Technical Appendix for "On Functional Central Limit Theorems for Dependent, Heterogenous Arrays with Applications to Tail Index and Tail Dependence Estimation"

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In this appendix we characterize sufficient conditions for weak convergence in $D[0,1]^k$ by compressing results due to Bickel and Wichura (1971) and Neuhaus (1971). See Section A.1. Section A.2 contains the omitted proofs of Lemmas 4.1, 4.2, 5.2, 5.3 and A.7-A.9.

We assume

(A.1) $\bar{F}_t(\lambda x)/\bar{F}_t(x) \rightarrow \lambda^{-\alpha}$ as $x \rightarrow \infty \quad \Leftrightarrow \quad \bar{F}_t(x) = x^{-\alpha}L(x), \quad x > 0,$

for some slowly varying function $L(x)$, and recall

(A.2) $(n/k_n)P(X_t > b_n) \rightarrow 1.$

Define $\{U_{n,t}, U^*_{n,t}(u),\}$

$$U_{n,t} := k_n^{-1/2}\left((\ln X_t/b_n)_+ - E[(\ln X_t/b_n)_+]\right)$$
$$U^*_{n,t}(u) := k_n^{-1/2}(I(X_t > b_n e^u) - P(X_t > b_n e^u)), \; u \in \mathbb{R}_+,$$

where $\{k_n, b_n\}$ satisfy (B.2).

A.1 WEAK CONVERGENCE IN $\mathbf{D[0,1]^k}$ Consider some $D([0,1]^k)$-valued process $X_n(\theta)$, $\theta \in [0,1]^k$. The following is necessarily brief. Classic sources are Bickel and Wichura (1971), Neuhaus (1971) and Billingsley (1968).

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Write $D_k := (D([0,1]^k), d_k(x,y))$, the space of right continuous functions $f : [0,1]^k \to \mathbb{R}$ with left limits metrized with $d_k(x,y)$, a multidimensional version of Billingsley's (1968) extended Skorokhod J_1-metric. Denote by $\Lambda_k = \Lambda \times \cdots \times \Lambda$ the collection of homeomorphisms[1] $\lambda = [\lambda_i]_{i=1}^k$ from $[0,1]^k$ to $[0,1]^k$, where $\lambda_i(0) = 0$, $\lambda_i(1) = 1$. Then

$$d_k(x,y) = \inf_{\lambda \in \Lambda, \varepsilon > 0} \left\{ \|\lambda\|^\circ \leq \varepsilon, \quad \sup_{\theta \in [0,1]^k} |x(\theta) - y(\lambda(\theta))| \leq \varepsilon \right\},$$

where

$$\|\lambda\|^\circ := \max_{1 \leq i \leq k} \left\{ \sup_{\theta_1 \neq \theta_2} \left| \ln \left([\lambda_i(\theta_1) - \lambda_i(\theta_2)] \, / \, [\theta_{1,i} - \theta_{2,i}] \right) \right| \right\}.$$

$D([0,1]^k$ is separable and complete under $d_k(x,y)$.

Let $\delta \in [0,1]$ and write $X_n(\cdot, \theta_i) := X_n(\{\theta_j\}_{j \neq i}, \theta_i)$. Define the moduli w and w''

$$w(X_n, \delta) := \sup_{|\theta - \theta'| \leq \delta} \left| X_n(\theta) - X_n(\theta') \right| \quad \text{and} \quad w''(X_n, \delta) := \max_{1 \leq i \leq k} \{ w_i''(X_n, \delta) \}$$

where

$$w_i''(X_n, \delta) := \sup_{\substack{\theta_{i,1} \leq \theta_i \leq \theta_{i,2} \\ \theta_{i,2} - \theta_{i,1} \leq \delta}} \{ |X_n(\cdot, \theta_i) - X_n(\cdot, \theta_{i,1})| \wedge |X_n(\cdot, \theta_i) - X_n(\cdot, \theta_{i,2})| \}.$$

Prokhorov's (1956) theorem equates relative compactness with tightness. The following claim can therefore be deduced from an Arzelá-Ascoli-type result relating stochastic equicontinuity to relative compactness, and standard inequality arguments. See the corollary to Theorem 2 in Bickel and Wichura (1971).

PROPOSITION 1 *Let $X_n(\xi, u) \to X(\xi, u)$ point-wise in finite dimensional distributions. If for each $i = 1...k$ and every $\varepsilon > 0$,*

$$\lim_{\delta \to 0} P\left(|X_n(\cdot, \theta_i = 1) - X_n(\cdot, \theta_i = 1 - \delta)| > \varepsilon \right) = 0,$$

and if for every $\varepsilon > 0$ and $\eta > 0$ there exists some $\delta \in (0,1)$ and $n_0 \in \mathbb{N}$ such that

$$P\left(w_i''(X_n, \delta) > \varepsilon / k \right) \leq \eta / k, \, \forall n \geq n_0,$$

then $X_n(\xi, u) \Rightarrow X(\xi, u)$ on D_k.

[1] Each $\lambda_i : [0,1] \to [0,1]$ is increasing, one-to-one, onto, continuous with a continuous inverse λ_i^{-1}.

A.2 OMITTED PROOFS Recall $g_n = o(K_n(\underline{\xi}))$, $r_n(\xi) = [K_n(\xi)/g_n]$ and $M_{n,i} := \max_{(i-1)g_n+1 \leq t \leq ig_n} c_{n,t}^2$ for some non-stochastic double array $\{c_{n,t}\}$.

LEMMA 4.1 *Let $K_n(\xi)/n \to a(\xi) > 0$ a finite constant function. There does* not *exist an array $\{c_{n,t}\}$ that satisfies $M_{n,i} = o(g_n^{-1/2})$ and $\sum_{i=1}^{r_n(\xi)} M_{n,i}^2 = O(g_n^{-1})$ such that $\{U_{n,t}/c_{n,t}\}$ or $\{U_{n,t}^*(u)/c_{n,t}\}$ are L_r-bounded for all t and any $r > 2$.*

Proof. By Lemma A.2 of Hill (2005)[2]

$$\lim_{n\to\infty}\left(\frac{n}{k_n}\right)^{1/r}\left\|I\left(X_t > b_n e^u\right) \ - \ P\left(X_t > b_n e^u\right)\right\|_r \leq Ke^{-\alpha u},\ \forall r \geq 1,$$

hence $\forall r \geq 1$

$$\left\|U_{n,t}^*(u)\right\|_r = O\left(k_n^{-1/2}(n/k_n)^{-1/r}\right).$$

Now $||U_{n,t}^*(u)/c_{n,t}||_r = O(1)$ $\forall t$ *if and only if* $c_{n,t} = O(k_n^{1/2}(n/k_n)^{1/r})$ $\forall t$. de Jong and Davidson (2000: Theorem 3.1) require $\sum_{i=1}^{r_n(\xi)} M_{n,i}^2 = O(g_n^{-1})$, hence $||U_{n,t}^*(u)/c_{n,t}||_r = O(1)$ $\forall t$ *and* $\sum_{i=1}^{r_n(\xi)} M_{n,i}^2 = O(g_n^{-1})$ *if and only if*

$$O(r_n(\xi)^{1/2}k_n^{-1/2}(n/k_n)^{1/r}) = O(g_n^{-1/2}).$$

But by construction

$$r_n(\xi)^{1/2}k_n^{-1/2}(n/k_n)^{1/r} \sim \left(K_n(\xi)/n\right)^{1/2} g_n^{-1/2}(n/k_n)^{1/2-1/r} = O(g_n^{-1/2})$$

if and only if $r = 2$ when $K_n(\xi)/n \to a(\xi)$, a finite positive function. An identical argument applies to $\{U_{n,t}/c_{n,t}\}$, cf. Lemma A.2 of Hill (2005b). ■

Recall $S_k := \sum_{t=1}^{k}[I(X_t > b_n e^u) - P(X_t > b_n e^u)]$ and $\sigma_n^2 := E[S_n^2]$.

LEMMA 4.2 *If X_t is iid with distribution tail (11) then $\sigma_n^2 \to \infty$, $||E(S_n|X_0)||_2 = 0$, and σ_n^2/n is regularly varying with index 1. Further, σ_n^2/k_n is slowly varying.*

Proof. Since X_t is iid $E(S_n|X_0) = E(S_n) = 0$, so $||E(S_n|X_0)||_2 = 0$.

Further, $X_t \overset{iid}{\sim}$ (A.1) and the construction of b_n imply

$$\begin{aligned}\sigma_n^2 &= \sum_{t=1}^{n} E\left(I(X_t > b_n e^u) - P(X_t > b_n e^u)\right)^2 \qquad \text{(A.3)}\\ &= n\left[P(X_t > b_n e^u) - P(X_t > b_n e^u)^2\right] \sim n \times \frac{k_n}{n} = k_n \to \infty.\end{aligned}$$

[2] This paper has since been published in Econometric Theory; see *Hill, J.B. (2010). On Tail Index Estimation for Dependent, Heterogeneous Data, Econometric Theory: in press.*

Define

$$l(n) := \frac{\sigma_n^2}{n}.$$

We now prove $l(\lambda n)/l(n) \to \lambda^{-1}$ for all $\lambda > 0$. First, use (A.1) and the construction

$$\frac{\lambda n}{k_n} P\left(X_t > b_{\lambda n}\right) \to 1$$

to deduce

$$\frac{\lambda n}{k_n} P\left(X_t > b_{\lambda n}\right) \sim \frac{\lambda n}{k_n} P\left(X_t > \lambda^{-1/\alpha} b_{\lambda n}\right) \to 1$$

hence

$$b_{\lambda n} = \lambda^{1/\alpha} b_n.$$

But this implies from (A.1) and (A.3)

$$\begin{aligned} l(\lambda n) &= \frac{\sigma_{\lambda n}^2}{\lambda n} \\ &= P\left(X_t > b_{\lambda n} e^u\right) - P\left(X_t > b_{\lambda n} e^u\right)^2 \\ &\sim \lambda^{-1} P\left(X_t > b_n e^u\right) - \lambda^{-2} P\left(X_t > \lambda^{1/\alpha} b_n e^u\right)^2 \end{aligned}$$

hence

$$\begin{aligned} \frac{n}{k_n} l(\lambda n) &\sim \lambda^{-1} \frac{n}{k_n} P\left(X_t > b_n e^u\right) - \lambda^{-2} \frac{n}{k_n} P\left(X_t > \lambda^{1/\alpha} b_n e^u\right)^2 \\ &\to \lambda^{-1} e^{-\alpha u} \end{aligned}$$

and

$$\frac{n}{k_n} l(n) = \frac{n}{k_n} \frac{\sigma_n^2}{n} = \frac{n}{k_n} P\left(X_t > b_n e^u\right) - \frac{n}{k_n} P\left(X_t > b_n e^u\right)^2 \to e^{-\alpha u}.$$

Therefore

$$\frac{l(\lambda n)}{l(n)} = \frac{(n/k_n)\, l(\lambda n)}{(n/k_n)\, l(n)} \to \lambda^{-1}.$$

Now define

$$\tilde{l}(n) := \frac{\sigma_n^2}{k_n}$$

and note by construction for any $\lambda > 0$

$$\begin{aligned} \tilde{l}(\lambda n) &= \frac{\sigma_{\lambda n}^2}{k_{\lambda n}} \\ &= \frac{\lambda n}{k_{\lambda n}} \left[P\left(X_t > b_{\lambda n} e^u\right) - P\left(X_t > b_{\lambda n} e^u\right)^2\right] \sim e^{-\alpha u} + o(1). \end{aligned}$$

Since λ is arbitrarily it follows instantly $\tilde{l}(\lambda n)/\tilde{l}(n) \to 1$, hence $\tilde{l}(n) := \sigma_n^2/k_n$ is slowly varying. ■

Recall

$$Z_{n,t}(\lambda, u, h) := k_n^{1/2}\lambda_i U^*_{i,n,t-h}(u_1) \times U^*_{j,n,t}(u_2).$$

LEMMA 5.2 *For all $r \geq 1$ and finite $h \geq 1$, $||Z_{n,t}(\lambda, u, h)||_r = O(k_n^{-1/2}(k_n/n)^{1/r})$ $= O(n^{-a(r)})$, $a(1) > 1/2$, where $a(2) = 1/2$, and $2a(2r) > a(r)$. Further, $a(r) > 1/r$ for all $r > 2$.*

Proof. Use the Minkowski and Cauchy-Schwartz inequalities to get for any $r \geq 1$

$$\|Z_{n,t}(\lambda, u, h)\|_r \leq k_n^{1/2}\sum_{i=1}^{h}|\lambda_i| \times \left\|U^*_{i,n,t-h}(u_1)\right\|_{2r} \times \left\|U^*_{j,n,t}(u_2)\right\|_{2r}.$$

By Lemma A.1 of Hill (2005b)

$$\left\{\left\|U^*_{i,n,t-h}(u_1)\right\|_{2r}, \left\|U^*_{j,n,t}(u_2)\right\|_{2r}\right\} = O((k_n/n)^{1/r}k_n^{-1/2}),$$

hence

$$\begin{aligned}\|Z_{n,t}(\lambda, u, h)\|_r &= O\left(k_n^{1/2}(k_n/n)^{1/2r}k_n^{-1/2}(k_n/n)^{1/2r}k_n^{-1/2}\right)\\ &= O\left((k_n/n)^{1/r}k_n^{-1/2}\right) = O\left(k_n^{-(1/2-1/r)}n^{-1/r}\right).\end{aligned}$$

The remaining argument $O(k_n^{-(1/2-1/r)}n^{-1/r}) = O(n^{-a(r)})$ simply mimics Lemma 3.1. ■

LEMMA 5.3 *Let $\{X_{1,t}, X_{2,t}\}$ satisfy Assumption 2.a with $q = 4$. Then $\{Z_{n,t}(\lambda, u, h)\}$ is L_2-NED on $\{\digamma_{n,t}\}$ with constants $d_{n,t}(\lambda, u)$, $\sup_{u\in[0,1]^2,\lambda'\lambda=1,t\geq 1} d_{n,t}(\lambda, u)$ $= O(k_n^{-1/2}(k_n/n)^{1/r})$ and coefficients $\varphi_{l_n} = o((k_n/n)^{1/2-1/r}l_n^{-1/2})$. Moreover, $E(Z_{n,t}(\lambda, u, h))^2 = O(1)$.*

Proof.

Step 1 (NED): Minkowski, Cauchy-Schwartz, and conditional Jensen's inequalities imply

$$\begin{aligned}&\left\|ZZ_{n,t}(u,h) - E[ZZ_{n,t}(u,h)|\digamma_{n,t-l_n}^{t+l_n}]\right\|_2\\ &\quad= k_n^{-1/2}\left\|Z_{1,n,t}(u_1)Z_{2,n,t-h}(u_2) - E[Z_{1,n,t}(u_1)Z_{2,n,t-h}(u_2)|\digamma_{n,t-l_n}^{t+l_n}]\right\|_2\\ &\quad\leq k_n^{-1/2}\left(\|Z_{1,n,t}(u_1)\|_4 + \left\|Z_{1,n,t}(u_1) - E[Z_{1,n,t}(u_1)|\digamma_{n,t-l_n}^{t+l_n}]\right\|_4\right)\\ &\qquad\quad\times\left\|Z_{2,n,t-h}(u_2) - E[Z_{2,n,t-h}(u_2)|\digamma_{n,t-l_n}^{t+l_n}]\right\|_4\\ &\qquad\quad+ \|Z_{2,n,t-h}(u_2)\|_4\, k_n^{-1/2}\left\|Z_{1,n,t}(u_2) - E[Z_{1,n,t}(u_1)|\digamma_{n,t-l_n}^{t+l_n}]\right\|_4\\ &\quad\leq 3\,\|Z_{1,n,t}(u_1)\|_4 \times d_{2,n,t-h}(u_1) \times \varphi_{2,l_n} + \|Z_{2,n,t-h}(u_2)\|_4 \times d_{1,n,t}(u_1) \times \varphi_{1,l_n}.\end{aligned}$$

From Lemma 3.1 each $\|Z_{i,n,t}(u_i)\|_4 = O(n^{-a(4)})$, $a(4) > 1/4$, and by construction $d_{i,n,t}(u) = O(n^{-a(r)})$. Thus, for some finite $K > 0$

$$\left\| ZZ_{n,t}(u,h) - E[ZZ_{n,t}(u,h)|\digamma_{n,t-l_n}^{t+l_n}] \right\|_2 \leq \widetilde{dd}_{n,t}(u,h) \times \widetilde{\varphi\varphi}_{l_n}$$

$$\widetilde{dd}_{n,t}(u,i) = K \times n^{-a(4)} \times \max\{d_{1,n,t}(u_1), d_{2,n,t-h}(u_2)\} = O(n^{-a(4)-a(r)})$$
$$\widetilde{\varphi\varphi}_{l_n} = \max_{i=1,2}\{\varphi_{i,l_n}\} = o(n^{a(r)-1/4} \times l_n^{-1/2}).$$

Hence

$$\left\| ZZ_{n,t}(u,h) - E[ZZ_{n,t}(u,h)|\digamma_{n,t-l_n}^{t+l_n}] \right\|_2 \ \leq d_{n,t}(u,h) \times \varphi_{l_n}$$

where $d_{n,t}(u,h) := n^{a(4)} \times \widetilde{dd}_{n,t}(u,h) = O(n^{-a(r)})$ and $\varphi_{l_n} := n^{-a(4)} \times \widetilde{\varphi\varphi}_{l_n}$ $= o(n^{a(r)-1/2} \times l_n^{-1/2})$ due to $a(4) + 1/4 > 1/2$, cf. Lemma 3. This proves $\{ZZ_{n,t}(u,h)\}$ is L_2-NED.

Under the maintained assumptions the conditions of Theorem 17.8 of Davidson (1994) are met. Thus, any linear combination

$$ZZ_{n,t}(u,\lambda,h) := \sum_{i=1}^{h} \lambda_i ZZ_{n,t}(u,i)$$

is L_2-NED with constants $d_{n,t}(u,\lambda,h) = \max_{1\leq i\leq h}\{\lambda_i d_{n,t}(u,i)\} = O(n^{-a(r)})$ and coefficients $h \times \varphi_{l_n} = O(n^{a(r)-1/2}) \times O(l_n^{-1/2-\iota})$.

Step 2 (Mixingale): Using Step 1, Theorem 17.5 of Davidson (1994) implies $\{ZZ_{n,t}(u,\lambda,h), \digamma_{n,t}\}$ is an L_2-mixingale with size $-1/2$ and constants $cc_{n,t}$ $= O(n^{-1/2})$. If the base $\{\epsilon_t\}$ is F-Strong Mixing, then for $r > 2$

$$\begin{aligned} &\|ZZ_{n,t}(u,\lambda,h) - E[ZZ_{n,t}(u,\lambda,h)|\digamma_{n,t-l_n}]\|_2 \\ &\quad \leq \max\{\|ZZ_{n,t}(u,\lambda,h)\|_r, d_{n,t}(u,\lambda,h)\} \times \max\{6\varepsilon_{l_n}^{1/2-1/r},\ \varphi_{l_n}\}. \end{aligned}$$

By Lemma 5.2 $||ZZ_{n,t}(u,\lambda,h)||_r = O(k_n^{-1/2}(k_n/n)^{1/r}) = O(n^{-a(r)})$, where $a(1)$ $> 1/2$, $a(2) = 1/2$, and $2a(2r) > a(r)$ in general, and $a(r) > 1/r$ for $r > 2$. Thus, for some finite $K > 0$

$$\begin{aligned} &\|ZZ_{n,t}(u,\lambda,h) - E[ZZ_{n,t}(u,\lambda,h)|\digamma_{n,t-l_n}]\|_2 \\ &\quad \leq \max\{\|ZZ_{n,t}(u,\lambda,h)\|_r, d_{n,t}(u,\lambda,h)\} \times \max\{6\varepsilon_{l_n}^{1/2-1/r},\ \varphi_{l_n}\} \\ &\quad \leq Kn^{-1/2} \times \max\left\{ \left(n^{(1/2-a(r))2r/(r-2)}\varepsilon_{l_n}\right)^{1/2-1/r},\ n^{1/2-a(r)}\varphi_{l_n} \right\} \\ &\quad = c_{n,t}(u,\lambda,h) \times \psi_{l_n}, \end{aligned}$$

say, where $c_{n,t}(u,\lambda,h) = O(n^{-1/2})$, and $\sup_n \psi_{l_n} = O(l_n^{-1/2-\iota})$ for some small ι > 0 follows from the F-Mixing and FE-NED coefficient properties. A similar argument holds for the other mixingale bound $||ZZ_{n,t}(u,\lambda,h) - E[ZZ_{n,t}(u,\lambda,h)|\digamma_{n,t+l_n}]||_2$

$\leq c_{n,t}(u,\lambda,h) \times \psi_{l_n+1}$ and in the F-Uniform Mixing case.
Step 3 (Bound): $E(\sum_{t=1}^{n} ZZ_{n,t}(u,\lambda,h))^2 = O(\sum_{t=1}^{n} cc_{n,t}^2(u,\lambda,h)) = O(1)$ now follows from McLeish's (1975a) bound for L_2-mixingales with size $-1/2$. ■

LEMMA A.7 *Under the conditions of Theorem 5.1 for any* $u \in [0,1]$

$$k_n^{1/2}\left(\hat{\alpha}_{k_n}^{-1}(\xi) - \alpha^{-1}\right) = \sum_{t=1}^{[n\xi]}\left(U_{n,t} - \alpha^{-1}U_{n,t}^{*}(u^{[k_n\xi]^{-1/2}})\right) + o_p(1).$$

Proof. Write

(A.4)

$$\begin{aligned}
&k_n^{1/2}\left(\hat{\alpha}_{k_n}^{-1}(\xi) - \alpha^{-1}\right)\\
&\quad= k_n^{1/2}\left(\frac{1}{k_n}\sum_{i=1}^{[k_n\xi]}\ln X_{(i)}/X_{([k_n\xi]+1)} - \alpha^{-1}\right)\\
&\quad= k_n^{1/2}\left(\frac{1}{k_n}\sum_{i=1}^{[k_n\xi]}\ln X_{(i)}/b_{[n\xi]} - E\left(\frac{1}{k_n}\sum_{t=1}^{[n\xi]}\left(\ln X_t/b_{[n\xi]}\right)_+\right)\right)\\
&\qquad\quad - k_n^{1/2}\ln X_{([k_n\xi]+1)}/b_{[n\xi]} + k_n^{1/2}\left[E\left(\frac{1}{k_n}\sum_{t=1}^{[n\xi]}\left(\ln X_t/b_{[n\xi]}\right)_+\right) - \alpha^{-1}\right].
\end{aligned}$$

Use Assumption C and arguments in Hsing (1991: p. 1554) to deduce

$$\text{(A.5)}\quad E\left(\frac{1}{k_n}\sum_{t=1}^{[n\xi]}\left(\ln X_t/b_{[n\xi]}\right)_+ - \alpha^{-1}\right) = o\left(1/k_n^{1/2}\right).$$

Moreover, from Corollary 3.3 we deduce

$$\left\{\sum_{t=1}^{[n\xi]}U_{n,t}, \sum_{t=1}^{[n\xi]}U_{n,t}^{*}\left(u^{[\xi k_n]^{-1/2}}\right)\right\} \Rightarrow \{W_1(\xi), W_2(\xi)\}$$

jointly on $D[\underline{\xi},1]$, where each $W_i(\xi)$ is Gaussian, and $|\ln X_{([\rho[k_n\xi]])} - \ln b_n(\rho[k_n\xi])|$ $\rightarrow 0$ for all ρ in an arbitrary neighborhood of 1 by Lemma 1 of Hill (2005b). Therefore an argument identical to Theorem 2.2 of Hsing (1991: eq. 2.4-2.7) applies:

(A.6)

$$k_n^{1/2}\left(1/k_n\sum_{i=1}^{[k_n\xi]}\ln X_{(i)}/b_{[n\xi]} - E\left(1/k_n\sum_{t=1}^{[n\xi]}\left(\ln X_t/b_{[n\xi]}\right)_+\right)\right) \Rightarrow W_1(\xi)$$

$$\alpha \times \sum_{t=1}^{[n\xi]} U_{n,t}^*\left(u^{k_n^{-1/2}}\right) \Rightarrow W_2(\xi).$$

Together, (A.4)-(A.6) imply

$$k_n^{1/2}\left(\hat{\alpha}_{k_n}^{-1}(\xi) - \alpha^{-1}\right) = \sum_{t=1}^{[n\xi]}\left(U_{n,t} - \alpha^{-1}U_{n,t}^*(u^{[\xi k_n]^{-1/2}})\right) + o_p(1).$$

■

LEMMA A.8 *Let $\{X_t\}$ be L_p-NED, $p \in (0,2]$, on $\{\digamma_{n,t}\}$ with constants d_t, $\sup_{t\geq 1} d_t < \infty$, and coefficients ϑ_{l_n} of size $\lambda > 0$. Then $\{X_t\}$ satisfies the $\bar{L}_q$-FE-NED property*

$$k_n^{-1/2}\left\|I\left(X_t > b_n/u\right) - P\left(X_t > b_n/u|\digamma_{n,t-l_n}^{t+l_n}\right)\right\|_q \leq d_{n,t}(u) \times \varphi_{l_n}$$

for any $q \geq 2$ and some displacement sequence $\{l_n\}$, $l_n \to \infty$, with constants $d_{n,t}(u)$ and coefficients φ_{l_n} of size $\lambda \times \min\{p,1\}/(2q)$. In particular, $d_{n,t}(u)$ is Lebesgue measurable, $\sup_{u\in[0,1],t\geq 1} d_{n,t}(u) = O((k_n/n)^{1/r}k_n^{-1/2})$ and $\int_0^1 u^{-1}d_{n,t}(u)du = O(k_n^{-1/2}(k_n/n)^{1/r})$ for some $r \geq q$. Further $\sup_{n\geq 1}\varphi_{l_n} \in [0,1)$, and $\varphi_{l_n} = o((k_n/n)^{1/p-1/r}l_n^{-\lambda})$.

Proof. Define $\bar{I}_{n,t}(u) = I(X_t > b_n/u)$ and $v := -\ln u$ such that $\bar{I}_{n,t}(u) = I(X_t > b_n e^v)$. For some $\eta > 0$ that may depend on n to be chosen below and any $q \geq 2$, since $|\bar{I}_{n,t}(u) - E[\bar{I}_{n,t}(u)|\digamma_{t-l_n}^{t+l_n}]| \leq 1$ it follows

$$\begin{aligned}
&E\left|\bar{I}_{n,t}(u) - E\left[\bar{I}_{n,t}(u)|\digamma_{t-l_n}^{t+l_n}\right]\right|^q \\
&\quad\leq E\left(\bar{I}_{n,t}(u) - E\left[\bar{I}_{n,t}(u)|\digamma_{t-l_n}^{t+l_n}\right]\right)^2 \\
&\quad\leq E\left(\bar{I}_{n,t}(u) - I\left(E\left[\bar{I}_{n,t}(u)|\digamma_{t-l_n}^{t+l_n}\right] > b_n\right)\right)^2 I\left(\left|X_t - E\left[X_t|\digamma_{t-l_n}^{t+l_n}\right]\right| \leq \eta\right) \\
&\qquad + E\left(\bar{I}_{n,t}(u) - I\left(E\left[\bar{I}_{n,t}(u)|\digamma_{t-l_n}^{t+l_n}\right] > b_n\right)\right)^2 I\left(\left|X_t - E\left[X_t|\digamma_{t-l_n}^{t+l_n}\right]\right| > \eta\right) \\
&\quad\leq E\left[I\left(b_n e^v - \eta < X_t < b_n e^v + \eta\right)\right] + P\left(\left|X_t - E\left[X_t|\digamma_{t-l_n}^{t+l_n}\right]\right| > \eta\right) \\
&\quad= \left[\bar{F}_t\left(b_n e^v - \eta\right) - \bar{F}_t\left(b_n e^v + \eta\right)\right] + P\left(\left|X_t - E\left[X_t|\digamma_{t-l_n}^{t+l_n}\right]\right| > \eta\right) \\
&\quad\leq \left[\bar{F}_t\left(b_n e^v - \eta\right) - \bar{F}_t\left(b_n e^v + \eta\right)\right] + E\left(\left|X_t - E\left[X_t|\digamma_{t-l_n}^{t+l_n}\right]\right|^p\right)/\eta^p \\
&\quad\leq \left[\bar{F}_t\left(b_n e^v - \eta\right) - \bar{F}_t\left(b_n e^v + \eta\right)\right] + d_t^p\vartheta_{l_n}^p/\eta^p
\end{aligned}$$

The second inequality is due to the conditional expectations minimizing the mean-squared-error, and a trivial identity. The third follows from basic logic, and a trivial inequality involving the indicator function. The fourth is Markov's inequality, and the fifth follows from the L_p-NED property.

Without loss of generality let $\sup_{l\in\mathbb{N}}\vartheta_l \in [0,1)$. Put $\eta = e^u b_n \vartheta_{l_n}^{1/2}$, and use $\lim_{z\to\infty} z^p \bar{F}_t(z) = 0$ and the mean-value-theorem to deduce

$$\begin{aligned}\bar{F}_t\left(b_n e^v \pm \eta\right) &= \bar{F}_t\left(b_n\left(1\pm\vartheta_{l_n}^{1/2}\right)e^v\right)\\ &= b_n^{-p}\left(1\pm\vartheta_{l_n}^{1/2}\right)^{-p} e^{-pv}\times o_p(1)\\ &= K b_n^{-p}\vartheta_{l_n}^{1/2}e^{-pv}\times o_p(1).\end{aligned}$$

Therefore

$$\begin{aligned}&E\left(\bar{I}_{n,t}(u) - E\left[\bar{I}_{n,t}(u)|\digamma_{t-l_n}^{t+l_n}\right]\right)^2\\ &\le \left[\bar{F}_t\left(b_n\left(1-\varphi_{l_n}^{1/2}\right)e^v\right) - \bar{F}_t\left(b_n\left(1+\varphi_{l_n}^{1/2}\right)e^v\right)\right] + e^{-pv}b_n^{-p}d_t^p\vartheta_{l_n}^{p/2}\\ &= \left[\bar{F}_t\left(b_n\left(1-\varphi_{l_n}^{1/2}\right)e^v\right) - \bar{F}_t\left(b_n\left(1+\varphi_{l_n}^{1/2}\right)e^v\right)\right] + e^{-pv}b_n^{-p}d_t^p\vartheta_{l_n}^{p/2}\\ &= K b_n^{-p}e^{-pv}\left(\varphi_{l_n}^{1/2}\times o_p(1) + d_t^p\vartheta_{l_n}^{p/2}\right)\\ &\le K b_n^{-p}e^{-pv}d_t^p\vartheta_{l_n}^{\min\{p,1\}/2},\end{aligned}$$

hence as $n\to\infty$

$$\begin{aligned}k_n^{-1/2}\left\|\bar{I}_{n,t}(u) - E\left[\bar{I}_{n,t}(u)|\digamma_{t-l_n}^{t+l_n}\right]\right\|_q &\le \left(K k_n^{-1/2}b_n^{-p/q}d_t^{p/q}e^{-vp/q}\right)\times\vartheta_{l_n}^{\min\{p,1\}/(2q)} \qquad (A.7)\\ &= d_{n,t}(u)\times\varphi_{l_n}\end{aligned}$$

where for arbitrary $r\ge q$ and some $a>0$

$$\begin{aligned}d_{n,t}(u) &= K k_n^{-1/2}(k_n/n)^{1/r}d_t^{p/q}e^{-vp/q} = K k_n^{-1/2}(k_n/n)^{1/r}d_t^{p/q}u^{p/q} \qquad (A.8)\\ \varphi_{l_n} &= ((k_n/n)^{1/q-1/r})\left[\left(\frac{n}{k_n}\right)^{1/q}b_n^{-p/q}l_n^{-a}\right]l_n^a\vartheta_{l_n}^{\min\{p,1\}/(2q)}.\end{aligned}$$

Clearly $\sup_{u\in[0,1],t\ge1} d_{n,t}(u) = O(k_n^{-1/2}(k_n/n)^{1/r})$ given $\sup_{t\ge1} d_t < \infty$, and $d_{n,t}(u)$ is Lebesgue measurable. In particular, by a change of variables

$$\begin{aligned}\int_0^1 u^{-1}d_{n,t}(u)du &= K k_n^{-1/2}(k_n/n)^{1/r}d_t^{p/q}\int_0^1 u^{p/q}\frac{1}{u}du\\ &= O\left(k_n^{-1/2}(k_n/n)^{1/r}\right)\times\int_0^\infty e^{-vp/q}dv = O\left(k_n^{-1/2}(k_n/n)^{1/r}\right).\end{aligned}$$

Finally, $\varphi_{l_n} = o((k_n/n)^{1/q-1/r}l_n^{-\lambda\min\{p,1\}/(2q)+a})$ if $\{l_n\}$ satisfies $n/[k_n b_n^p l_n^{qa}] = O(1)$ for some $a > \lambda\min\{p,1\}/(2q)$. Since $\{l_n\}$ is arbitrarily outside of $l_n\to\infty$, we can choose $l_n\to\infty$ sufficiently fast and a to be sufficiently tiny such that both $n/[k_n b_n^p l_n^{qa}] = O(1)$ and $(n/k_n)^{1/q-1/r}l_n^{\lambda\min\{p,1\}/(2q)}\varphi_{l_n}\to 0$, hence the size is $\lambda\min\{p,1\}/(2q)$. ■

LEMMA A.9 *Let $\{X_t\}$ be L_p-NED on $\{\digamma_{n,t}\}$ with constants d_t, $\sup_{t\geq 1} d_t < \infty$, and coefficients ϑ_{l_n} of size $\lambda > 0$. Then $\{X_t\}$ satisfies the L_q-FE-NED property*

$$k_n^{-1/2} \left\| I_{n,t}(u_1, u_2) - E\left[I_{n,t}(u_1, u_2) | \digamma_{n,t-l_n}^{t+l_n} \right] \right\|_p \leq (\tilde{d}_{n,t} \times \max_{1\leq i\leq k-1} |u_{2,i} - u_{1,i}|^{1/p}) \times \varphi_{l_n}$$

for any $q \geq 2$ with constants $\tilde{d}_{n,t}$, $\sup_{t\geq 1} \tilde{d}_{n,t} = O((k_n/n)^{1/r} k_n^{-1/2})$ for some $r \geq q$ and coefficients φ_{l_n}.

Proof. Let $0 < u_1 < u_2 < 1$, define $v_i := -\ln u_i$ and define

$$I_{n,t}(u_1, u_2) := I\left(b_n e^{v_2} - \eta < X_t < b_n e^{v} + \eta\right).$$

We will first show

$$E\left(I_{n,t}(u_1, u_2) - E\left[I_{n,t}(u_1, u_2) | \digamma_{t-l_n}^{t+l_n} \right] \right)^2 \leq K d_t^p (u_2 - u_1) b_n^{-p} \vartheta_{l_n}^{\min\{p,1\}/2}.$$

Suppose $p \in (0, 1]$ and choose

$$\eta = \left(\frac{1}{e^{v_1} - e^{v_2}} \right) b_n \vartheta_{l_n}^{1/2}$$
$$\eta_1 = \left(\frac{e^{v_1}}{e^{v_1} - e^{v_2}} \right) b_n \vartheta_{l_n}^{1/2} \text{ and } \eta_2 = \left(\frac{e^{v_2}}{e^{v_1} - e^{v_2}} \right) b_n \vartheta_{l_n}^{1/2}.$$

Use $\eta_i > \eta$ to deduce by an argument similar to Lemma A.8

$$\begin{aligned}
&E\left(I_{n,t}(u_1, u_2) - E\left[I_{n,t}(u_1, u_2) | \digamma_{t-l_n}^{t+l_n} \right] \right)^2 \\
&\quad \leq \left[\bar{F}_t(b_n e^{v_1} - \eta) - \bar{F}_t(b_n e^{v_2} + \eta) \right] + d_t^p \vartheta_{l_n}^p / \eta^p \\
&\quad \leq \left[\bar{F}_t(b_n e^{v_1} - \eta_1) - \bar{F}_t(b_n e^{v_2} + \eta_2) \right] + d_t^p \vartheta_{l_n}^p / \eta^p \\
&\quad \leq K b_n^{-p} \vartheta_{l_n}^{1/2} \left(\frac{e^{-pv_1}}{(e^{v_1} - e^{v_2})^{-p}} - \frac{e^{-pv_2}}{(e^{v_1} - e^{v_2})^{-p}} \right) + d_t^p \left(\frac{1}{e^{v_1} - e^{v_2}} \right)^{-p} b_n^{-p} \vartheta_{l_n}^{p/2} \\
&\quad = K d_t^p \frac{1}{(e^{v_1} - e^{v_2})^{-p}} b_n^{-p} \vartheta_{l_n}^{\min\{p,1\}/2} \\
&\quad \leq K d_t^p (u_2 - u_1)^p b_n^{-p} \vartheta_{l_n}^{\min\{p,1\}/2} \\
&\quad \leq K d_t^p (u_2 - u_1) b_n^{-p} \vartheta_{l_n}^{\min\{p,1\}/2}
\end{aligned}$$

where the second to last line follows from $0 < \bar{u} \leq \{u_1, u_2\}$ since

$$\left(\frac{1}{u_1} - \frac{1}{u_2} \right)^p = \left(\frac{1}{u_1} \frac{1}{u_2} \right)^p (u_2 - u_1)^p \leq \frac{1}{\bar{u}^{2p}} (u_2 - u_1)^p,$$

and the last line exploits $u_2 - u_1 \in (0, 1)$ and $p \in (0, 1]$.

Now suppose $p > 1$ and choose

$$\eta = \left(\frac{1}{e^{v_1} - e^{v_2}}\right)^{1/p} b_n \vartheta_{l_n}^{1/2}$$

$$\eta_1 = \left(\frac{e^{v_1}}{e^{v_1} - e^{v_2}}\right) b_n \vartheta_{l_n}^{1/2} \text{ and } \eta_2 = \left(\frac{e^{v_2}}{e^{v_1} - e^{v_2}}\right) b_n \vartheta_{l_n}^{1/2}.$$

such that $\eta_i > \eta$ since $e^{v_i} > (e^{v_1} - e^{v_2})^{(p-1)/p}$ and

$$\begin{aligned}
&E\left(I_{n,t}\left(u_1, u_2\right) - E\left[I_{n,t}\left(u_1, u_2\right) | F_{t-l_n}^{t+l_n}\right]\right)^2 \\
&\quad \leq K b_n^{-p} \vartheta_{l_n}^{1/2} \left(\frac{e^{-pv_2}}{\left(e^{v_1} - e^{v_2}\right)^{-p}} - \frac{e^{-pv_1}}{\left(e^{v_1} - e^{v_2}\right)^{-p}}\right) + d_t^p \left(\frac{1}{e^{v_1} - e^{v_2}}\right)^{-1} b_n^{-p} \vartheta_{l_n}^{p/2} \\
&\quad \leq K d_t^p \times \left(\frac{1}{u_1} - \frac{1}{u_2}\right) \times b_n^{-p} \vartheta_{l_n}^{\min\{p,1\}/2} \\
&\quad \leq K d_t^p \times (u_2 - u_1) \times b_n^{-p} \vartheta_{l_n}^{\min\{p,1\}/2}.
\end{aligned}$$

The remainder of the proof simply mimics (A.7)-(A.8):

$$\begin{aligned}
&k_n^{-1/2} \left\| \bar{I}_{n,t}(u_1, u_2) - E\left[\bar{I}_{n,t}(u_1, u_2) | F_{t-l_n}^{t+l_n}\right] \right\|_2 \\
&\quad \leq K k_n^{-1/2} d_t^{p/2} \times \left(u_2 - u_1\right)^{1/2} \times b_n^{-p/2} \vartheta_{l_n}^{\min\{p,1\}/4} \\
&\quad = \left(K (k_n/n)^{1/r} k_n^{-1/2} d_t^{p/2}\right) \times \left(u_2 - u_1\right)^{1/2} \times (n/k_n)^{1/r} b_n^{-p/2} \vartheta_{l_n}^{\min\{p,1\}/4} \\
&\quad = \tilde{d}_{n,t} \times \left(u_2 - u_1\right)^{1/2} \times \varphi_{l_n},
\end{aligned}$$

where $\sup_{t \geq 1} \tilde{d}_{n,t} = O(k_n^{-1/2} (k_n/n)^{1/r})$ and $\varphi_{l_n} = o((k_n/n)^{1/q - 1/r} l_n^{-\lambda \min\{p,1\}/(2q)})$ for some $\{l_n\}$, $l_n \to \infty$. ■