Solutions

Homework 5

Math 232 section 006

Due: Tuesday, October 23rd

1. Determine if the following integrals converge or diverge. Evaluate those that converge.

4 pts each

(a) $\displaystyle\int_{-\infty}^{\infty} e^{-r^2} r\,dr = \lim_{t\to\infty}\left(\int_{-t}^{0} e^{-r^2} r\,dr + \int_{0}^{t} e^{-r^2} r\,dr\right)$

$$= \lim_{t\to\infty}\left(e^{-r^2}\Big|_{-t}^{0} + e^{-r^2}\Big|_{0}^{t}\right)$$

$$= \lim_{t\to\infty}\left(\cancel{1} - e^{-t^2} + e^{-t^2} - \cancel{1}\right)$$

$$= 0$$

(b) $\displaystyle\int_{10}^{\infty} \frac{\ln x}{x^2}\,dx$

$$= \lim_{t\to\infty}\int_{10}^{t}\frac{\ln x}{x^2}\,dx$$

$$= \frac{1}{10} + \frac{1}{10}\ln 10 - \lim_{t\to\infty}\left(\frac{1}{t} + \frac{1}{t}\ln t\right)$$

$$\boxed{= \frac{1}{10} + \frac{1}{10}\ln 10}$$

$u = \ln x \qquad v = -\frac{1}{x}$

$du = \frac{1}{x} \qquad dv = \frac{1}{x^2}\,dx$

$$\int_{10}^{t}\frac{\ln x}{x^2}\,dx = -\frac{1}{x}\ln x\Big|_{10}^{t} + \int_{10}^{t}\frac{1}{x^2}\,dx$$

$$= -\frac{1}{x}\ln x\Big|_{10}^{t} - \frac{1}{x}\Big|_{10}^{t}$$

$$= -\frac{1}{t}\ln t + \frac{1}{10}\ln 10$$

$$-\frac{1}{t} + \frac{1}{10}$$

(c) $\int_0^1 \frac{x}{\sqrt{1-x^2}}dx$ $= \lim_{s \to 0^+} -\frac{1}{2}\int_s^1 \frac{-2x\,dx}{\sqrt{1-x^2}}$

$= \lim_{s \to 0^+} -\frac{1}{\cancel{2}} \cancel{2}\sqrt{1-x^2}\Big|_s^1$

$= \lim_{s \to 0^+} +\sqrt{1-s^2}$

$= +1$

2. Evaluate the following integral: $\int \frac{5x^2+4x+3}{x(x+1)^2}dx$ $= \int \left(\frac{3}{x} + \frac{2}{x+1} - \frac{4}{(x+1)^2}\right)dx$

$= 3\ln|x| + 2\cdot\ln|x+1|$

$+\frac{4}{x+1} + C$

$\frac{5x^2+4x+3}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

$= \frac{A(x+1)^2 + Bx(x+1) + Cx}{x(x+1)^2}$

$= (A+B)x^2 + (B+2A+C)x + A$

$A=3$

$B+2A+C=4$

$A+B=5$

$B=2$

$C=-4$

6 pts

3. Prove that $\int_{-\infty}^{0} e^x \cos x \, dx$ is convergent.

5 pts

$$e^x \cos x \le e^x \quad \forall x \le 0$$

so $$\int_{-\infty}^{0} e^x \cos x \, dx \le \int_{-\infty}^{0} e^x \, dx$$

$$\le e^x \Big|_{-\infty}^{0}$$

$$\le 1$$

and $$e^x \cos x \geq -e^x \quad \forall x \le 0$$

$$\int_{-\infty}^{0} e^x \cos x \, dx \geq \int_{-\infty}^{0} -e^x \, dx$$

$$\geq -1$$

so $$\left| \int_{-\infty}^{0} e^x \cos x \, dx \right| \le 1$$ ie is convergent.

8 pts

4. Use the integral test to determine if the following series converges or diverges: $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$

let $f(x) = \frac{x}{(x^2+1)^2}$

(1) $f(x) \geq 0 \quad \forall x \geq 0$

$f'(x) = \frac{1}{(x^2+1)^2} - \frac{2x \cdot 2x}{(x^2+1)^3}$

$= \frac{1}{(x^2+1)^2} - \frac{4x^2}{(x^2+1)^3}$

$= \frac{x^2+1-4x^2}{(x^2+1)^3}$

$= \frac{1-3x^2}{(x^2+1)^3}$

(2) $f'(x) < 0 \quad \forall x \geq 1$

There fore f is decreasing

$\int_1^{\infty} \frac{x}{(x^2+1)^2}\,dx$

$= \lim_{t \to \infty} \frac{1}{2} \int_1^t \frac{2x\,dx}{(x^2+1)^2}$

$= \lim_{t \to \infty} -\frac{1}{2} \left(\frac{1}{x^2+1}\right) \Big|_1^t$

$= \frac{1}{4} - \lim_{t \to \infty} \frac{1}{2(t^2+1)}$

$= \frac{1}{4}$

since $f \geq 0$ and decreasing the integral comparison test tells us $\sum \frac{n}{(1+n^2)^2}$ converges since $\int_1^{\infty} \frac{x}{(x^2+1)^2}\,dx$ converges.

6 pts

5. Find the Taylor series (about $x = 0$) for the function:

$$f(x) = \frac{3x^2}{(x+4)(2-x)} = 3x^2\left(\frac{1}{6(x+4)} + \frac{1}{6(2-x)}\right)$$

$$\frac{1}{(x+4)(2-x)} = \frac{A}{x+4} + \frac{B}{2-x}$$

$$= \frac{2A - Ax + Bx + 4B}{(x+4)(2-x)}$$

$B - A = 0$

$2A + 4B = 1$

$6B = 1$

$B = \frac{1}{6}$

$A = \frac{1}{6}$

$$= \frac{x^2}{2}\left(\frac{1}{4\left(1-\left(\frac{-x}{4}\right)\right)} + \frac{1}{2\left(1-\frac{x}{2}\right)}\right)$$

$$= \frac{x^2}{8}\left(1 + \left(\frac{-x}{4}\right) + \left(\frac{-x}{4}\right)^2 + \cdots\right)$$

$$+ \frac{x^2}{4}\left(1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \cdots\right)$$

$$= \frac{x^2}{8}\sum_{n=0}^{\infty}\left(\frac{-x}{4}\right)^n + \frac{x^2}{4}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$$

$$= \sum_{n=0}^{\infty}\left(\frac{(-1)^n x^{n+2}}{2^{2n+3}} + \frac{x^{n+2}}{2^{n+2}}\right)$$