

ECON 770  
Introduction to the Mathematical  
and Statistical Foundations of Econometrics  
Homework 1 (for 10/16/2007)  
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## 1 Exercise 1

Consider a probability space  $(\Omega, F, P)$  and a sequence of events  $A_i$ ,  $i = 1, 2, \dots$ , in the  $\sigma$ -field  $F$ .

1. Show that:

$$P[A_1 \cup A_2] = P(A_1) + P(A_2) - P[A_1 \cap A_2]$$

2. We want to show by recursion that:

$$P\left[\bigcup_{1 \leq i \leq n} A_i\right] = S_1 - S_2 + S_3 + \dots + (-1)^{n-1} S_n$$

where for  $k = 1, \dots, n$ :

$$S_k = \sum_{i_1 < i_2 < \dots < i_k} P[A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}]$$

where the sum is computed over the  $\binom{n}{k}$  subsets  $(i_1, i_2, \dots, i_k)$  of  $\{1, 2, \dots, n\}$ . Show that if the formula is true for any family of  $2, 3, \dots, n$  events, it is also true for any family of  $(n + 1)$  events.

3. A family of  $n$  events  $A_i$ ,  $i = 1, 2, \dots, n$  are said to be exchangeable if for any  $k = 1, 2, \dots, n$  there exists a number  $p_k$  such that for all subset  $(i_1, i_2, \dots, i_k)$  of  $\{1, 2, \dots, n\}$ , we have:

$$P[A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}] = p_k$$

Show that in this case:

$$P\left[\bigcup_{1 \leq i \leq n} A_i\right] = \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} p_k$$

4. We assume that all possible assignments of the 52 cards of a game of bridge to the 4 players have the same probability.

We denote by  $A_i$ ,  $i = 1, 2, 3, 4$ , the event:

"the player number  $i$  gets all her cards in only one suit"

Show that the events  $A_i$ ,  $i = 1, 2, 3, 4$ , are exchangeable and compute  $p_i$ ,  $i = 1, 2, 3, 4$ .

Deduce  $P\left[\bigcup_{1 \leq i \leq 4} A_i\right]$ .

## 2 Exercise 2

$X_n$  denotes a random variable with binomial distribution  $B(n, p)$ ,  $q = 1 - p$ , and  $r$  is an integer smaller than  $n$  but larger than  $np$ .

1. Show that for any integer  $k$  larger than  $r$  :

$$\frac{P[X_n = k]}{P[X_n = k - 1]} < 1 - \frac{r - np}{rq}$$

2. Deduce:

$$P[X_n \geq r] \leq P[X_n = r] \frac{rq}{r - np}$$

3. Show that:

$$P[X_n = r] \leq \frac{1}{r - np}$$

*Indication:* Think about the number of possible values between the mode of the distribution and the value  $r$ .

4. Deduce:

$$P[X_n \geq r] \leq \frac{rq}{(r - np)^2}$$

5. Show without any computation that for a positive integer  $s$  smaller than  $np$  :

$$P[X_n \leq s] \leq \frac{(n - s)p}{(np - s)^2}$$

6. Deduce that for any  $\varepsilon > 0$  :

$$\lim_{n \rightarrow \infty} P\left[\left|\frac{X_n}{n} - p\right| \geq \varepsilon\right] = 0$$

7. Interpret this result.

## 3 Exercise 3

We consider a sample space  $\Omega$ ,  $F$  a family of subsets of  $\Omega$  and  $\Omega' \subset \Omega$ .

1. Consider the mapping  $f$  from  $\Omega'$  to  $\Omega$  defined by

$$\forall \omega \in \Omega', f(\omega) = \omega$$

Characterize  $f^{-1}(F)$ .

2. We denote by  $\Omega' \cap F$  the sets of all  $\Omega' \cap A$  for  $A \in F$ .

Show that the smallest  $\sigma$ -field on  $\Omega'$  containing  $\Omega' \cap F$  is  $\Omega' \cap \sigma(F)$  where  $\sigma(F)$  denotes the smallest  $\sigma$ -field on  $\Omega$  containing  $F$ .

## 4 Exercise 4

An urn contains  $b$  black and  $r$  red balls.

A ball is drawn at random. It is replaced and, moreover,  $c$  balls of the color drawn and  $d$  balls of the opposite color are added.

A new random drawing is made from the urn (now containing  $r + b + c + d$  balls)

and this procedure is repeated.

1. Compute the probability that of  $n = n_1 + n_2$  drawings the first  $n_1$  ones result in black balls and the remaining ones in red balls.
2. In all the rest of the exercise, we assume  $d = 0$ . Show that all possible sequences of  $n_1$  black and  $n_2$  red balls have the same probability.
3. Deduce that the probability  $p(n_1, n)$  that  $n$  drawings result in  $n_1$  black and  $n_2$  red balls can be written:

$$p(n_1, n) = \frac{\binom{n_1-1+b/c}{n_1} \binom{n_2-1+r/c}{n_2}}{\binom{n-1+(b+r)/c}{n}}$$

4. Show that:

$$p(n_1, n) = \frac{\binom{-b/c}{n_1} \binom{-r/c}{n_2}}{\binom{-(b+r)/c}{n}}$$

5. Check by direct computation that:

$$\sum_{n_1=0}^n p(n_1, n) = 1$$