Department of Economics Fall 2007
UNC-Chapel Hill

ECON 770
Introduction to Econometrics
Homework 3 (due for December 4)
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EXERCISE 1:

Let X_n , n= 1,2,… be a sequence of real valued random variables. X_n converges almost surely towards a variable X ($X_n \rightarrow_{as} X$) if P { ω; X(ω) = $\text{Lim}_{n=\infty} X_n(\omega)$} = 1. We have seen in class that this is equivalent to:

$$\forall \varepsilon > 0 \text{ , } \text{Lim}_{N=\infty} P[\ \text{Sup}_{n \geq N} |X_n - X| \leq \varepsilon] = 1.$$

1.

a) Show that for any α>0:

$$\varepsilon^{\alpha}\ \mathbf{1}[\ |X_n - X| \geq \varepsilon] \leq |X_n - X|^{\alpha} .$$

b) Deduce that:

$$P[\ |X_n - X| \geq \varepsilon] \leq (1/\varepsilon^{\alpha})E\ |X_n - X|^{\alpha} .$$

c) Deduce that for $\sigma(X) = [\text{Var}(X)]^{1/2}$;

$$P[\ |X - E(X)| < 2\sigma(X)] \geq 0.75 .$$

2. We say that X_n converges in probability towards a variable X ($X_n \rightarrow_P X$) if:

$$\forall \varepsilon > 0 \text{ , } \text{Lim}_{n=\infty} P[\ |X_n - X| > \varepsilon] = 0.$$

a) Show that:

$$(X_n \rightarrow_{as} X) \Rightarrow (X_n \rightarrow_P X).$$

b) Show that:

$$\sum_{n=1}^{\infty} P[\ |X_n - X| > \varepsilon] < \infty \Rightarrow (X_n \rightarrow_{as} X).$$

3. We say that X_n converges L^2 (or in Mean Squared Error) towards a variable X ($X_n \rightarrow_{MSE} X$) if:

$$\text{Lim}_{n=\infty} E[(X_n - X)^2] = 0.$$

a) Show that:

$$(X_n \rightarrow_{MSE} X) \Rightarrow (X_n \rightarrow_P X).$$

b) Show that:

$$\sum_{n=1}^{\infty} E[(X_n - X)^2] < \infty \Rightarrow (X_n \rightarrow_{as} X).$$

c) Show that:

$$X_n \rightarrow_{MSE} a \text{ , constant number,}$$

If and only if:

$$E(X_n) \rightarrow a \text{ and } \text{Var}(X_n) \rightarrow 0.$$

4. In this question we assume that:

$$n \neq q \Rightarrow Cov[X_n , X_q] = 0,$$

and:

$$(1/n^2) \Sigma_{q=1}^{n} Var(X_q) \rightarrow_{n=\infty} 0.$$

We denote Y_n the sample mean:

$$Y_n = (1/n) \Sigma_{q=1}^{n} X_q .$$

a) Show that the above assumption about the asymptotic behavior of variance is fulfilled in particular if:

$$Var(X_n) = \sigma^2 \text{ , independent of n.}$$

b) Show that:

$$Y_n \rightarrow_P E(X).$$

c) Show that:

$$\Sigma_{n=1}^{\infty} (1/n^2) \Sigma_{q=1}^{n} Var(X_q) < \infty \Rightarrow Y_n \rightarrow_{as} E(X).$$

Does it help in this respect to know that $Var(X_n) = \sigma^2$, independent of n?

EXERCISE 2:

Let X_n , n= 1,2,… be a sequence of real valued random variables. We assume in all the exercise that X_n is a martingale, that is:

$$\forall n, \ E[X_{n+1} / B_n] = X_n ,$$

where $B_n = \sigma [X_q ; q \leq n]$.

1. Interpret the martingale property and show that:

$$E[X_{n+k} / B_n] = X_n \text{ for any positive integer k.}$$

2. We assume in this question that:

$$(1/n^2) Var(X_n) \rightarrow_{n=\infty} 0.$$

Show that:

$$(1/n) \Sigma_{q=1}^{n} (X_{q+1} - X_q) \rightarrow_{MSE} 0 .$$

3. Show that:

$$\forall n, \ E[X_n^2] \leq E[X_{n+1}^2].$$

We assume throughout that $\forall n, \ E[X_n^2] \leq c$ for some given number c.

4. Show that for n<q:

$$E[(X_q - X_n)^2] = E[X_q^2] - E[X_n^2].$$

5. Deduce that:

$$Lim_{n=\infty} Sup_{q>n} E[(X_q - X_n)^2] = 0.$$

We admit that this allows us to conclude that for some variable X_∞ ,

$$X_n \rightarrow_{MSE} X_\infty .$$

6. We want to show in this question that the result of question 1 even allows to conclude that:

$$E[X_\infty / B_n] = X_n .$$

a) Show that for every $A \in B_n$ and for any positive integer k:

$$E[\mathbf{1}_A X_{n+k}] = E[\mathbf{1}_A X_n]$$

b) Conclude.