Technical Appendix for
"On Functional Central Limit Theorems for
Dependent, Heterogenous Arrays with
Applications to Tail Index and Tail Dependence
Estimation"

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In this appendix we characterize sufficient conditions for weak convergence in
$D[0, 1]^k$ by compressing results due to Bickel and Wichura (1971) and Neuhaus
(1971). See Section A.1. Section A.2 contains the omitted proofs of Lemmas
4.1, 4.2, 5.2, 5.3 and A.7-A.9.

We assume

(A.1) $\bar{F}_t(\lambda x)/\bar{F}_t(x) \to \lambda^{-\alpha}$ as $x \to \infty \iff \bar{F}_t(x) = x^{-\alpha}L(x)$, $x > 0$,

for some slowly varying function $L(x)$, and recall

(A.2) $(n/k_n)P(X_t > b_n) \to 1$.

Define $\{U_{n,t}, U_{n,t}^*(u), \}$

\[
U_{n,t} := k_n^{-1/2}((\ln X_t/b_n)_+ - E[\ln X_t/b_n]), \\
U_{n,t}^*(u) := k_n^{-1/2}(I(X_t > b_n e^u) - P(X_t > b_n e^u)), \ u \in \mathbb{R}_+,
\]

where $\{k_n, b_n\}$ satisfy (B.2).

A.1 WEAK CONVERGENCE IN $D[0,1]^k$ Consider some $D([0, 1]^k)$-
valued process $X_n(\theta)$, $\theta \in [0,1]^k$. The following is necessarily brief. Classic
sources are Bickel and Wichura (1971), Neuhaus (1971) and Billingsley (1968).

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Write $D_k := (D([0,1]^k), d_k(x,y))$, the space of right continuous functions $f : [0,1]^k \to \mathbb{R}$ with left limits metrized with $d_k(x,y)$, a multidimensional version of Billingsley’s (1968) extended Skorokhod $J_1$-metric. Denote by $\Lambda_k = \Lambda \times \cdots \times \Lambda$ the collection of homeomorphisms $\lambda = [\lambda_i]_{i=1}^k$ from $[0,1]^k$ to $[0,1]^k$, where $\lambda_i(0) = 0$, $\lambda_i(1) = 1$. Then

$$
d_k(x,y) = \inf_{\lambda \in \Lambda, \varepsilon > 0} \left\{ \|\lambda\|^0 \leq \varepsilon, \sup_{\theta \in [0,1]^k} |x(\theta) - y(\lambda(\theta))| \leq \varepsilon \right\},
$$

where

$$
\|\lambda\|^0 := \max_{1 \leq i \leq k} \left\{ \sup_{\theta_1 \neq \theta_2} \left| \ln \left( \frac{\lambda_i(\theta_1) - \lambda_i(\theta_2)}{[\theta_1,i] - [\theta_2,i]} \right) \right| \right\}.
$$

$D([0,1]^k)$ is separable and complete under $d_k(x,y)$.

Let $\delta \in [0,1]$ and write $X_n(\cdot, \theta_i) := X_n(\{\theta_j\}_{j \neq i}, \theta_i)$. Define the moduli $w$ and $w''$

$$
w(X_n, \delta) := \sup_{|\theta - \theta'| \leq \delta} \left| X_n(\theta) - X_n(\theta') \right| \quad \text{and} \quad w''(X_n, \delta) := \max_{1 \leq i \leq k} \left\{ w''_i(X_n, \delta) \right\}
$$

where

$$
w''_i(X_n, \delta) := \sup_{\theta_{i,1} \leq \theta \leq \theta_{i,2}, \theta_{i,2} - \theta_{i,1} \leq \delta} \left\{ |X_n(\cdot, \theta_i) - X_n(\cdot, \theta_{i,1})| \wedge |X_n(\cdot, \theta_i) - X_n(\cdot, \theta_{i,2})| \right\}.
$$

Prokhorov’s (1956) theorem equates relative compactness with tightness. The following claim can therefore be deduced from an Arzelá-Ascoli-type result relating stochastic equicontinuity to relative compactness, and standard inequality arguments. See the corollary to Theorem 2 in Bickel and Wichura (1971).

**Proposition 1** Let $X_n(\xi, u) \to X(\xi, u)$ point-wise in finite dimensional distributions. If for each $i = 1, \ldots, k$ and every $\varepsilon > 0$,

$$
\lim_{\delta \to 0} P(|X_n(\cdot, \theta_i) = 1 - X_n(\cdot, \theta_i - \delta)| > \varepsilon) = 0,
$$

and if for every $\varepsilon > 0$ and $\eta > 0$ there exists some $\delta \in (0,1)$ and $n_0 \in \mathbb{N}$ such that

$$
P\left( w''(X_n, \delta) > \varepsilon/k \right) \leq \eta/k, \forall n \geq n_0,
$$

then $X_n(\xi, u) \Rightarrow X(\xi, u)$ on $D_k$.

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1 Each $\lambda_i : [0,1] \to [0,1]$ is increasing, one-to-one, onto, continuous with a continuous inverse $\lambda_i^{-1}$. 

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A.2 OMITTED PROOFS

Recall $g_n = o(K_n(\xi))$, $r_n(\xi) = [K_n(\xi)/g_n]$ and $M_{n,i} := \max_{(i-1)g_n+1 \leq t \leq ig_n} c_{n,t}^2$ for some non-stochastic double array $\{c_{n,t}\}$.

**Lemma 4.1** Let $K_n(\xi)/n \rightarrow a(\xi) > 0$ a finite constant function. There does not exist an array $\{c_{n,t}\}$ that satisfies $M_{n,i} = o(g_n^{-1/2})$ and $\sum_{i=1}^{r_n(\xi)} M_{n,i}^2 = O(g_n^{-1})$ such that $\{U_{n,t}/c_{n,t}\}$ or $\{U_{n,t}^*(u)/c_{n,t}\}$ are $L_r$-bounded for all $t$ and any $r > 2$.

**Proof.** By Lemma A.2 of Hill (2005)\(^2\)

$$
\lim_{n \rightarrow \infty} \left( \frac{n}{k_n} \right)^{1/r} \|I(X_t > b_n e^n) - P(X_t > b_n e^n)\|_r \leq K e^{-\alpha n}, \forall r \geq 1,
$$

hence $\forall r \geq 1$

$$
\|U_{n,t}^*(u)\|_r = O \left( k_n^{-1/2}(n/k_n)^{-1/r} \right).
$$

Now $\|U_{n,t}^*(u)/c_{n,t}\|_r = O(1)$ $\forall t$ if and only if $c_{n,t} = O(k_n^{1/2}(n/k_n)^{1/r})$ $\forall t$. de Jong and Davidson (2000: Theorem 3.1) require $\sum_{i=1}^{r_n(\xi)} M_{n,i}^2 = O(g_n^{-1})$, hence $\|U_{n,t}^*(u)/c_{n,t}\|_r = O(1)$ $\forall t$ and $\sum_{i=1}^{r_n(\xi)} M_{n,i}^2 = O(g_n^{-1})$ if and only if

$$
O(r_n(\xi)^{1/2}k_n^{-1/2}(n/k_n)^{1/r}) = O(g_n^{-1/2}).
$$

But by construction

$$
r_n(\xi)^{1/2}k_n^{-1/2}(n/k_n)^{1/r} \sim (K_n(\xi)/n)^{1/2} g_n^{-1/2}(n/k_n)^{1/2-1/r} = O(g_n^{-1/2})
$$

if and only if $r = 2$ when $K_n(\xi)/n \rightarrow a(\xi)$, a finite positive function. An identical argument applies to $\{U_{n,t}/c_{n,t}\}$, cf. Lemma A.2 of Hill (2005b).∎

Recall $S_k := \sum_{t=1}^{k} I(X_t > b_n e^n) - P(X_t > b_n e^n)$ and $\sigma_n^2 := E[S_n^2]$.

**Lemma 4.2** If $X_t$ is iid with distribution tail (11) then $\sigma_n^2 \rightarrow \infty$, $\|E(S_n|X_0)\|_2 = 0$, and $\sigma_n^2/n$ is regularly varying with index 1. Further, $\sigma_n^2/k_n$ is slowly varying.

**Proof.** Since $X_t$ is iid $E(S_n|X_0) = E(S_n) = 0$, so $\|E(S_n|X_0)\|_2 = 0$.

Further, $X_t \overset{iid}{\sim} (A.1)$ and the construction of $b_n$ imply

$$
\sigma_n^2 = \sum_{t=1}^{n} E(I(X_t > b_n e^n) - P(X_t > b_n e^n))^2 = n \left[ P(X_t > b_n e^n) - P(X_t > b_n e^n)^2 \right] \sim n \times \frac{k_n}{n} = k_n \rightarrow \infty.
$$

\(^2\)This paper has since been published in Econometric Theory; see Hill, J.B. (2010). On Tail Index Estimation for Dependent, Heterogeneous Data, Econometric Theory: in press.
Define
\[ l(n) := \frac{\sigma_n^2}{n}. \]

We now prove \( l(\lambda n)/l(n) \rightarrow \lambda^{-1} \) for all \( \lambda > 0 \). First, use (A.1) and the construction
\[
\frac{\lambda_n}{k_n} P(X_t > b_{\lambda n}) \rightarrow 1
\]
to deduce
\[
\frac{\lambda_n}{k_n} P(X_t > b_{\lambda n}) \sim \frac{\lambda_n}{k_n} P\left(X_t > \lambda^{-1/\alpha} b_{\lambda n}\right) \rightarrow 1
\]
hence
\[
b_{\lambda n} = \lambda^{1/\alpha} b_n.
\]
But this implies from (A.1) and (A.3)
\[
l(\lambda n) = \frac{\sigma_{\lambda n}}{\lambda n}
\]
\[
= P(X_t > b_{\lambda n}e^u) - P(X_t > b_{\lambda n}e^u)^2
\]
\[
\sim \lambda^{-1} P(X_t > b_n e^u) - \lambda^{-2} P\left(X_t > \lambda^{1/\alpha} b_n e^u\right)^2
\]
hence
\[
\frac{n}{k_n} l(\lambda n) \sim \lambda^{-1} \frac{n}{k_n} P(X_t > b_n e^u) - \lambda^{-2} \frac{n}{k_n} P\left(X_t > \lambda^{1/\alpha} b_n e^u\right)^2
\]
\[
\rightarrow \lambda^{-1} e^{-\alpha u}
\]
and
\[
\frac{n}{k_n} l(n) = \frac{n}{k_n} \frac{\sigma_n^2}{n} = \frac{n}{k_n} P(X_t > b_n e^u) - \frac{n}{k_n} P(X_t > b_n e^u)^2 \rightarrow e^{-\alpha u}.
\]
Therefore
\[
\frac{l(\lambda n)}{l(n)} = \frac{(n/k_n) l(\lambda n)}{(n/k_n) l(n)} \rightarrow \lambda^{-1}.
\]

Now define
\[ \tilde{l}(n) := \frac{\sigma_n^2}{k_n} \]
and note by construction for any \( \lambda > 0 \)
\[
\tilde{l}(\lambda n) = \frac{\sigma_{\lambda n}}{k_{\lambda n}}
\]
\[
= \frac{\lambda_n}{k_{\lambda n}} \left[ P(X_t > b_{\lambda n} e^u) - P(X_t > b_{\lambda n} e^u)^2 \right] \sim e^{-\alpha u} + o(1).
\]
Since λ is arbitrarily small, it follows instantly \( \tilde{I}(\lambda n) / \tilde{I}(n) \to 1 \), hence \( \tilde{I}(n) := \sigma_n^2 / k_n \) is slowly varying.

Recall

\[ Z_{n,t}(\lambda, u, h) := k_n^{1/2} \lambda_i U_{i,n,t-h}^*(u_1) \times U_{j,n,t}^*(u_2). \]

**Lemma 5.2** For all \( r \geq 1 \) and finite \( h \geq 1 \), \( \|Z_{n,t}(\lambda, u, h)\|_r = O(k_n^{-1/2}(k_n/n)^{1/r}) = O(n^{-a(r)}) \), where \( a(1) > 1/2 \), and \( 2a(2r) > a(r) \). Further, \( a(r) > 1/r \) for all \( r > 2 \).

**Proof.** Use the Minkowski and Cauchy-Schwartz inequalities to get for any \( r \geq 1 \)

\[ \|Z_{n,t}(\lambda, u, h)\|_r \leq k_n^{1/2} \sum_{i=1}^h |\lambda_i| \times \|U_{i,n,t-h}^*(u_1)\|_2 \times \|U_{j,n,t}^*(u_2)\|_2. \]

By Lemma A.1 of Hill (2005b)

\[ \left\{ \|U_{i,n,t-h}^*(u_1)\|_2, \|U_{j,n,t}^*(u_2)\|_2 \right\} = O((k_n/n)^{1/r} k_n^{-1/2}), \]

hence

\[ \|Z_{n,t}(\lambda, u, h)\|_r = O \left( k_n^{1/2}(k_n/n)^{1/2r} k_n^{-1/2}(k_n/n)^{1/2r} k_n^{-1/2} \right) = O \left( (k_n/n)^{1/r} k_n^{-1/2} \right) = O \left( (k_n/n)^{1/(2-1/r)} n^{-1/r} \right). \]

The remaining argument \( O(k_n^{-1/(2-1/r)} n^{-1/r}) = O(n^{-a(r)}) \) simply mimics Lemma 3.1.

**Lemma 5.3** Let \( \{X_{1,t}, X_{2,t}\} \) satisfy Assumption 2.a with \( q = 4 \). Then \( \{Z_{n,t}(\lambda, u, h)\} \)

is L2-NED on \( \{F_{n,t}\} \) with constants \( d_{n,t}(\lambda, u), \sup_{u \in [0,1]} \lambda \lambda_{t-1} \geq 1 \) \( d_{n,t}(\lambda, u) \)

\( = O(k_n^{-1/2}(k_n/n)^{1/r}) \) and coefficients \( \varphi_{1,n} = o((k_n/n)^{1/2-1/r} n^{-1/2}) \). Moreover, \( E(Z_{n,t}(\lambda, u, h))^2 = O(1) \).

**Proof.**

**Step 1 (NED):** Minkowski, Cauchy-Schwartz, and conditional Jensen’s inequalities imply

\[
\|Z_{n,t}(u,h) - E[Z_{n,t}(u,h)]F_{n,t-l_n}^{t+l_n}\|_2 \\
= k_n^{-1/2} \left[ \|Z_{1,n,t}(u_1)Z_{2,n,t-h}(u_2) - E[Z_{1,n,t}(u_1)Z_{2,n,t-h}(u_2)]F_{n,t-l_n}^{t+l_n}\|_2 \right] \\
\leq k_n^{-1/2} \left( \|Z_{1,n,t}(u_1)\|_4 + \|Z_{1,n,t}(u_1) - E[Z_{1,n,t}(u_1)]F_{n,t-l_n}^{t+l_n}\|_4 \right) \\
\times \|Z_{2,n,t-h}(u_2) - E[Z_{2,n,t-h}(u_2)]F_{n,t-l_n}^{t+l_n}\|_4 \\
+ \|Z_{2,n,t-h}(u_2)\|_4 k_n^{-1/2} \|Z_{1,n,t}(u_2) - E[Z_{1,n,t}(u_1)]F_{n,t-l_n}^{t+l_n}\|_4 \\
\leq 3 \|Z_{1,n,t}(u_1)\|_4 \times d_{2,n,t-h}(u_1) \times \varphi_{2,l_n} + \|Z_{2,n,t-h}(u_2)\|_4 \times d_{1,n,t}(u_1) \times \varphi_{1,l_n}.
\]
From Lemma 3.1 each $\|Z_{i,n,t}(u_i)\|_1 = O(n^{-\alpha(4)})$, $\alpha(4) > 1/4$, and by construction $d_{i,n,t}(u) = O(n^{-\alpha(r)})$. Thus, for some finite $K > 0$

$$
\left\|ZZ_{n,t}(u,h) - E[ZZ_{n,t}(u,h)|F_{n,t-\ell_n}]\right\|_2 \leq \tilde{d}d_{n,t}(u,h) \times \tilde{\varphi}_{l_n}
$$

$$
\tilde{d}d_{n,t}(u,i) = K \times n^{-\alpha(4)} \times \max\{d_{1,n,t}(u_1), d_{2,n,t-h}(u_2)\} = O(n^{-\alpha(4)} - \alpha(r))
$$

$$
\tilde{\varphi}_{l_n} = \max\{\varphi_{i,l_n}\} = o(n^{\alpha(r) - 1/4} \times l_n^{-1/2}).
$$

Hence

$$
\left\|ZZ_{n,t}(u,h) - E[ZZ_{n,t}(u,h)|F_{n,t-\ell_n}]\right\|_2 \leq d_{n,t}(u,h) \times \varphi_{l_n}
$$

where $d_{n,t}(u,h) := n^{\alpha(4)} \times \tilde{d}d_{n,t}(u,h) = O(n^{-\alpha(r)})$ and $\varphi_{l_n} := n^{-\alpha(4)} \times \tilde{\varphi}_{l_n} = o(n^{\alpha(r) - 1/2} \times l_n^{-1/2})$ due to $\alpha(4) + 1/4 > 1/2$, cf. Lemma 3. This proves \{\{ZZ_{n,t}(u,h)\}\} is $L_2$-NED.

Under the maintained assumptions the conditions of Theorem 17.8 of Davidson (1994) are met. Thus, any linear combination

$$
ZZ_{n,t}(u,\lambda) := \sum_{i=1}^{h} \lambda_i ZZ_{n,t}(u,i)
$$

is $L_2$-NED with constants $d_{n,t}(u,\lambda, h) = \max_{1 \leq i \leq h}\{\lambda_i d_{n,t}(u,i)\} = O(n^{-\alpha(r)})$ and coefficients $h \times \varphi_{l_n} = O(n^{\alpha(r) - 1/2}) \times O(l_n^{-1/2})$.

**Step 2 (Mixingale):** Using Step 1, Theorem 17.5 of Davidson (1994) implies \{\{ZZ_{n,t}(u,\lambda, h), F_{n,t}\}\} is an $L_2$-mixingale with size $-1/2$ and constants $c_{n,t} = O(n^{-1/2})$. If the base \{\{\epsilon_i\}\} is F-Strong Mixing, then for $r > 2$

$$
\left\|ZZ_{n,t}(u,\lambda, h) - E[ZZ_{n,t}(u,\lambda, h)|F_{n,t-\ell_n}]\right\|_2 \leq \max\{\|ZZ_{n,t}(u,\lambda, h)\|_r, d_{n,t}(u,\lambda, h)\} \times \max\{6\varepsilon_{l_n}^{1/2 - 1/r}, \varphi_{l_n}\}.
$$

By Lemma 5.2 \|ZZ_{n,t}(u,\lambda, h)\| = O(k_n^{-1/2}(k_n/n) ^ {1/r}) = O(n^{-\alpha(r)})$, where $a(1) > 1/2$, $a(2) = 1/2$, and $2a(2)r > a(r)$ in general, and $\alpha(r) = 1/2$ for $r > 2$.

Thus, for some finite $K > 0$

$$
\left\|ZZ_{n,t}(u,\lambda, h) - E[ZZ_{n,t}(u,\lambda, h)|F_{n,t-\ell_n}]\right\|_2 \leq K n^{-1/2} \times \max\{\left(\frac{1}{2} - a(r)\right)2^{2r/(r-2)\varepsilon_{l_n}}\}^{1/2 - 1/r}, n^{1/2 - a(r)} \varphi_{l_n}\}
$$

$$
= c_{n,t}(u,\lambda, h) \times \psi_{l_n},
$$

say, where $c_{n,t}(u,\lambda, h) = O(n^{-1/2})$, and $\sup_n \psi_{l_n} = O(l_n^{-1/2 - \epsilon})$ for some small $\epsilon > 0$ follows from the F-Mixing and FE-NED coefficient properties. A similar argument holds for the other mixingale bound \|ZZ_{n,t}(u,\lambda, h) - E[ZZ_{n,t}(u,\lambda, h)|F_{n,t+\ell_n}]\|_2
\[ \leq c_{n,t}(u, \lambda, h) \times \psi_{l_n+1} \text{ and in the F-Uniform Mixing case.} \]

**Step 3 (Bound):**

\[ E(\sum_{t=1}^{n} ZZ_{n,t}(u, \lambda, h))^2 = O(\sum_{t=1}^{n} cc_{n,t}(u, \lambda, h)) = O(1) \]

Now follows from McLeish's (1975a) bound for \( L_2 \)-mixingales with size \(-1/2\).

**Lemma A.7**

Under the conditions of Theorem 5.1 for any \( u \in [0, 1] \)

\[
k_n^{1/2} \left( \hat{a}_{k_n}^{-1}(\xi) - \alpha^{-1} \right) = \sum_{t=1}^{[n\xi]} \left( U_{n,t} - \alpha^{-1} U_{n,t}^* \left( u^{[k_n \xi]^{-1/2}} \right) \right) + o_p(1).
\]

**Proof.** Write

(A.4)

\[
k_n^{1/2} \left( \hat{a}_{k_n}^{-1}(\xi) - \alpha^{-1} \right)
= k_n^{1/2} \left( \frac{1}{k_n} \sum_{i=1}^{[k_n \xi]} \ln X_{(i)}/X_{([k_n \xi]+1)} - \alpha^{-1} \right)
= k_n^{1/2} \left( \frac{1}{k_n} \sum_{i=1}^{[k_n \xi]} \ln X_{(i)}/b_{[n\xi]} - E \left( \frac{1}{k_n} \sum_{t=1}^{[n\xi]} \ln X_t/b_{[n\xi]} \right) \right)
- k_n^{1/2} \ln X_{([k_n \xi]+1)}/b_{[n\xi]} + k_n^{1/2} \left[ E \left( \frac{1}{k_n} \sum_{t=1}^{[n\xi]} \ln X_t/b_{[n\xi]} \right) - \alpha^{-1} \right].
\]

Use Assumption C and arguments in Hsing (1991: p. 1554) to deduce

(A.5)

\[
E \left( \frac{1}{k_n} \sum_{t=1}^{[n\xi]} \ln X_t/b_{[n\xi]} \right) = o \left( 1/k_n^{1/2} \right).
\]

Moreover, from Corollary 3.3 we deduce

\[
\sum_{t=1}^{[n\xi]} U_{n,t}^* \sum_{t=1}^{[n\xi]} U_{n,t} \left( u^{[k_n \xi]^{-1/2}} \right) \xrightarrow{p} \{ W_1(\xi), W_2(\xi) \}
\]

Jointly on \( D[\xi, 1] \), where each \( W_i(\xi) \) is Gaussian, and \( \ln X_t / \rho[b_{[k_n \xi]}] - \ln b_{n}(\rho[b_{[k_n \xi]}]) \to 0 \) for all \( \rho \) in an arbitrary neighborhood of 1 by Lemma 1 of Hill (2005b). Therefore an argument identical to Theorem 2.2 of Hsing (1991: eq. 2.4-2.7) applies:

(A.6)
\[ k_n^{1/2} \left( \frac{1}{k_n} \sum_{i=1}^{[k_n \xi]} \ln X_{(i)}/b_{[n\xi]} - E \left( \frac{1}{k_n} \sum_{t=1}^{[n\xi]} (\ln X_t/b_{[n\xi]})_+ \right) \right) \Rightarrow W_1(\xi) \]

\[ \alpha \times \sum_{t=1}^{[n\xi]} U_{n,t}^* \left( u^{k_n^{-1/2}} \right) \Rightarrow W_2(\xi). \]

Together, (A.4)-(A.6) imply

\[ k_n^{1/2} \left( \alpha_{k_n}^{-1}(\xi) - \alpha^{-1} \right) = \sum_{t=1}^{[n\xi]} \left( U_{n,t} - \alpha^{-1} U_{n,t}^* (u^{[k_n]^{-1/2}}) \right) + o_p(1). \]

**Lemma A.8** Let \( \{X_t\} \) be \( L_p\)-NED, \( p \in (0,2] \), on \( \{F_{n,t}\} \) with constants \( d_t \), \( \sup_{t \geq 1} d_t < \infty \), and coefficients \( d_{1,n} \) of size \( \lambda > 0 \). Then \( \{X_t\} \) satisfies the \( L_q\)-FE-NED property

\[ k_n^{-1/2} \left\| I(\xi > b_n/u) - P \left( X_t > b_n/u | F_{n,t} \right) \right\|_q \leq d_n,t(u) \times \varphi_{1,n} \]

for any \( q \geq 2 \) and some displacement sequence \( \{l_n\}, l_n \to \infty \), with constants \( d_{n,t}(u) \) and coefficients \( \varphi_{1,n} \) of size \( \lambda \times \min\{p,1\}/(2q) \). In particular, \( d_{n,t}(u) \) is Lebesgue measurable, \( \sup_{u \in [0,1], t \geq 1} d_{n,t}(u) = O((k_n/n)^{1/r}k_n^{-1/2}) \) and \( \int_0^1 u^{-1} d_{n,t}(u) du = O((k_n/n)^{1/2}k_n^{-1}) \) for some \( r \geq q \). Further \( \sup_{t \geq 1} \varphi_{1,n} \in [0,1] \), and \( \varphi_{1,n} = o((k_n/n)^{1/p-1/2}) \).

**Proof.** Define \( \bar{I}_{n,t}(u) = I(X_t > b_n/u) \) and \( v := -\ln u \) such that \( \bar{I}_{n,t}(u) = I(X_t > b_n e^v) \). For some \( \eta > 0 \) that may depend on \( n \) to be chosen below and any \( q \geq 2 \), since \( |\bar{I}_{n,t}(u) - E[\bar{I}_{n,t}(u) | F_{t-l_n}^{t+l_n}]| \leq 1 \) it follows

\[ E \left[ \bar{I}_{n,t}(u) - E \left[ \bar{I}_{n,t}(u) | F_{t-l_n}^{t+l_n} \right] \right]^q \]

\[ \leq E \left( \bar{I}_{n,t}(u) - E \left[ \bar{I}_{n,t}(u) | F_{t-l_n}^{t+l_n} \right] \right)^2 \]

\[ \leq E \left( \bar{I}_{n,t}(u) - I \left( E \left[ \bar{I}_{n,t}(u) | F_{t-l_n}^{t+l_n} \right] > b_n \right) \right)^2 \left( X_t - E \left[ X_t | F_{t-l_n}^{t+l_n} \right] < \eta \right) \]

\[ + E \left( \bar{I}_{n,t}(u) - I \left( E \left[ \bar{I}_{n,t}(u) | F_{t-l_n}^{t+l_n} \right] > b_n \right) \right)^2 \left( X_t - E \left[ X_t | F_{t-l_n}^{t+l_n} \right] > \eta \right) \]

\[ \leq E \left[ I(b_ne^v - \eta < X_t < b_ne^v + \eta) \right] + P \left( X_t - E \left[ X_t | F_{t-l_n}^{t+l_n} \right] > \eta \right) \]

\[ = \left[ \bar{F}_t(b_ne^v - \eta) - \bar{F}_t(b_ne^v + \eta) \right] + P \left( X_t - E \left[ X_t | F_{t-l_n}^{t+l_n} \right] > \eta \right) \]

\[ \leq \left[ \bar{F}_t(b_ne^v - \eta) - \bar{F}_t(b_ne^v + \eta) \right] + E \left( X_t - E \left[ X_t | F_{t-l_n}^{t+l_n} \right] \right)^p /\eta^p \]

\[ \leq \left[ \bar{F}_t(b_ne^v - \eta) - \bar{F}_t(b_ne^v + \eta) \right] + d^{p}\eta^{p}/\eta^{p} \]

The second inequality is due to the conditional expectations minimizing the mean-squared-error, and a trivial identity. The third follows from basic logic.
and a trivial inequality involving the indicator function. The fourth is Markov’s inequality, and the fifth follows from the $L_p$-NED property.

Without loss of generality let sup$_{t \in [\theta_t]} \theta_t \in [0, 1)$. Put $\eta = e^u b_n \theta_{t_n}^{1/2}$, and use lim$_{z \to -\infty} z^p \bar{F}(z) = 0$ and the mean-value-theorem to deduce

$$
\bar{F}(b_n e^v \pm \eta) = \bar{F}(b_n \left(1 \pm \theta_{t_n}^{1/2}\right) e^v)
= b_n^p \left(1 \pm \theta_{t_n}^{1/2}\right)^{-p} e^{-pv} \times o_p(1)
= K b_n^p \theta_{t_n}^{1/2} e^{-pv} \times o_p(1).
$$

Therefore

$$
E \left( \bar{I}_{n,t}(u) - E \left[ \bar{I}_{n,t}(u) | \bar{F}_{t^-q} \right] \right)^2
\leq \left[ \bar{F}(b_n \left(1 - \theta_{t_n}^{1/2}\right) e^v) - \bar{F}(b_n \left(1 + \theta_{t_n}^{1/2}\right) e^v) \right]
+ e^{-pv} b_n^p d_{t_n} \theta_{t_n}^{p/2}
= \left[ \bar{F}(b_n \left(1 - \theta_{t_n}^{1/2}\right) e^v) - \bar{F}(b_n \left(1 + \theta_{t_n}^{1/2}\right) e^v) \right]
+ e^{-pv} b_n^p d_{t_n} \theta_{t_n}^{p/2}
= K b_n^p e^{-pv} \left( \theta_{t_n}^{1/2} \times o_p(1) + d_{t_n} \theta_{t_n}^{p/2} \right)
\leq K b_n^p e^{-pv} d_{t_n} \theta_{t_n}^{\min(p,1)/2},
$$
hence as $n \to \infty$

$$
k_n^{-1/2} \left\| \bar{I}_{n,t}(u) - E \left[ \bar{I}_{n,t}(u) | \bar{F}_{t^-q} \right] \right\|_q
\leq \left( K k_n^{-1/2} b_n^p q d_{t_n} \theta_{t_n}^{p/q} e^{-vp/q} \right) \times \theta_{t_n}^{\min(p,1)/2}.
$$

where for arbitrary $r \geq q$ and some $a > 0$

$$
d_{n,t}(u) = K k_n^{-1/2} (k_n/n)^{1/r} d_{t_n}^{1/r} q e^{-vp/q} = K k_n^{-1/2} (k_n/n)^{1/r} d_{t_n}^{1/r} q u
$$

Clearly sup$_{u \in [0,1], t \geq 1} d_{n,t}(u) = O(k_n^{-1/2} (k_n/n)^{1/r})$ given sup$_{t \geq 1} d_t < \infty$, and $d_{n,t}(u)$ is Lebesgue measurable. In particular, by a change of variables

$$
\int_0^1 u^{-1} d_{n,t}(u) du = K k_n^{-1/2} (k_n/n)^{1/r} d_{t_n}^{1/r} q \int_0^1 u^{p/q} \frac{1}{u} du
= O \left( k_n^{-1/2} (k_n/n)^{1/r} \right) \times \int_0^\infty e^{-vp/q} dv = O \left( k_n^{-1/2} (k_n/n)^{1/r} \right).
$$

Finally, $\varphi_{t_n} = o((k_n/n)^{1/q - 1/r} l_n^{-\lambda \min(p,1)/(2q)})$ if $\{l_n\}$ satisfies $n/[k_n b_n^{\lambda \min(p,1)/(2q)}] = O(1)$ for some $a > \lambda \min(p,1)/(2q)$. Since $\{l_n\}$ is arbitrarily outside of $l_n \to \infty$, we can choose $l_n \to \infty$ sufficiently fast and a to be sufficiently tiny such that both $n/[k_n b_n^{\lambda \min(p,1)/(2q)}] = O(1)$ and $(n/k_n)^{1/q - 1/r} l_n^{-\lambda \min(p,1)/(2q)} \varphi_{t_n} \to 0$, hence the size is $\lambda \min(p,1)/(2q)$.
LEMA A.9 Let \( \{X_t\} \) be \( L_p\)-NED on \( \{F_{n,t}\} \) with constants \( d_t, \sup_{t \geq 1} d_t < \infty \), and coefficients \( \varphi_{tn} \) of size \( \lambda > 0 \). Then \( \{X_t\} \) satisfies the \( L_q\)-FE-NED property
\[
-k_n^{-1/2} \left\| I_{n,t} (u_1, u_2) - E \left[ I_{n,t} (u_1, u_2) | F_{n,t-l_n} \right] \right\|_p \\
\leq (\tilde{d}_{n,t} \times \max_{1 \leq i \leq k-1} |u_{2,i} - u_{1,i}|^{1/p}) \times \varphi_{tn},
\]
for any \( q \geq 2 \) with constants \( \tilde{d}_{n,t}, \sup_{t \geq 1} \tilde{d}_{n,t} = O((k_n/n)^{1/r} k_n^{-1/2}) \) for some \( r \geq q \) and coefficients \( \varphi_{tn} \).

Proof. Let \( 0 < u_1 < u_2 < 1 \), define \( v_i := -\ln u_i \) and define
\[
I_{n,t} (u_1, u_2) := I (b_n e^{v_1} - \eta < X_t < b_n e^{v} + \eta).
\]
We will first show
\[
E \left( I_{n,t} (u_1, u_2) - E \left[ I_{n,t} (u_1, u_2) | F_{n,t-l_n} \right] \right)^2 \leq K d_t^p (u_2 - u_1) b_n^{-p} \varphi_{tn}^{\min\{p,1\}/2}.
\]
Suppose \( p \in (0,1] \) and choose
\[
\eta = \left( \frac{1}{e^{v_1} - e^{v_2}} \right) b_n \varphi_{tn}^{1/2},
\]
\[
\eta_1 = \left( \frac{e^{v_1}}{e^{v_1} - e^{v_2}} \right) b_n \varphi_{tn}^{1/2} \quad \text{and} \quad \eta_2 = \left( \frac{e^{v_2}}{e^{v_1} - e^{v_2}} \right) b_n \varphi_{tn}^{1/2}.
\]
Use \( \eta_i > \eta \) to deduce by an argument similar to Lemma A.8
\[
E \left( I_{n,t} (u_1, u_2) - E \left[ I_{n,t} (u_1, u_2) | F_{n,t-l_n} \right] \right)^2 \leq \left[ \bar{F}_t (b_n e^{v_1} - \eta) - \bar{F}_t (b_n e^{v_2} + \eta) \right] + d_t^p \varphi_{tn}^p / \eta^p \leq \left[ \bar{F}_t (b_n e^{v_1} - \eta_1) - \bar{F}_t (b_n e^{v_2} + \eta_2) \right] + d_t^p \varphi_{tn}^p / \eta^p \leq K b_n^{-p} \varphi_{tn}^{1/2} \left( \frac{e^{-pv_1}}{e^{v_1} - e^{v_2}} - \frac{e^{-pv_2}}{e^{v_1} - e^{v_2}} \right) + d_t^p \left( \frac{1}{e^{v_1} - e^{v_2}} \right) \frac{1}{e^{-pv_1} \varphi_{tn}^{\min\{p,1\}/2}} + d_t^p \frac{1}{e^{v_1} - e^{v_2}} b_n^{-p} \varphi_{tn}^{\min\{p,1\}/2} \leq K d_t^p (u_2 - u_1) b_n^{-p} \varphi_{tn}^{\min\{p,1\}/2} \leq K d_t^p (u_2 - u_1) b_n^{-p} \varphi_{tn}^{\min\{p,1\}/2}
\]
where the second to last line follows from \( 0 < \bar{u} \leq \{u_1, u_2\} \) since
\[
\left( \frac{1}{u_1} - \frac{1}{u_2} \right)^p = \left( \frac{1}{u_1} \frac{1}{u_2} \right)^p (u_2 - u_1)^p \leq \frac{1}{\bar{u}^2} (u_2 - u_1)^p,
\]
and the last line exploits \( u_2 - u_1 \in (0,1) \) and \( p \in (0,1] \).
Now suppose $p > 1$ and choose 
\[ \eta = \left( \frac{1}{e^{v_1} - e^{v_2}} \right)^{1/p} b_n \vartheta_{l_n}^{1/2} \]
\[ \eta_1 = \left( \frac{e^{v_1}}{e^{v_1} - e^{v_2}} \right) b_n \vartheta_{l_n}^{1/2} \]
and \[ \eta_2 = \left( \frac{e^{v_2}}{e^{v_1} - e^{v_2}} \right) b_n \vartheta_{l_n}^{1/2} . \]
such that $\eta_1 > \eta$ since $e^{v_1} > (e^{v_1} - e^{v_2})^{(p-1)/p}$ and 
\[
E \left( I_{n,t} (u_1, u_2) - E \left[ I_{n,t} (u_1, u_2) \left| F_{l-n}^{t + l_n} \right. \right] \right)^2 
\leq K b_n^{-p} \vartheta_{l_n}^{1/2} \left( \frac{e^{-p v_2}}{(e^{v_1} - e^{v_2})^p} - \frac{e^{-p v_1}}{(e^{v_1} - e^{v_2})^p} \right) + d_t^{p} \left( \frac{1}{e^{v_1} - e^{v_2}} \right)^{-1} b_n^{-p} \vartheta_{l_n}^{p/2} 
\leq K d_t^{p} \times \left( \frac{1}{u_1} - \frac{1}{u_2} \right) \times b_n^{-p} \vartheta_{l_n}^{\min\{p,1\}/2} 
\leq K d_t^{p} \times (u_2 - u_1) \times b_n^{-p} \vartheta_{l_n}^{\min\{p,1\}/2} .
\]
The remainder of the proof simply mimics (A.7)-(A.8):
\[
k_n^{-1/2} \left\| I_{n,t} (u_1, u_2) - E \left[ I_{n,t} (u_1, u_2) \left| F_{l-n}^{t + l_n} \right. \right] \right\|_2 
\leq K k_n^{-1/2} d_t^{p/2} \times (u_2 - u_1)^{1/2} \times b_n^{-p/2} \vartheta_{l_n}^{\min\{p,1\}/4} 
= \left( K (k_n/n)^{1/2} k_n^{-1/2} d_t^{p/2} \right) \times (u_2 - u_1)^{1/2} \times (n/k_n)^{1/2} b_n^{-p/2} \vartheta_{l_n}^{\min\{p,1\}/4} 
= d_{n,t} \times (u_2 - u_1)^{1/2} \times \varphi_{l_n},
\]
where $\sup_{t \geq 1} d_{n,t} = O(k_n^{-1/2} (k_n/n)^{1/r})$ and $\varphi_{l_n} = o((k_n/n)^{1/q - 1/r} \Lambda_{l_n}^{\min\{p,1\}/(2q)})$ for some $\{l_n\}$, $l_n \to \infty$. ■