Microeconomic Theory October 20, 2000
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2. An application of the multi-variable IFT

We need to find $\frac{\partial k^*}{\partial r}$ and in order to do so, we will make use of the IFT.

We know that k^* (the optimal amount of capital) is determined by solving the following maximization problem:

$\max_{k,L,z} pf(k,L) - rk - w(z)L - cz$ (Note: we also choose z, to maximize)

F.O.C.:

$$pf_1(k,L) - r = 0$$
$$pf_2(k,L) - w(z) = 0$$
$$-w'(z)L - c = 0$$

S.O.C.: The Hessian Matrix H is negative definite (assuming there exists a regular interior optimum)

$$H = \begin{pmatrix} pf_{11} & pf_{12} & 0 \\ pf_{21} & pf_{22} & -w'(z) \\ 0 & -w'(z) & -w''(z) \end{pmatrix}$$

Now, we can use the FOCs to determine $\frac{\partial k^*}{\partial r}$:

(Making the usual technical assumptions,) By the IFT we know that there exists a k^* which is determined by p, r and c – hence k^* is a function of p,r and c, such that $k^* = k^*(p,r,c)$. Also we know by the IFT, that

$$H \cdot \begin{pmatrix} \frac{\partial k^*}{\partial r} \\ \frac{\partial L^*}{\partial r} \\ \frac{\partial z^*}{\partial r} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

. By Cramer's rule we find that $\frac{\partial k^*}{\partial r} = \frac{-pf_{22}w''L - (w')^2}{\det H}$.

Note, in class, I assumed a different order of the inputs in the production function. I defined f as f(L,k) instead of f(k,L), so the result given in class was $\frac{\partial k^*}{\partial r} = \frac{-pf_{11}w''L - (w')^2}{\det H}$. Both results are equivalent and correct.

2. (Cont.'d) Contrary to what I said in class, it is possible to sign $\frac{\partial k^*}{\partial r}$: The denominator is negative (following the SOC). The numerator is going to be positve, even though we cannot determine the sign of each element of the numerator (for example, we do not know the sign of w"). However,the numerator as a whole is equal to the principal minor of order 2, which we get by eliminating the 1st row & column. That is,

numerator = det $\begin{pmatrix} pf_{22} & -w' \\ -w' & -w''\cdot L \end{pmatrix}$= principal minor of order 2 of the Hessian matrix H.

We know if H is negative definite, that all principal minor of order 2 must be positive. Hence,

det $\begin{pmatrix} pf_{22} & -w' \\ -w' & -w''\cdot L \end{pmatrix} > 0$ or $-pf_{11}w''L-(w')^2 > 0$.

Using this information the sign of $\frac{\partial k^*}{\partial r}$ turns out to be negative.