\end{aligned}
$$

where

$$
\begin{aligned}
&\varepsilon^\alpha P \left(u_{t-h} \left[1 \pm \phi^h \prod_{j=1}^{h} I(|u_{t-j}| > v_n) \right] > \varepsilon \right) \\
&\qquad = \varepsilon^\alpha P \left(u_{t-h} \left(1 \pm \phi^h \right) > \max\{\varepsilon, v_n\} \right) \times \bar{P}_n^{h-1} \\
&\qquad\qquad + \varepsilon^\alpha P \left(u_{t-h} > \max\{\varepsilon, v_n\} \right) \times \left(1 - \bar{P}_n^{h-1} \right) \\
&\qquad\qquad + \varepsilon^\alpha P \left(v_n \geq u_{t-h} > \varepsilon \right) \\
&\qquad \sim \varepsilon^\alpha \left(1 \pm \phi^h \right)^\alpha P \left(u_{t-h} > \max\{\varepsilon, v_n\} \right) \times \bar{P}_n^{h-1} \\
&\qquad\qquad + \varepsilon^\alpha P \left(u_{t-h} > \max\{\varepsilon, v_n\} \right) \times \left(1 - \bar{P}_n^{h-1} \right) \\
&\qquad\qquad + \varepsilon^\alpha P \left(v_n \geq u_{t-h} > \varepsilon \right) \\
&\qquad \to \left(1 \pm \phi^h \right)^\alpha \times \bar{P}_n^{h-1} + \left(1 - \bar{P}_n^{h-1} \right) + 0 \quad \text{as } \varepsilon \to \infty,
\end{aligned}
$$

and

$$
\begin{aligned}
&\varepsilon^\alpha P \left(\sum_{i=1}^{h-1} \phi^i u_{t-i} \prod_{j=1}^{i} I(|u_{t-j}| > v_n) > \varepsilon \right) \\
&\quad \sim \varepsilon^\alpha P \left(u_t > \max\{\varepsilon, v_n\} \right) \times P_n \times \sum_{i=1}^{h-2} \sum_{j=1}^{i} \bar{P}_n^i \phi^{j\alpha} + \varepsilon^\alpha P \left(u_t > \max\{\varepsilon, v_n\} \right) \times \bar{P}_n^h \sum_{i=1}^{h} \phi^{i\alpha} \\
&\quad = \varepsilon^\alpha P \left(u_{t-i} > \max\{\varepsilon, v_n\} \right) \times \left[P_n \phi^\alpha \sum_{i=1}^{h-2} \bar{P}_n^i \left(\frac{1 - \phi^{i\alpha}}{1 - \phi^\alpha} \right) + \bar{P}_n^h \left(\frac{1 - \phi^{h\alpha}}{1 - \phi^\alpha} \right) \right] \\
&\quad \to P_n \phi^\alpha \sum_{i=1}^{h-2} \bar{P}_n^i \left(\frac{1 - \phi^{i\alpha}}{1 - \phi^\alpha} \right) + \bar{P}_n^h \left(\frac{1 - \phi^{h\alpha}}{1 - \phi^\alpha} \right) \quad \text{as } \varepsilon \to \infty \\
&\quad = \frac{\phi^\alpha}{1 - \phi^\alpha} \bar{P}_n \left(1 - \bar{P}_n^{h-2} \right) - \frac{\phi^\alpha P_n}{1 - \phi^\alpha} \left(\frac{\bar{P}_n \phi^\alpha - \bar{P}_n^{h-1} \phi^{\alpha(h-1)}}{1 - \bar{P}_n \phi^\alpha} \right) + \bar{P}_n^h \left(\frac{1 - \phi^{h\alpha}}{1 - \phi^\alpha} \right)
\end{aligned}
$$

and

$$\varepsilon^{\alpha} P\left(\sum_{i=1}^{\infty} \phi^{i} u_{t-h-i} \prod_{j=1}^{i} I(|u_{t-h-j}| > v_n) \left[1 \pm \phi^{h} \prod_{j=1}^{h} I(|u_{t-j}| > v_n)\right] > \varepsilon\right)$$

$$= \varepsilon^{\alpha} \sum_{i=1}^{\infty} P\left(\sum_{j=1}^{i} \phi^{j} u_{t-h-j}\left[1 \pm \phi^{h}\right] > \varepsilon \cap \{u_{t-h-j} > v_n\}_{j=1}^{i}\right) \times \bar{P}_n^{h-1} \times P_n$$

$$+ \varepsilon^{\alpha} \sum_{i=1}^{\infty} P\left(\sum_{j=1}^{i} \phi^{j} u_{t-h-j} > \varepsilon \cap \{u_{t-h-j} > v_n\}_{j=1}^{i}\right) \times \left(1 - \bar{P}_n^{h-1}\right) \times P_n$$

$$\sim \left[1 \pm \phi^{h}\right]^{\alpha} P_n \bar{P}_n^{h-1} \sum_{i=1}^{\infty} \bar{P}_n^{i} \sum_{j=1}^{i} \phi^{j\alpha} \varepsilon^{\alpha} P\left(u_{t-h-j} > \varepsilon\right)$$

$$+ \left(1 - \bar{P}_n^{h-1}\right) P_n \sum_{i=1}^{\infty} \bar{P}_n^{i} \sum_{j=1}^{i} \phi^{j\alpha} \varepsilon^{\alpha} P\left(u_{t-h-j} > \varepsilon\right)$$

$$\to \phi^{\alpha}\left[1 \pm \phi^{h}\right]^{\alpha} \bar{P}_n^{h} \frac{1}{\left(1 - \bar{P}_n \phi^{\alpha}\right)} + \phi^{\alpha}\left(1 - \bar{P}_n^{h-1}\right) \bar{P}_n \frac{1}{\left(1 - \bar{P}_n \phi^{\alpha}\right)}$$

$$= \frac{\phi^{\alpha}}{1 - \bar{P}_n \phi^{\alpha}}\left[\left(1 \pm \phi^{h}\right)^{\alpha} \bar{P}_n^{h} + \bar{P}_n - \bar{P}_n^{h}\right].$$

Therefore

$$\lim_{\varepsilon \to \infty} \varepsilon^{\alpha} P\left(W_{n,t} \pm W_{n,t-h} > \varepsilon\right)$$

$$= 1 + \left(1 \pm \phi^{h}\right)^{\alpha} \times \bar{P}_n^{h-1} + \left(1 - \bar{P}_n^{h-1}\right)$$

$$+ \frac{\phi^{\alpha}}{1 - \phi^{\alpha}} \bar{P}_n\left(1 - \bar{P}_n^{h-2}\right) - \frac{\phi^{\alpha} P_n}{1 - \phi^{\alpha}}\left(\frac{\bar{P}_n \phi^{\alpha} - \bar{P}_n^{h-1} \phi^{\alpha(h-1)}}{1 - \bar{P}_n \phi^{\alpha}}\right) + \bar{P}_n^{h}\left(\frac{1 - \phi^{h\alpha}}{1 - \phi^{\alpha}}\right)$$

$$+ \frac{\phi^{\alpha}}{1 - \bar{P}_n \phi^{\alpha}}\left(\left(1 \pm \phi^{h}\right)^{\alpha} \bar{P}_n^{h} + \bar{P}_n - \bar{P}_n^{h}\right).$$

Claim (ii): Use Steps 1 and 2 of Claim (i) above to deduce

$$\lim_{n \to \infty} \rho_n^{(0)}(X_{n,t}, X_{n,t-h}) = 1 - \frac{4}{2 \times 2} = 0, \ \ \forall h \geq 0.$$

Finally, Steps 1 and 3 of Claim (i) imply

$$\begin{aligned}
&\rho_n^{(0)}(W_{n,t}, W_{n,t-h}) \\
&= \frac{1}{2}\left[\bar{P}_n^{h-1} + \phi^\alpha \bar{P}_n \frac{\phi^\alpha \bar{P}_n^h \left(1-\phi^{\alpha h}\right)}{(1-\phi^\alpha)} - \left(1-\phi^h\right)^\alpha \bar{P}_n^{h-1}\right] \\
&= \frac{1}{2}\left[\bar{P}_n^{h-1} + \phi^\alpha \bar{P}_n \frac{\phi^\alpha \bar{P}_n^h \left(1-\phi^{\alpha h}\right)}{(1-\phi^\alpha)} - \left(1-\phi^h\right)^\alpha \bar{P}_n^{h-1}\right] \\
&\to \frac{1}{2}\left[1-(1-\phi)^\alpha\right] \quad \text{if } h = 1 \\
&\to 0 \quad \text{if } h > 1.
\end{aligned}$$

■

Proof of Lemma B.4.

Claims (i) and (ii): The process $\{X_t\}$ has a convergent series representation

$$X_t = \sum_{j=0}^{\infty} \beta^j X_t^{(j)}, \quad \text{where } X_t^{(0)} = u_t \text{ and } X_t^{(j)} = u_{t-j}^2 \left(\prod_{i=1}^{j-1} u_{t-i}\right), j \geq 1.$$

See Davis and Resnick (1996) and Resnick and Stărică (1998). Corollary 2.4 of Davis and Resnick (1996) implies $X_t^{(j)}$ and X_t have regularly varying tails (B,1), where $X_t^{(j)}$, $j \geq 1$, and X_t have tail index $\alpha/2$. In particular, only $X_t^{(j)} = u_{t-j}^2(\sqcap_{i=1}^{j-1} u_{t-i})$ provides relevant tail information, and each $X_t^{(i)}$ is independent of any other $X_t^{(j)}$, $i < j$. Therefore Lemma B.5.v delivers the desired result. ■

REFERENCES cited only in this appendix)

[1] Davis, R., and S. Resnick, 1996, Limit Theory for Bilinear Processes with Heavy-Tailed Noise, Annals of Applied Probability 6, 1191–1210.

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[3] Resnick, S. and C. Stărică, 1998, Tail Index Estimation for Dependent Data, The Annals of Applied Probability 8, 1156-1183.