Why Gaussian Macro-Finance Term Structure Models Are (Nearly) Unconstrained Factor-VARs *

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Abstract

This paper explores the impact of simultaneously enforcing the no-arbitrage structure of a Gaussian macro-finance term structure model (*MTSM*) and accommodating measurement errors through filtering on the maximum likelihood estimates of the model-implied conditional distributions of the macro risk factors and bond yields. For the typical yield curves and macro variables studied in this literature, the estimated joint distribution within a canonical *MTSM* is nearly identical to the estimate from an economic-model-free factor vector-autoregression (factor-VAR), even when measurement errors are large. It follows that a canonical *MTSM* does not offer any new insights into economic questions regarding the historical distribution of the macro risk factors and yields, over and above what is learned from a factor-VAR. In particular, the discipline of a canonical *MTSM* is empirically inconsequential for analyses of impulse response functions of bond yields and macro factors or empirical studies of term premiums. These results are rotation-invariant and, therefore, apply to many of the specifications of risk factors in the literature. In deriving these results we develop a new canonical form for *MTSM*s that is particularly revealing about the nature of the over-identifying restrictions implied by *MTSM*s relative to yield-based factor models.

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1 Introduction

Gaussian macro-dynamic term structure models (*MTSM*s) typically feature three key ingredients: (i) a low-dimensional factor-structure in which the risk factors are both macroeconomic and yield-based variables; (ii) the assumption of no arbitrage opportunities in bond markets; and (iii) accommodation of measurement errors in bond markets owing to the presence of microstructure noise or errors introduced by the bootstrapping of zero-coupon yields. The low-dimensional factor structure is motivated by the observation that most of the variation in bond yields is explained by a small number of principal components (PCs).[1] The overlay of an arbitrage-free *MTSM* brings information about the entire yield curve to bear on the links between macroeconomic shocks and bond yields, in a consistent structured way. Thirdly, with measurement errors on bond yields,[2] *MTSM*s are formulated as state-space models and estimation proceeds using filtering.

This paper takes the low-dimensional factor structure of bond yields and macro factors imposed in *MTSM*s as given and explores the implications of no-arbitrage and the presence of measurement errors for the Kalman filter estimator of the joint distribution of these variables. Initially we follow the extant literature and assume that macro risk factors are measured without errors. We derive sufficient and easily verified theoretical conditions for the Kalman filter estimator within a canonical[3] *MTSM* to be (nearly) identical to the ordinary least-squared (OLS) estimator of an unconstrained factor-VAR. We show that these conditions are very nearly satisfied by the canonical versions of several prominent *MTSM*s. The practical implication of our analysis is that canonical *MTSM*s typically do not offer any new insights into economic questions regarding the historical distribution of macro variables and yields, over and above what one can learn from an economics-free factor-VAR.

Our theoretical propositions focus on the *entire conditional distribution* of the risk factors and bond yields in models where *all bond yields are measured with errors* and so filtering must be used in estimation. Both of these ingredients are essential for exploring what *MTSM*s teach us about the impulse responses (IRs) of bond yields to macro shocks.[4] The theoretical propositions and empirical illustrations about the role of no-arbitrage restrictions in Joslin, Singleton, and Zhu (2011) (JSZ) and Duffee (2011a) are largely silent on this issue, because they focus on the conditional means (forecasts) of yield-based risk factors within models

[1] This has been widely documented for U.S. Treasury yields (e.g., Litterman and Scheinkman (1991)). Ang, Piazzesi, and Wei (2006) and Bikbov and Chernov (2010) are among the many studies of *MTSM*s that base their selection of a small number of risk factors (typically three or four) on similar PC evidence.

[2] Virtually the entire literature on *MTSM*s assumes that the macro factors are measured *without* errors. See Duffee (1996) for a discussion of measurement issues at the short end of the Treasury curve. The use of splines to extract zero-coupon yields from coupon yield curves and the differing degrees of liquidity of individual bonds along the yield curve introduce errors in yields along the entire maturity spectrum.

[3] A canonical model is maximally flexible (in the sense that each member of the family of *MTSM*s is represented) and enforces a minimal set of normalizations to ensure econometric identification.

[4] Recent analyses of IRs within *MTSM*s include Ang and Piazzesi (2003) who examine the responses of bond yields to their macro risk factors; Bikbov and Chernov (2010) who quantify the proportion of bond yield variation attributable to macro risk factors; and Joslin, Priebsch, and Singleton (2011) who quantify the effects of unspanned macro risks on forward term premiums.

which maintain a good cross-sectional fit to the yield curve. In contrast, we examine whether the imposition of the structure of a *MTSM* affects features of the risk factors that depend on *both* the conditional mean and variance parameters (as do *IR*s and term premiums). Moreover, we allow all of the individual yields to be priced imperfectly, possibly with large errors. Filtering often has little effect on *ML* estimators in Gaussian models with latent or yield-based risk factors (*YTSM*s), in large part because the standard deviations of these errors are typically small (only a few basis points).[5] In contrast, pricing errors on individual bond yields in *MTSM*s exceed 100 basis points in some prominent *MTSM*s.

A key condition for a *MTSM* and its factor-*VAR* counterpart to produce (nearly) identical conditional distributions of the risk factors when there are pricing errors of this magnitude is that the ratio of the average pricing errors to their variances for the yield-based risk factors be approximately zero. Historical and *MTSM*-implied low-order *PC*s track each other very closely, even though the pricing errors on individual bonds are at times large, and this is what drives our empirical findings of irrelevance. Our propositions also provide a theoretical underpinning for the findings in JSZ and Duffee (2011b) that higher-order *PC*s are not accurately priced in five-factor *YTSM*s.

To derive our irrelevance results we develop a canonical form for the family of $\mathcal{N}$-factor *MTSM*s in which $\mathcal{M}$ of the factors are the macro variables M_t and the remaining $\mathcal{L} = \mathcal{N} - \mathcal{M}$ risk factors are the first $\mathcal{L}$ principal components (*PC*s) of bond yields, $\mathcal{P}_t^{\mathcal{L}}$. This form provides an organizing framework within which it is easy to determine whether a *MTSM* is econometrically identified. It also leads directly to a formal characterization of the added flexibility of a *MTSM* (relative to an $\mathcal{N}$-factor model with no macro risk factors) in terms of a theoretical spanning condition of M_t by the first $\mathcal{N}$ *PC*s of yields.

Using this canonical form we show that our irrelevancy propositions are fully rotation invariant:[6] if our sufficient conditions are satisfied, then all choices of individual yields or *PC*s of yields as elements of $\mathcal{P}_t^{\mathcal{L}}$ necessarily result in identical (inconsequential) effects of no-arbitrage restrictions. Moreover when $\mathcal{P}_t^{\mathcal{L}}$ is normalized to be $\mathcal{L}$ low-order *PC*s, then the model-implied joint distribution of $Z_t' \equiv (M_t', \mathcal{P}_t^{\mathcal{L}}{}')$ is virtually identical to the one implied by a standard unconstrained VAR model of the *observed* risk factors Z_t^o.

Another important insight that emerges from this analysis is that, in *MTSM*s in which certain portfolios of yields are assumed to be priced perfectly, the choice of these portfolios is not innocuous. For instance, in otherwise identical *MTSM*s that assume $\mathcal{P}_t^{\mathcal{L}} = \mathcal{P}_t^o$, the *IR*s of bond yields to macro shocks can vary substantially across alternative choices of $\mathcal{P}_t^{\mathcal{L}}$ (e.g., a *PC* or an individual bond yield). Such *IR*s are fully invariant to the choice of $\mathcal{P}^{\mathcal{L}}$ when all bonds are priced with error.

We explore the empirical relevance of our propositions within a three-factor *MTSM*– model $GM_3(g, \pi)$– in which the risk factors are output growth (g), inflation (π), and the first

[5] This is documented in JSZ for estimates of the conditional mean parameters in *YTSM*s, and in Duffee (2011a) for the loadings that link the yield-based risk factors to the prices of individual bonds.

[6] See Dai and Singleton (2000) for the definition of invariant affine transformations. Such transformations lead to equivalent models in which the pricing factors $\tilde{\mathcal{P}}_t^{\mathcal{N}}$ are obtained by applying affine transformations of the form $\tilde{\mathcal{P}}_t^{\mathcal{N}} = C + D\mathcal{P}_t^{\mathcal{N}}$, for nonsingular $\mathcal{N} \times \mathcal{N}$ matrix D.

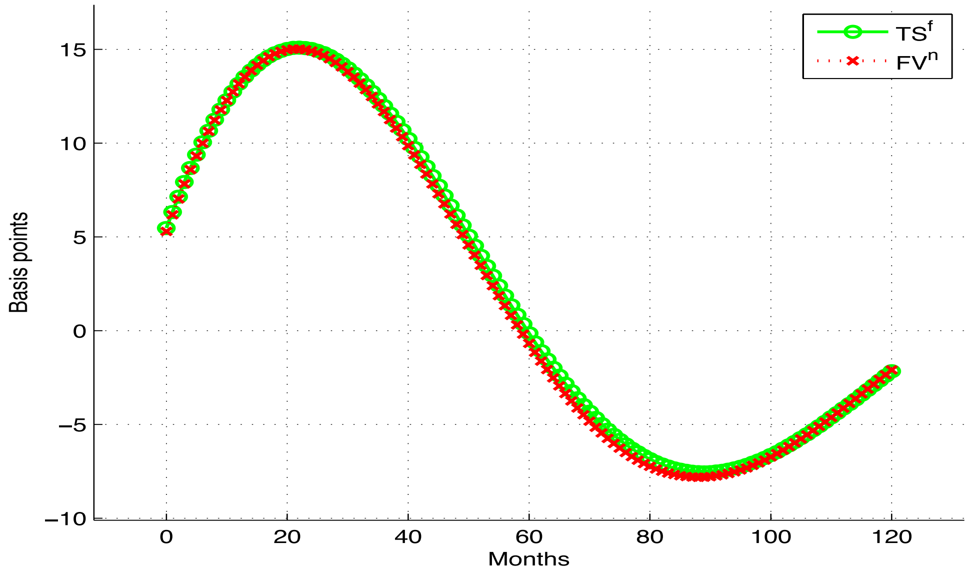


Figure 1: Impulse responses in basis points of $PC1$ to a shock to inflation in model $GM_3(g,\pi)$ (TS^f) and its corresponding factor-VAR (FV^n).

PC of bond yields ($PC1$).[7] The no-arbitrage structure of $GM_3(g,\pi)$ implies over-identifying restrictions on the distribution of bond yields, and its Kalman filter estimates imply root mean-squared pricing errors on the order of forty basis points (see Section 4.1). Nevertheless, the IRs of $PC1$ to a shock to inflation implied by $GM_3(g,\pi)$ and by its corresponding factor-VAR (FV^n) are virtually indistinguishable (Figure 1).

Up to this point we follow the literature on macro-finance term structure models in assuming that the macro risk factors are measured without error. To break the asymmetric treatment of yield and macro factors we next allow for all of the observed variables– y_t and M_t alike– to be measured with errors. For *MTSM*s with a small number of macro factors– typical of this literature– we obtain the striking result that the filtered risk factors closely resemble the low dimensional PCs of bond yields. That is, once measurement errors on M_t are accommodated, the likelihood function largely gives up on fitting the observed macro factors in favor of more accurate pricing of bonds.

These results suggest that caution is in order when drawing conclusions about the joint distribution of macro risks and bond yields from *MTSM*s in which macro variables appear as risk factors. If, as is typically the case, the macro factors are assumed to be measured without errors, then low dimensional *MTSM*s may price individual bond yields very poorly. In such circumstances, conclusions drawn about the nature of risk premiums in bond markets, and in particular their relationships with macro risks, may be unreliable. Allowing for measurement

[7] Full details of the data and estimation results are provided in Section 4.3. Unless otherwise noted, the loadings for $PC1$ are rescaled so that they add up to one.

errors on the macro factors improves the fits for individual bond yields. However, *MTSM*s that accommodate filtering on M_t may attribute implausibly large percentages of variation in M_t to measurement errors, thereby rendering such *MTSM*s largely uninformative about the nature of say *IR*s of yields on bond portfolios to macro shocks.

Our irrelevance results extend to canonical versions of *MTSM*s in which the macro factors M_t are unspanned by bond yields (Joslin, Priebsch, and Singleton (2011)). However, the implications of accommodating measurement errors on M are very different across *MTSM*s with spanned versus unspanned macro risks. Extending model $GM_3(g, \pi)$ to a *MTSM* in which $M_t' = (g_t, \pi_t)$ is unspanned by the yield PCs, keeping measurement errors on both bond yields and M, we obtain the striking result that M re-emerges as a significant predictor of risk premiums in bond markets, *over and above yield curve information.* That is, the model with unspanned macro risks shows that filtered macro factors embody important information about risk in bond markets, whereas M is largely "filtered out" of *MTSM*s with spanned macro risks in favor of matching yields with PC-like risk factors.

Finally, throughout this analysis our focus is on canonical models. Certain types of restrictions, when imposed in combination with the no-arbitrage restrictions of a *MTSM*, may overturn our irrelevancy results and increase the efficiency of ML estimators relative to those of the unconstrained VAR. Most studies of *MTSM*s have left open the question of whether their particular formulations led to materially different estimates of historical distributions relative to those from a VAR.[8] We show that our irrelevancy results are robust to several often-imposed constraints on the $\mathbb{P}$ distribution of Z_t.

To fix notation, suppose that a *MTSM* is evaluated using a set of J yields $y_t = (y_t^{m_1}, \ldots, y_t^{m_J})'$ with maturities $(m_1, \ldots, m_J)$ with $J \geq \mathcal{N}$, where $\mathcal{N}$ is the number of pricing factors. We introduce a fixed, full-rank matrix of portfolio weights $W \in \mathbb{R}^{J \times J}$ and define the "portfolios" of yields $\mathcal{P}_t = W y_t$ and, for any $j \leq J$, we let $\mathcal{P}_t^j$ and W^j denote the first j portfolios and their associated weights. The modeler's choice of W will determine which portfolios of yields enter the *MTSM* as risk factors and which additional portfolios are used in estimation. Throughout, we assume a flat prior on the initial observed data.

2 A Canonical *MTSM*

This section gives a heuristic construction of our canonical form; formal regularity conditions and a proof that our form is canonical are presented in Appendix A. Suppose that $\mathcal{M}$ macroeconomic variables M_t enter a *MTSM* as risk factors and that the one-period interest rate r_t is an affine function of M_t and an additional $\mathcal{L}$ pricing factors $\mathcal{P}_t^{\mathcal{L}}$,

$$r_t = \rho_0 + \rho_{1M} \cdot M_t + \rho_{1\mathcal{P}} \cdot \mathcal{P}_t^{\mathcal{L}} \equiv \rho_0 + \rho_1 \cdot Z_t, \quad Z_t = (M_t', \mathcal{P}_t^{\mathcal{L}\prime})'. \tag{1}$$

For now, we will suppose that all of the elements in Z_t incrementally affect bond prices; see Section 2.5 for a relaxation of this assumption. Some treat $\mathcal{P}_t^{\mathcal{L}}$ in (1) as a set of $\mathcal{L}$ latent

[8] JSZ and Duffee (2011a) explore empirically whether various constraints on the $\mathbb{P}$ distribution of the risk factors in *YTSM*s improve out-of-sample forecasts of these factors. We look beyond their focus on conditional means and perfectly priced risk factors to the new issues that arise in *MTSM*s.

risk factors,[9] while others include portfolios of yields as risk factors.[10] Fixing M_t and the dimension $\mathcal{L}$ of $\mathcal{P}_t^{\mathcal{L}}$, these two theoretical formulations are observationally equivalent. In fact, as we show, we are free to rotate (see Appendix C) the entire vector Z_t to express bond prices in terms of $\mathcal{P}_t^{\mathcal{N}}$, the first $\mathcal{N} = \mathcal{M} + \mathcal{L}$ entries of the modeler's chosen portfolios of yields. This is an implication of affine pricing of $\mathcal{P}_t^{\mathcal{N}}$ in terms of Z_t. Accordingly, in characterizing a canonical form for the family of *MTSM*s with short-rate processes of the form (1), we are free to start with either interpretation of $\mathcal{P}_t^{\mathcal{L}}$ (latent or yield-based) and to use any of these rotations of the risk factors Z_t.

We select a rotation of Z_t and its associated risk-neutral ($\mathbb{Q}$) distribution so that our maximally flexible canonical form is particularly revealing about the joint distribution of Z_t and bond yields implied by *MTSM*s with $\mathcal{N}$ pricing factors that include M_t.

2.1 The Canonical Form

Consider a *MTSM* with risk factors Z_t and short rate as in (1), with Z_t following a Gaussian process under the risk-neutral distribution,

$$\Delta Z_t = K_0^{\mathbb{Q}} + K_1^{\mathbb{Q}} Z_{t-1} + \sqrt{\Sigma}\epsilon_t^{\mathbb{Q}},\ \epsilon_t^{\mathbb{Q}} \sim N(0, I). \tag{2}$$

Absent arbitrage opportunities in this bond market, (1) and (2) imply affine pricing of bonds of all maturities (Duffie and Kan (1996)). The yield portfolios $\mathcal{P}_t$ can be expressed as

$$\mathcal{P}_t = A_{TS} + B_{TS} Z_t, \tag{3}$$

where the loadings (A_{TS}, B_{TS}) are known functions of the parameters $(K_0^{\mathbb{Q}}, K_1^{\mathbb{Q}}, \rho_0, \rho_1, \Sigma)$ governing the risk neutral distribution of yields, and hereafter "TS" denotes features of a *MTSM*. A canonical version of this model is obtained by imposing normalizations that ensure that the only admissible rotation of Z_t that leaves the distribution of r_t unaffected is the identity matrix. To arrive at our canonical form we observe that from the first $\mathcal{N}$ entries of (3), Z_t, and hence all bond yields y_t, can be expressed as affine functions of $\mathcal{P}_t^{\mathcal{N}}$.[11] After rotating to a pricing model with risk factors $\mathcal{P}_t^{\mathcal{N}}$, we adopt the canonical form of JSZ. What is distinctive about their canonical form is that the risk-neutral distribution of $\mathcal{P}_t^{\mathcal{N}}$ is fully characterized by the covariance matrix Σ, the long-run $\mathbb{Q}$-mean $r_\infty^{\mathbb{Q}} \equiv E^{\mathbb{Q}}[r_t]$ of r_t, and the rotation invariant (and hence economically interpretable) $\mathcal{N}$-vector $\lambda^{\mathbb{Q}}$ of distinct real eigenvalues of the feedback matrix $K_1^{\mathbb{Q}}$.[12]

[9]Studies with this formulation include Ang and Piazzesi (2003), Ang, Dong, and Piazzesi (2007), Bikbov and Chernov (2010), Chernov and Mueller (2011), and Smith and Taylor (2009).

[10]Examples include Ang, Piazzesi, and Wei (2006) and Jardet, Monfort, and Pegoraro (2010).

[11]This inversion presumes that the $\mathcal{N}$-factor *MTSM* is non-degenerate in the sense that all $\mathcal{M}$ macro factors distinctly contribute to the pricing of bonds after accounting for the remaining $\mathcal{L}$ factors. Formal regularity conditions are provided in Appendix A.

[12]Extensions to the more general case of $K_1^{\mathbb{Q}}$ being in ordered real Jordan form, or to a zero root in the $\mathbb{Q}$ process of Z_t, are straightforward along the lines of Theorem 1 in JSZ.

A key implication of (3) is that, within any *MTSM* that includes M_t as pricing factors in (1), these macro factors must be spanned by $\mathcal{P}_t^{\mathcal{N}}$:

$$M_t = \gamma_0 + \gamma_1 \mathcal{P}_t^{\mathcal{N}}, \tag{4}$$

for some conformable γ_0 and γ_1 that implicitly depend on W. Using (4), we apply the rotation

$$Z_t = \begin{pmatrix} M_t \\ \mathcal{P}_t^{\mathcal{L}} \end{pmatrix} = \begin{pmatrix} \gamma_0 \\ 0 \end{pmatrix} + \begin{pmatrix} & \gamma_1 & \\ I_{\mathcal{L}} & 0_{\mathcal{L}\times(\mathcal{N}-\mathcal{L})} \end{pmatrix} \mathcal{P}_t^{\mathcal{N}} \tag{5}$$

to the canonical form in terms to $\mathcal{P}_t^{\mathcal{N}}$ to obtain an equivalent model in which the risk factors are M_t and $\mathcal{P}_t^{\mathcal{L}}$, r_t satisfies (1), and Z_t follows the Gaussian $\mathbb{Q}$ process (2). Our specification is completed by assuming that, under the historical distribution $\mathbb{P}$, Z_t follows the process

$$\Delta Z_t = K_0^{\mathbb{P}} + K_1^{\mathbb{P}} Z_{t-1} + \sqrt{\Sigma}\epsilon_t^{\mathbb{P}},\ \ \epsilon_t^{\mathbb{P}} \sim N(0, I). \tag{6}$$

Summarizing, in our canonical form the first $\mathcal{M}$ components of the pricing factors Z_t are the macro variables M_t, and without loss of generality the risk factors are rotated so that the remaining $\mathcal{L}$ components of Z_t are the "state yield portfolios" $\mathcal{P}_t^{\mathcal{L}}$ (the first $\mathcal{L}$ components of $\mathcal{P}_t^{\mathcal{N}}$); r_t is given by (1); M_t is related to $\mathcal{P}_t$ through (4); and Z_t follows the Gaussian $\mathbb{Q}$ and $\mathbb{P}$ processes (2) and (6). Moreover, for given W, the risk-neutral parameters $(\rho_0, \rho_1, K_0^{\mathbb{Q}}, K_1^{\mathbb{Q}})$ are explicit functions of $\Theta_{TS}^{\mathbb{Q}} \equiv (r_\infty^{\mathbb{Q}}, \lambda^{\mathbb{Q}}, \gamma_0, \gamma_1, \Sigma)$.

Our canonical construction reveals the essential difference between term structure models based entirely on yield-based pricing factors $\mathcal{P}_t^{\mathcal{N}}$ and those that include macro risk factors. A *MTSM* with pricing factors $(M_t, \mathcal{P}_t^{\mathcal{L}})$ offers more flexibility in fitting the joint distribution of bond yields than a pure latent factor model (one in which $\mathcal{N} = \mathcal{L}$), because the "rotation problem" of the risk factors is most severe in the latter setting. In the JSZ canonical form with pricing factors $\mathcal{P}_t^{\mathcal{N}}$, the underlying parameter set is $(\lambda^{\mathbb{Q}}, r_\infty^{\mathbb{Q}}, K_0^{\mathbb{P}}, K_1^{\mathbb{P}}, \Sigma)$. A *MTSM* adds the spanning property (4) with its $\mathcal{M}(\mathcal{N}+1)$ free parameters. Thus, any canonical $\mathcal{N}$-factor *MTSM* with macro factors M_t gains $\mathcal{M}(\mathcal{N}+1)$ free parameters relative to pure latent-factor Gaussian models. Of course this added flexibility (by parameter count) of a *MTSM* is gained at a cost: the realizations of M_t are linked to the yield-based risk factors by (4).

In taking the model to the data, we accommodate the fact that the observed data $\{M_t^o, \mathcal{P}_t^o\}$ will not be perfectly matched by a theoretical no-arbitrage model. Accordingly we suppose that the observed yield portfolios $\mathcal{P}_t^o$ are equal to their theoretical values plus a mean-zero measurement error. Absent any guidance from economic theory, and consistent with the literature, we presume that the measurement errors are *i.i.d.* normal, thereby giving rise to a Kalman filtering problem.[13] The observation equation is then (3) adjusted for these errors:

$$\mathcal{P}_t^o = A_{TS}(\Theta_{TS}^{\mathbb{Q}}) + B_{TS}(\Theta_{TS}^{\mathbb{Q}})\, Z_t + e_t, \quad e_t \sim N(0, \Sigma_e), \tag{7}$$

the state equation is (6), and (A_{TS}, B_{TS}) are functions of the parameters $\Theta_{TS}^{\mathbb{Q}}$ of our normalization. Until Section 5 we follow the literature and assume that the observed macro factors M_t^o coincide with their theoretical counterparts M_t. Together (6) and (7) comprise the state space representation of the *MTSM*. The full parameter set is $\Theta_{TS} = (\Theta_{TS}^{\mathbb{Q}}, K_0^{\mathbb{P}}, K_1^{\mathbb{P}}, \Sigma_e)$.

[13] This formulation subsumes the case of cross-sectionally uncorrelated pricing errors (Σ_e is diagonal) adopted by Ang, Dong, and Piazzesi (2007) and Bikbov and Chernov (2010), as well as the case where Σ_e is singular with the first $\mathcal{L}$ rows and columns of Σ_e equal to zero. In the latter case, $\mathcal{P}_t^{\mathcal{L}} = \mathcal{P}_t^{\mathcal{L}o}$.

2.2 State-Space Formulations Under Alternative Hypotheses

Throughout our subsequent analysis we compare the *MTSM*s characterized by (6) and (7) to their "unconstrained alternatives." Since a *MTSM* involves multiple over-identifying restrictions, the relevant alternative model depends on which of these restrictions one is interested in relaxing.

The FV ("factor-VAR") alternative follows Duffee (2011a) and maintains the state equation (6), but generalizes the observation equation to

$$\mathcal{P}_t^o = A_{FV} + B_{FV}\, Z_t + e_t, \tag{8}$$

for conformable matrices A_{FV} and B_{FV}, with e_t normally distributed from the same family as the *MTSM*. For identification we normalize the first $\mathcal{L}$ entries of A_{FV} to zero and the first $\mathcal{L}$ rows of B_{FV} to the corresponding standard basis vectors. Except for this, A_{FV} and B_{FV} are free from any restrictions.[14] The full parameter set is $\Theta_{FV} = (A_{FV}, B_{FV}, K_0^{\mathbb{P}}, K_1^{\mathbb{P}}, \Sigma, \Sigma_e)$. Since all bonds are priced with errors, the FV model is estimated using the Kalman filter.

A less constrained alternative would have $\mathcal{P}_t^o$ (or y_t) following a full J-dimensional VAR as, for instance, in Ang and Piazzesi (2003). However, comparing a *MTSM* to this alternative confounds the restrictions that bond yields lie in a low-dimensional factor space with the no-arbitrage constraints. Moving from an unconstrained J-dimensional VAR to the factor-VAR in (8) can improve the precision of estimates of the historical distribution of bond yields and, in fact, this is illustrated by the findings in Ang, Piazzesi, and Wei (2006). Such an improvement may arise even if no-arbitrage restrictions have no impact on fit. By taking (8) as our alternative, we home in on the roles of no-arbitrage and filtering in studies of *MTSM*s.

2.3 Model Specifications

Since we examine a wide variety of measurement assumptions about Z_t in both arbitrage-free *MTSM*s (TS) and unconstrained factor-VARs (FV), it is instructive to summarize the cases examined into Table 1. The superscript n indicates no errors in measuring the risk factors ($M_t^o = M_t$ and $\mathcal{P}_t^{\mathcal{L}o} = \mathcal{P}_t^{\mathcal{L}}$). When the yield factors are filtered ($\mathcal{P}_t^{\mathcal{L}o} \neq \mathcal{P}_t^{\mathcal{L}}$ and $M_t^o = M_t$) we use the superscript f; and when both the yields and macro variables are filtered we use the superscript fm. Finally, $TS_{\mathcal{P}}^n$ and $FV_{\mathcal{P}}^n$ refer to *MTSM*s in which $\mathcal{P}_t^{\mathcal{N}} = \mathcal{P}_t^{\mathcal{N}o}$ and the macro-variables are measured with error ($M_t^o \neq M_t$). TS and FV alone refer to the generic features of $A_{TS} + B_{TS} Z_t$ and $A_{FV} + B_{FV} Z_t$, respectively, along with (6).

Relative to model TS^f, model FV^f relaxes the over-identifying restrictions implied by the assumption of no arbitrage, but maintains the low-dimensional factor structure of returns and the presumption of measurement errors on bond yields. Thus, in assessing whether these two models imply nearly identical joint distributions for (y_t, M_t), the focus is on whether the no arbitrage restrictions induce a difference. On the other hand, differences between the TS^f and TS^n models, which both maintain a similar no-arbitrage structure, should arise

[14] A subtle issue is that this is slightly over-identifying since it implies that a relationship of the form $\alpha + \beta \cdot \mathcal{P}_t^{\mathcal{L}} = 0$ cannot hold in the model. Certainly this would be rejected in the data for typical choices of W. However, the ODE theory implies this normalization is just-identifying in the no-arbitrage model.

Model Name	No Arbitrage Imposed	Measurement Errors for Yield Factors	Measurement Errors for Macro Variables
TS^f	X	X	
TS^n	X		
FV^f		X	
FV^n			
TS^{fm}	X	X	X
FV^{fm}		X	X
$TS^n_{\mathcal{P}}$	X		X
$FV^n_{\mathcal{P}}$			X

Table 1: Summary of the notations for different model specifications.

mainly out of the different treatments of measurement errors of the pricing factors. Finally, in moving from model TS^f to model FV^n one is relaxing both the no arbitrage restrictions and adding the presumption that the entire state vector Z_t is measured without errors ($\mathcal{P}_t^{\mathcal{L}o} = \mathcal{P}_t^{\mathcal{L}}$ and $M_t^o = M_t$ in model FV^n), while maintaining the low-dimensional factor structure.

2.4 Discussion

A key feature of our normalization is that it imposes "pricing consistency" in the sense that the state yield portfolios recovered from the pricing equation (3) always agree with their theoretical values. Ang, Piazzesi, and Wei (2006) and Jardet, Monfort, and Pegoraro (2010) enforce pricing consistency by minimizing sums of squared pricing errors subject to a consistency constraint. Their approach requires that their state yield portfolios are priced perfectly by the *MTSM*, and their two-step estimation strategy is asymptotically inefficient. In this section we show that our choice of canonical form automatically enforces pricing consistency even when all bonds are priced imperfectly by the *MTSM* and, accordingly, Kalman filter estimators are fully efficient.

Equally importantly, our canonical forms for the TS^f and FV^f models are invariant with respect to the modeler's choice of W. That is, all admissible choices of W– e.g., choices that set the state yield factors to individual yields or to low-order PCs of bond yields– lead to exactly the same Kalman filter estimates of the parameters of the joint distribution of (y_t^o, M_t^o). In fact, so long as one enforces the model-implied spanning condition (4), representations of model TS^f in which the risk factors are all yield portfolios (e.g., $Z_t = \mathcal{P}_t^{\mathcal{N}}$) or the mix $(M_t, \mathcal{P}_t^{\mathcal{L}})$ of macro and yield-based factors lead to identical fitted moments of (y_t^o, M_t^o) regardless of the choice of admissible W.

The remainder of this section discusses each of these points in turn.

Pricing Consistency

To illustrate the consistency issue, consider the *MTSM* with a single macro variable ($\mathcal{M} = 1$), and two pricing factors ($\mathcal{L} = 2$) with W chosen so that the two state yield portfolios are the short rate and the two-year (twenty-four month) rate: $Z_t = (M_t, r_t, y_t^{24})$. Pricing consistency requires that when one computes the loadings for the two-year yield from (3) by solving the recurrence relation given in Appendix B, it must be that the intercept is 0 and the loadings on Z_t are $(0, 0, 1)$. The two-year rate, up to convexity, is the average of expected future short rates. Since our model is Gaussian, the convexity term is constant. Thus, for a monthly sampling frequency, we require

$$y_t^{24} = \frac{1}{24} E_t^{\mathbb{Q}} \left[\sum_{\tau=0}^{23} r_{t+\tau} \right] + \text{constant}. \tag{9}$$

The $\mathbb{Q}$-expectations in (9) can be computed according to the dynamics in (2) which give

$$E_t^{\mathbb{Q}}[r_{t+\tau}] = (0, 1, 0) E_t^{\mathbb{Q}}[Z_{t+\tau}] = (0, 1, 0)(I + K_1^{\mathbb{Q}})^{\tau} Z_t + \text{constant}.$$

Thus pricing consistency– the requirement that the loadings on Z_t be $(0, 0, 1)$– imposes non-linear restrictions on the $\mathbb{Q}$ parameters $K_1^{\mathbb{Q}}$ and ρ_1.[15] Analysis of the constant term leads to additional nonlinear restrictions on the parameters $(K_0^{\mathbb{Q}}, \Sigma, \rho_0)$.

We specify the $\mathbb{Q}$ distribution in terms of the primitive parameters $\Theta_{TS}^{\mathbb{Q}}$. As such, the associated mapping from $\Theta_{TS}^{\mathbb{Q}}$ to the loadings on Z_t in the observation equation (7) automatically embeds these nonlinear constraints, thereby ensuring that pricing consistency always holds exactly.

Invariance of the theoretical model

Changing from one choice of the weight matrix W to another W^* has no impact on the distribution of the theoretical yields or macro-variables in a *MTSM* when the parameters are transformed appropriately. That is, consider the TS model and fix a portfolio matrix W and parameter vector $\Theta_{TS}(W) = (r_\infty^{\mathbb{Q}}, \lambda^{\mathbb{Q}}, \gamma_0, \gamma_1, \Sigma, K_0^{\mathbb{P}}, K_1^{\mathbb{P}})$. ($\Sigma_e$ has no role in this discussion.) For any other admissible weighting matrix W^*, the TS model with parameter vector $\Theta_{TS}^*(W^*) = (r_\infty^{\mathbb{Q}}, \lambda^{\mathbb{Q}}, \gamma_0^*, \gamma_1^*, \Sigma^*, K_0^{\mathbb{P}*}, K_1^{\mathbb{P}*})$, where for example $\gamma_1^* = \gamma_1 (W^{*\mathcal{N}} W^{-1} B_{TS})^{-1}$, implies *exactly the same joint distribution for* (M_t, y_t). This analysis holds equally well for the FV model, provided W maintains non-singularity among the state yield portfolios.

Our framework and its invariance property extend immediately to the case where the risk factors are linear combinations of both the yields and macro variables. That is, we can recast our entire analysis in terms of the first $\mathcal{N}$ elements of the vector $\widetilde{W}(M_t', y_t')'$, where $\widetilde{W}$ is a full-rank $(\mathcal{M} + J) \times (\mathcal{M} + J)$ matrix. Our chosen normalization is the special case in which $\widetilde{W}$ is block diagonal with the first diagonal block being the $\mathcal{M} \times \mathcal{M}$ identity matrix and

[15] Specifically, we need $(0, 1, 0) \sum_{\tau=0}^{23} (I + K_1^{\mathbb{Q}})^{\tau} = (0, 0, 1)$ since the loadings from (9) must recover $y_t^{24} = (0, 0, 1) Z_t$.

the second diagonal block being W. Exactly as above, any other canonical form based on a different choice $\widetilde{W}^*$ can be re-expressed in terms of our canonical form. Thus, once again, the joint distribution of (M_t, y_t) is not affected by the modeler's choice of $\widetilde{W}$.

Invariance with measurement errors

Importantly, whether or not the invariance of theoretical *MTSM*s and factor-*VAR*s to the choice of W carries over to their econometric implementations depends on one's assumptions about measurement errors on the yields and risk factors. Consider first the case where all J y_t are priced with errors. So long as the measurement error variance Σ_e for model TS^f based on yield weights W is transformed to $\Sigma_e^* = A\Sigma_e A'$ when this model is reparametrized in terms of the weights $W^* = AW$, Kalman filtering will produce *identical* fitted distributions for (y_t^o, M_t^o). Thus, canonical models based on different choices of W give rise to observationally equivalent representations of bond yields. The same is true for model FV^f. Thus, comparisons between models TS^f and FV^f are fully invariant to the modeler's choice of W.

This invariance is robust to the imposition of restrictions provided that the restrictions are properly adjusted when rotating to risk factors based on a different W^*. For example, a common assumption in the literature is that the measurement errors are independent and of equal variance: $\Sigma_e = \sigma_e^2 I$. This form would be preserved by any orthogonal re-weighting matrix A. If in one model $W = I_J$, so that the portfolios are individual yields, and in the second model W^* is given by the loadings of the yield PCs (an orthogonal matrix), then identical Kalman filter estimates will be obtained for the distribution of (y_t^o, M_t^o). For a general re-weighting A, identical estimates are obtained so long as Σ_e is replaced by $\sigma_e^2 AA'$.

In contrast, comparisons between model TS^f (with full rank Σ_e) and the associated model TS^n (with $\mathcal{P}_t^{\mathcal{L}o} = \mathcal{P}_t^{\mathcal{L}}$) will depend on the modeler's choice of W. For instance, assuming that $\mathcal{L}$ of the yields y_t are measured perfectly, as for example in Ang, Piazzesi, and Wei (2006), may lead to very different impulse responses than those obtained assuming that $(PC1_t, \ldots, PC\mathcal{L}_t)$ are measured perfectly. This is simply a consequence of the fact that the distribution of the error on any mismeasured portfolio yield will no longer be invariant to the choice of W. We illustrate the practical implications of this point in Section 4.

The same logic of observational equivalence applies to the standard assumption that the macro-variables are observed without errors. The estimates of the joint distribution of (M_t^o, y_t^o) under this assumption will in general differ from those obtained when M_t is presumed to be measured with error. On the other hand, when both (M_t, y_t) are observed with measurement errors, observationally equivalent models will be obtained for arbitrary choices of the W used to construct $\mathcal{P}_t^{\mathcal{L}}$, so long as the joint distribution of the measurement errors for the yields and macro variables is properly matched to one's choice of W.

Verifying econometric identification and pricing consistency in practice

Verification that one has a well-specified *MTSM* is greatly facilitated by specifying a canonical form and then, within this form, imposing sufficient normalizations and restrictions to ensure econometric identification and internal (pricing) consistency. Instead, many studies of *MTSM*s impose a mix of zero restrictions on the $\mathbb{P}$, $\mathbb{Q}$, and market price of risk parameters without explicitly mapping their models into a canonical form and verifying that it is identified.[16]

Our canonical form reveals that a necessary "order" condition for identification is that the dimension of our Θ_{TS} (excluding Σ_e)– $1 + 2\mathcal{N} + \mathcal{N}^2 + \mathcal{M}(\mathcal{N}+1) + \mathcal{N}(\mathcal{N}+1)/2$– must be at least as large as the number of free parameters in any *MTSM* with $\mathcal{N}$ risk factors, $\mathcal{M}$ of which are macro variables. It also leads to an easily imposed set of normalizations that ensure identification and pricing consistency. To our knowledge, ours is the only formally developed canonical form for the complete family of *MTSM*s.[17]

2.5 Models with Unspanned Risk Factors

The *MTSM*s considered so far have the macro variables entering directly as risk factors determining interest rates, as is the case with the large majority of the extant literature. Joslin, Priebsch, and Singleton (2011) propose a different class of models that allow for unspanned macro risks– risks that cannot be replicated by linear combinations of bond yields.[18] Their canonical model with unspanned risks shares two important properties with *MTSM*s with spanned risks: (1) except for the volatility parameter (Σ), the $\mathbb{P}$-parameters are distinct from the $\mathbb{Q}$-parameters; and (2) Σ only affects yield levels and not the loadings of yields on the risk factors. It follows that our subsequent results on the near equivalence of models TS^f and FV^n with spanned macro risks apply with equal force to canonical settings with unspanned macro risks.[19] Also, importantly, within the class of factor-VAR models, our normalization encompasses the case of unspanned models when the appropriate entries of B_{FV} in (8) are set to zero. In this sense, we encompass the entire literature on *MTSM*s.

3 Conditions for the (Near) Observational Equivalence of *MTSM*s and Factor-VARs

To derive sufficient conditions for the general agreement of Kalman filter estimators of models TS^f and FV^f, we fix a choice of W and derive (stronger) sufficient conditions for the Kalman filter estimators of the distribution of Z_t from models TS^f and FV^f to be (nearly) identical to those implied by model FV^n, the factor-VAR with observed risk factors ($Z_t^o = Z_t$).

Importantly, as long as there exists one W^* such that the estimated distributions of Z_t (nearly) agree in models TS^f and FV^n, it *must* follow that models TS^f and FV^f imply (nearly)

[16] Recent examples include the *MTSM*s examined by Bikbov and Chernov (2010) and the constant parameter case in Ang, Boivin, Dong, and Loo-Kung (2010). The following necessary condition for identification suggests that the first of these models is in fact under-identified, while the second may be over-identified.

[17] Pericoli and Taboga (2008) attempt an adaptation of the canonical form for yield-only models in Dai and Singleton (2000) to *MTSM*s, but their forms are not identified models (Hamilton and Wu (2011)).

[18] For additional applications of their framework, see Wright (2010) and Barillas (2010). Duffee (2011b) discusses a complementary model of unspanned risks in yield-only models.

[19] In the case that yields or macro variables are forecastable by variables not in their joint span, this applies only to the comparison of the no arbitrage model and the factor-VAR which are estimated by Kalman filtering. This is because in this case the assumption that $\mathcal{P}_t = \mathcal{P}_t^o$ cannot hold by construction.

identical distributions of Z_t *for all admissible portfolio matrices* W. This is true despite the fact that bilateral comparisons of the models (TS^f, FV^n) or the models (FV^f, FV^n) are rotation-dependent on W. Equally importantly, for such a W^*, everything that one can learn about the $\mathbb{P}$ distribution of Z_t from a canonical $MTSM$ in which all bonds are measured with errors can be equally learned from analysis of the corresponding economics-free model FV^n.

The filtering problem in both models TS^f and FV^f is one of estimating the true values of $\mathcal{P}_t^{\mathcal{L}}$, the first $\mathcal{L}$ PCs of the bond yields y_t. Intuitively, a key condition for the Kalman filter estimates of models (TS^f, FV^f) to match the OLS estimates of model FV^n is that the filtered pricing factors equal their observed counterparts. However, this observation begs the more fundamental question of when this approximation holds. Additionally, this matching is not sufficient for the Kalman filter estimates of the drift or the volatility of Z_t to match their OLS counterparts from model FV^n. The remainder of this section derives sufficient conditions for the efficient estimates of models TS^f and FV^n to (nearly) coincide.

To fix the notation, let $X_t^f = E[X_t|\mathcal{F}_t]$ and $X_t^s = E[X_t|\mathcal{F}_T]$ denote the filtered and smoothed version of any random variable X_t, where $\mathcal{F}_t$ is the observable information known at time t: $(y_1^o, M_1^o, \ldots, y_t^o, M_t^o)$.

3.1 When do the Filtered Yields Differ from the Observed Yields?

The filtered yields will agree closely with the observed yields when the filtered yield factors $\mathcal{P}_t^{\mathcal{L}f}$ are close to their observed counterparts $\mathcal{P}_t^{\mathcal{L}o}$. Consider the errors $e_t^{\mathcal{L}} \equiv \mathcal{P}_t^{\mathcal{L}} - \mathcal{P}_t^{\mathcal{L}o}$ and and let $\mathcal{I}_t$ denote the information generated by $\mathcal{F}_{t-1}$ and the current information $(M_t^o, \mathcal{P}_t^{-\mathcal{L}o})$, where $\mathcal{P}_t^{-\mathcal{L}o}$ denotes the last $J-\mathcal{L}$ of the observed yield portfolios $\mathcal{P}_t^o$. Conditional on $\mathcal{I}_t$, $e_t^{\mathcal{L}}$ and $\mathcal{P}_t^{\mathcal{L}o}$ are jointly normal and, therefore, the filtering error $FE_t^{\mathcal{L}} = \mathcal{P}_t^{\mathcal{L}f} - \mathcal{P}_t^{\mathcal{L}o}$ is[20]

$$FE_t^{\mathcal{L}} = E[e_t^{\mathcal{L}}|\mathcal{P}_t^{\mathcal{L}o}, \mathcal{I}_t] = \Sigma_{e^{\mathcal{L}}} S^{-1}(\mathcal{P}_t^{\mathcal{L}o} - E[\mathcal{P}_t^{\mathcal{L}o}|\mathcal{I}_t]), \tag{10}$$

where $\Sigma_{e^{\mathcal{L}}}$ is the covariance matrix of $e_t^{\mathcal{L}}$ and $S = Var(\mathcal{P}_t^{\mathcal{L}o}|\mathcal{I}_t)$ is the forecast-error variance of $\mathcal{P}_t^{\mathcal{L}o}$ based on $\mathcal{I}_t$. To assess the magnitude of $FE_t^{\mathcal{L}}$ we use the filtering mean-squared errors

$$RMSFE^2 \equiv diag\, E\left[FE_t^{\mathcal{L}} FE_t^{\mathcal{L}\prime}\right] = diag\left[\Sigma_{e^{\mathcal{L}}} S^{-1} \Sigma_{e^{\mathcal{L}}}\right]. \tag{11}$$

By this metric filtering errors will be small, $\mathcal{P}_t^{\mathcal{L}o} \approx \mathcal{P}_t^{\mathcal{L}f}$, when the magnitudes of the measurement errors on the yields (measured by $\Sigma_{e^{\mathcal{L}}}$) are small relative to the uncertainty about $\mathcal{P}_t^{\mathcal{L}o}$ given the information set $\mathcal{I}_t$ (measured by S).[21]

$\Sigma_{e^{\mathcal{L}}}$ is determined by the pricing errors on individual yields, the correlations among these errors, and the choice of W. Importantly, there is a diversification effect from constructing

[20] When random vectors (X, Y) follow a multivariate normal distribution, $E[X|Y] = \mu_X + \Sigma_{XY}\Sigma_Y^{-1}(Y - \mu_Y)$, where μ_X and μ_Y are the mean of X and Y, Σ_Y is the variance of Y and Σ_{XY} is the covariance of X and Y. Here $X = e_t^{\mathcal{L}}$ and $Y = \mathcal{P}_t^{\mathcal{L}o}$, and Σ_{XY} is simply the variance of the errors by independence.

[21] Note that when the measurement errors are serially uncorrelated, as has typically been assumed in the literature, S is always at least as large as $\Sigma_{e^{\mathcal{L}}}$ ($S - \Sigma_{e^{\mathcal{L}}}$ is positive semi-definite). This follows from the observations that $\mathcal{P}_t^{\mathcal{L}o} = \mathcal{P}_t^{\mathcal{L}} + e_t^{\mathcal{L}}$, the pair $(\mathcal{P}_t^{\mathcal{L}}, e_t^{\mathcal{L}})$ are independent, and $e_t^{\mathcal{L}}$ is independent of $\mathcal{I}_t$. So even if the theoretical $\mathcal{P}_t^{\mathcal{L}}$ were perfectly forecastable based on $\mathcal{I}_t$, it would still be the case that we would have a forecast variance of $\Sigma_{e^{\mathcal{L}}}$ when forecasting $\mathcal{P}_t^{\mathcal{L}o}$ because $e_t^{\mathcal{L}}$ is unforecastable based on $\mathcal{I}_t$.

$\mathcal{P}_t = W y_t$ that typically leads to the diagonal elements of $\Sigma_{e^{\mathcal{L}}}$ being smaller than the corresponding RMSEs for individual yields. For example, if the individual yield errors are cross-sectionally independent and if the first row of W weights yields equally (corresponding to a level factor), then the RMSE of the yield portfolio will be reduced by a factor of $1/\sqrt{J}$.[22] Owing to this diversification effect, even if individual bonds are priced with sizable errors, the elements of $\Sigma_{e^{\mathcal{L}}}$ can still be relatively small.

In particular, increasing the number of yields used in the estimation is likely to reduce the measurement error for the level portfolio and increase the match between the observed and filtered level. Furthermore, S will be much larger than $\Sigma_{e^{\mathcal{L}}}$ when there is substantial uncertainty about $\mathcal{P}_t^{\mathcal{L}o}$ based on the information in $\mathcal{I}_t$. This uncertainty is likely to rise as the sampling frequency decreases.

Thus $FE_t^{\mathcal{L}}$ will tend to decline when W is chosen so that (i) there is cancelation of measurement errors across maturities, (ii) more cross-sectional information is used in estimation (J is large), and (iii) the variance of the error in forecasting $\mathcal{P}_t^{\mathcal{L}o}$ based on $\mathcal{I}_t$ is large. This dependence of $FE_t^{\mathcal{L}}$ on W means that, for a given model, some choices of W may imply that $\mathcal{P}_t^{\mathcal{L}o} \approx \mathcal{P}_t^{\mathcal{L}f}$, while for other choices the differences may be large. Choices of W that select individual yields are inherently handicapped in this regard, because they forego the diversification benefits of nontrivial portfolios.

To assign a (rough) magnitude to $RMSFE$ suppose that all yields are observed with $i.i.d.$ measurement errors of equal variance σ_y^2 and there is a single yield portfolio ($\mathcal{L} = 1$) which is a level factor with equal weights $(1/J)$. Then $RMSFE = (\sigma_y/J) \times (\sigma_y/\sqrt{S})$. If, for example, $\sigma_y = 10$ basis points, $\sqrt{S} = 20$ basis points, and there are $J = 10$ yields used in estimation, then $RMSFE$ is about half a basis point. Quadrupling σ_y to 40 basis points, and increasing $\sqrt{S}$ to 50 basis points, holding J at 10, increases $RMSFE$ to only about two and one-half basis points.

These results also provide a context for interpreting previous work with large numbers of latent or yield-based risk factors. The reported large differences between the filtered and observed values of the high-order PCs in the five-factor $YTSM$s studied by Duffee (2011b) and JSZ may be attributable to the smaller forecast-error variances of the higher-order PCs. Under the typical assumption of $i.i.d.$ measurement errors and normalized loadings, the measurement error variances are the same for all PCs. However, the sample standard deviations of the fourth and fifth PCs, about 19 and 13 basis points respectively for our data, are much smaller than those for the first three PCs. Since the forecast-error variances of the fourth and fifth PCs must be smaller than their respective unconditional variances, it is likely that the elements of $\Sigma_{e^{\mathcal{L}}} S^{-1}$ corresponding to these PCs are relatively large. Whence, the Kalman filter will emphasize measurement error reduction over fitting the cross-section of yields, resulting in large differences between the higher-order PC^o and PC^f.

[22] Typically PCs are normalized so that the sum of the squares of the weights is one. This condition also ensures the observational equivalence of Section 2.4 if one supposes that the individual yield measurement errors are independent with equal variances. For ease of interpretation, and without loss of generality, it is convenient to rescale the PCs so that the sum of the weights is equal to one for the first PC.

3.2 ML Estimation of the Conditional Distribution of (M_t^o, y_t^o)

With sufficient conditions for $\mathcal{P}_t^{\mathcal{L}o} \approx \mathcal{P}_t^{\mathcal{L}f}$ in hand, we turn next to establishing sufficient conditions for the Kalman filter estimators of models TS^f and FV^f to (nearly) coincide. For either of the models TS^f or FV^f, the observed data, $\{M_t^o, y_t^o\}$ follow a multivariate normal distribution that can be computed efficiently by using the Kalman filter. From a theoretical perspective, we can think of building the likelihood of the data by integrating the joint density $f_m^{\mathbb{P}}(\vec{Z} = z, \vec{\mathcal{P}}^o, \vec{M}^o; \Theta_m)$ over the missing data $\vec{Z}$:

$$f_m^{\mathbb{P}}(\vec{\mathcal{P}}^o, \vec{M}^o; \Theta_m) = \int_z f_m^{\mathbb{P}}(\vec{Z} = z, \vec{\mathcal{P}}^o, \vec{M}^o; \Theta_m) dz, \tag{12}$$

for $m =$TS^f or FV^f, with $\vec{X}$ denoting the full sample: $\vec{X} = (X_1, X_2, \ldots, X_T)$. For ease of notation, we omit the subscript m from $f_m^{\mathbb{P}}$ and Θ_m in all expressions that apply to both the *MTSM*s and the factor-*VAR*s.

The density $\log f^{\mathbb{P}}(\vec{Z}, \vec{\mathcal{P}}^o, \vec{M}^o)$ in (12) is equal to

$$\sum_{t=1}^{T} \log f^{\mathbb{P}}\left(\mathcal{P}_t^o | Z_t; \Theta^{\mathbb{Q}}, \Sigma_e\right) + \sum_{t=1}^{T} \log f^{\mathbb{P}}\left(Z_t | Z_{t-1}; K_1^{\mathbb{P}}, K_0^{\mathbb{P}}, \Sigma\right). \tag{13}$$

This construction reveals that the conditional distribution of the risk factors Z_t depends only on $(K_1^{\mathbb{P}}, K_0^{\mathbb{P}}, \Sigma)$, *and* $(K_1^{\mathbb{P}}, K_0^{\mathbb{P}})$ enter only $f^{\mathbb{P}}(Z_t|Z_{t-1})$ and not $f^{\mathbb{P}}(\mathcal{P}_t^o|Z_t)$. This shared property of the null model TS^f and the alternative model FV^f is immediately apparent in our canonical form, while being largely obscured in the standard identification schemes of *MTSM*s such as the one based on Dai and Singleton (2000).

A key difference between models TS^f and FV^f is how Σ enters (13). The functional dependence of $f^{\mathbb{P}}(Z_t|Z_{t-1})$ on Σ is identical for these two models. However, owing to the diffusion invariance property of the no-arbitrage model, Σ only affects $f_{TS}^{\mathbb{P}}(\mathcal{P}_t^o|Z_t)$ and not $f_{FV}^{\mathbb{P}}(\mathcal{P}_t^o|Z_t)$. Nevertheless, for our canonical form, this difference turns out to be largely inconsequential for Kalman filter estimates of Σ.

The factorization of the conditional likelihood function (13) implies that for model FV^n with $Z_t^o = Z_t$ estimation reduces to two sets of *OLS* regressions. Estimation of a VAR for the observed risk factors Z_t^o gives the ML estimates of the parameters $(K_1^{\mathbb{P}}, K_0^{\mathbb{P}}, \Sigma)$ in (6). A linear projection of $\mathcal{P}_t^o$ onto Z_t^o recovers the ML estimates of the parameters in (8).

Taking the derivative of (12) with respect to Θ and setting this equal to zero, and dividing by the marginal density of $(\vec{\mathcal{P}}^o, \vec{M}^o)$, gives the first-order conditions (e.g., Dempster, Laird, and Rubin (1977)):

$$0 = E\left[\partial_\Theta \log f^{\mathbb{P}}(\vec{Z}, \vec{\mathcal{P}}^o, \vec{M}^o; \hat{\Theta}) \middle| \mathcal{F}_T\right], \tag{14}$$

where T is the sample size and, in model FV^n with our choice of W, (14) holds without the conditional expectation. Using the fact that $f(\mathcal{P}_t^o|Z_t)$ does not depend on $(K_0^{\mathbb{P}}, K_1^{\mathbb{P}})$, the ML estimators of the conditional mean parameters $(K_0^{\mathbb{P}}, K_1^{\mathbb{P}})$ satisfy

$$[\hat{K}_0^{\mathbb{P}}, \hat{K}_1^{\mathbb{P}}]' = \left(\left(\tilde{Z}'\tilde{Z}\right)^{\mathrm{s}}\right)^{-1} \left(\tilde{Z}'\Delta Z\right)^{\mathrm{s}}, \tag{15}$$

where the "hats" indicate ML estimators, the superscript "s" denotes the smoothed version of the object in parentheses, $\tilde{Z}_t = [1, Z_t']'$, and Z and $\tilde{Z}$ are matrices with rows corresponding to Z_t and $\tilde{Z}_t$, respectively, for t ranging from 1 to T.

From (15) it is seen that a key ingredient for Kalman filter estimates of $(K_0^{\mathbb{P}}, K_1^{\mathbb{P}})$ from models TS^f and FV^f to agree with each other and with those from model FV^n is that $(\tilde{Z}_t\tilde{Z}_t')^{\text{s}}$ be close to $\tilde{Z}_t^o\tilde{Z}_t^{o\prime}$, period-by-period. Equation (15) is *almost* the estimator of $[K_0^{\mathbb{P}}, K_1^{\mathbb{P}}]$ obtained from OLS estimation of a VAR on the smoothed risk factors Z_t^{s}. Underlying the difference between (15) and the latter estimator is the fact that

$$(Z_tZ_t')^{\text{s}} = \text{Var}(Z_t|\mathcal{F}_T) + Z_t^{\text{s}}Z_t^{\text{s}\prime}. \tag{16}$$

This equation and the analogous extensions to $(Z_tZ_{t+1}')^{\text{s}}$ reveal that, provided the smoothed state is close to the observed state and $\text{Var}(Z_t|\mathcal{F}_T)$ is small, the ML estimates of $(K_0^{\mathbb{P}}, K_1^{\mathbb{P}})$ from model FV^n will be similar to those obtained by Kalman filtering within a $MTSM$. In Section 3.1 we derived conditions under which $\mathcal{P}_t^{\mathcal{L}o} \approx \mathcal{P}_t^{\mathcal{L}f}$. In Appendix D, we show that these same conditions (with a few mild additional assumptions) imply that $\text{Var}(Z_t|\mathcal{F}_T)$ is small as well. As with the approximation $\mathcal{P}_t^{\mathcal{L}o} \approx \mathcal{P}_t^{\mathcal{L}f}$, the near equality of the ML estimates of $[K_0^{\mathbb{P}}, K_1^{\mathbb{P}}]$ across the three models TS^f, FV^f, and FV^n may arise even in the presence of large pricing errors on the individual bond yields.

Turning to estimation of Σ, in model FV^f there is no diffusion invariance and $f_{FV^f}^{\mathbb{P}}(\mathcal{P}_t^o|Z_t)$ does not depend on Σ. Therefore, the first-order conditions for maximizing the likelihood function depend only on $\log f_{FV^f}^{\mathbb{P}}(Z_t|Z_{t-1}; \hat{\Theta}_{FV})$. This leads to the first-order condition

$$E\left[vec\left((\hat{\Sigma}_{FV^f})^{-1} - (\hat{\Sigma}_{FV^f})^{-1}\hat{\Sigma}_{FV^f}^u(\hat{\Sigma}_{FV^f})^{-1}\right)\Big|\,\mathcal{F}_T\right] = 0, \tag{17}$$

where the sample covariance matrix $\hat{\Sigma}_{FV^f}^u$ is based on the residuals $\hat{i}_{FV^f,t}^u = \Delta Z_t - (\hat{K}_{0FV^f}^{\mathbb{P}} + \hat{K}_{1FV^f}^{\mathbb{P}}Z_{t-1})$ that are partially observed owing to their dependence on $\vec{Z}$. From (17), we obtain $\hat{\Sigma}_{FV^f} = (\hat{\Sigma}_{FV^f}^u)^{\text{s}}$. Using the logic of our discussion of the conditional mean, as long as the estimated model FV^f accurately prices the risk factors, then $(\Sigma_{FV^f}^u)^{\text{s}}$ will be nearly identical to the OLS estimator of Σ from the VAR model FV^n.

The ML estimator of Σ in model TS^f will in general be more efficient than in model FV^n and this is true even when there is no measurement error in the state yield portfolios. The first-order conditions for Σ in model TS^f have an additional term since the density $f_{TS^f}^{\mathbb{P}}(\mathcal{P}_t^o|Z_t;\Theta)$ also depends on Σ. Combining this term, derived in Appendix E as (A53), with (17) gives

$$E\left[vec\left(\frac{1}{2}\left[(\hat{\Sigma}_{TS^f})^{-1} - (\hat{\Sigma}_{TS^f})^{-1}\hat{\Sigma}_{TS^f}^u(\hat{\Sigma}_{TS^f})^{-1}\right]\right) - \hat{\beta}_Z'(\hat{\Sigma}_{e,TS^f})^{-1}\frac{1}{T}\sum_t \hat{e}_{TS^f,t}^u\Bigg|\,\mathcal{F}_T\right] = 0,$$

where $\hat{\Sigma}_{TS^f}^u$ is the sample covariance of the residuals $\hat{i}_{TS^f,t}^u = \Delta Z_t - (\hat{K}_{0TS}^{\mathbb{P}} + \hat{K}_{1TS}^{\mathbb{P}}Z_t)$, $\hat{\beta}_Z$ is the vector defined in Appendix E, and the unobserved pricing errors $\hat{e}_{TS^f,t}^u$ from (7) are evaluated at the ML estimators and depend on the partially observed $\vec{Z}$.

The following two conditions are sufficient for the Kalman filter estimators of Σ in models TS^f and FV^f to be approximately equal. First, we require that the risk factors be priced sufficiently accurately for

$$\hat{\Sigma}_{FV^f} = \left(\hat{\Sigma}^u_{FV^f}\right)^{\text{s}} \approx \hat{\Sigma}_{FV^n}. \tag{18}$$

To guarantee that the right hand side of (18) is close to the estimate of Σ in the *MTSM*, our second requirement is that the average-to-variance ratio $(\hat{\Sigma}_e)^{-1}(T^{-1}\sum \hat{e}^o_t)$ of pricing errors be close to zero, where $\hat{e}^o_t$ is computed from (7) evaluated at the ML estimates and using $\vec{Z}^o$. When both conditions are satisfied, $(\hat{\Sigma}_e)^{-1}(T^{-1}\sum \hat{e}^u_t)^{\text{s}}$ will be close to zero as well, ensuring that $\hat{\Sigma}_{TS^f} \approx (\hat{\Sigma}^u_{FV})^{\text{s}}$ and, hence, that the estimators of Σ from all three models TS^f, FV^f, and FV^n approximately agree with each other.

3.3 Discussion

Summarizing, we have just shown that the same conditions derived in Section 3.1 for $\mathcal{P}^{\mathcal{L}o}_t \approx \mathcal{P}^{\mathcal{L}f}_t$ also ensure that the ML estimators of the conditional mean parameters of the state process Z_t approximately coincide for all three models TS^f, FV^f, and FV^n. When, in addition, the sample average of the fitted pricing errors for $\mathcal{P}^o_t$, $T^{-1}\sum \hat{e}^o_t$, is small relative to the estimated covariance matrix $\hat{\Sigma}_e$ of these errors, the ML estimates of the conditional variance Σ of Z_t will also approximately coincide in these models.

These observations regarding the conditional distribution of Z_t extend to individual bond yields with one additional requirement. Specifically, the factor loadings from OLS projections of y^o_t onto Z^o_t need to be close to their model-based counterparts estimated using the Kalman filter. By the same reasoning as above, if $\mathcal{P}^{\mathcal{L}}_t$ is reasonably accurately priced, the OLS loadings are likely to be close to those implied by model FV^f.[23] Nevertheless, large errors in the pricing of individual bonds might lead to large efficiency gains from ML estimation of the loadings within a *MTSM*. This is an empirical question that we take up subsequently.

Further intuition for our results comes from exploring two restrictive special cases: the state yield portfolios are observed without measurement error in the *MTSM* ($\mathcal{P}^{\mathcal{L}o}_t = \mathcal{P}^{\mathcal{L}}_t$) and, on top of this, the *MTSM* is just-identified in the sense that the restriction of no arbitrage is non-binding on the factor-VAR model for the risk factors. We discuss each of these in turn.

A stark version of our results is obtained under the assumption $\mathcal{P}^{\mathcal{L}o}_t = \mathcal{P}^{\mathcal{L}}_t$, in which case the relevant comparison is between models TS^n and FV^n. With exact pricing of $\mathcal{P}^{\mathcal{L}o}_t$, the

[23]To see this, first note that the loadings of y_t on Z_t are simply the loadings of $\mathcal{P}_t$ on Z_t, premultiplied by the inverse of W. Second note that, for the FV^f model, the loadings of $\mathcal{P}_t$ on Z_t are given by:

$$(\hat{A}_{FV^f}, \hat{B}_{FV^f}) = \left(\frac{1}{T}\sum_t \left[\mathcal{P}^o_t\left(\tilde{Z}'_t\right)^{\text{s}}\right]\right)\left(\frac{1}{T}\sum_t \left[\left(\tilde{Z}_t\tilde{Z}'_t\right)^{\text{s}}\right]\right)^{-1},$$

which should be close to the loadings from projecting $\mathcal{P}^o_t$ on Z^o_t if $\mathcal{P}^{\mathcal{L}o}_t$ is accurately priced. Within the context of *YTSM*s in which $\mathcal{P}^{\mathcal{N}}_t$ is priced perfectly and measurement errors on yields are relatively small, Duffee (2011a) documents this point using Monte Carlo methods.

ML estimates of $(K_0^{\mathbb{P}}, K_1^{\mathbb{P}})$ from model TS^n exactly coincide with the OLS estimates from model FV^n, regardless of the magnitude of the mean-to-variance ratios of pricing errors.[24] Therefore, a sufficient condition for the conditional distribution of the risk factors Z_t^o in a *MTSM* to be fully invariant to the imposition of the no-arbitrage restrictions is that the ratio $(\hat{\Sigma}_e)^{-1}(T^{-1}\sum \hat{e}^o_{TS,t})$ is zero. Owing to the Gaussian property, these invariance results extend to the *unconditional* distributions of $\{Z_t\}$ as well.

The first-order conditions with respect to the "constant terms" $(r^{\mathbb{Q}}_{\infty}, \gamma_0)$ in model TS^f set $\mathcal{M}+1$ linear combinations of the filtered means $(T^{-1}\sum \hat{e}^u_{TS,t})^{\mathrm{s}}$ to zero. Therefore, if the number of yields used in estimation (J) equals $\mathcal{N}+1$ within an $\mathcal{N}$-factor *MTSM* with $\mathcal{M}$ macro factors– equivalently, exactly $\mathcal{M}+1$ portfolios of yields are included with measurement errors– then the mean-to-variance ratios will be optimized at zeros.[25]

The first-order conditions of the ML estimators in our general setup (an over-identified *MTSM* with $J > \mathcal{N}+1$ imperfectly priced bond portfolios) do not set the sample mean of the pricing error $\hat{e}^u_{TS^f,t}$ to zero. Nevertheless, the likelihood function has $\mathcal{M}+1$ degrees of freedom to use in making the mean-to-variance ratios close to zero. Our analysis shows that much of the intuition from just-identified *MTSM*s will carry over to over-identified *MTSM*s whenever the *MTSM* accurately prices the yield-based factors $\mathcal{P}_t^{\mathcal{L}}$, and this may be true even when the *MTSM*-implied errors in pricing individual bonds are quite large.

4 Empirical Comparisons of *MTSM*s and Factor-*VAR*s

We now turn to assess the empirical relevance of the theory we developed in Section 3. We examine, step-by-step, to what extent our sufficient conditions for the observational equivalence of *MTSM*s and factor-*VAR*s hold in practice.

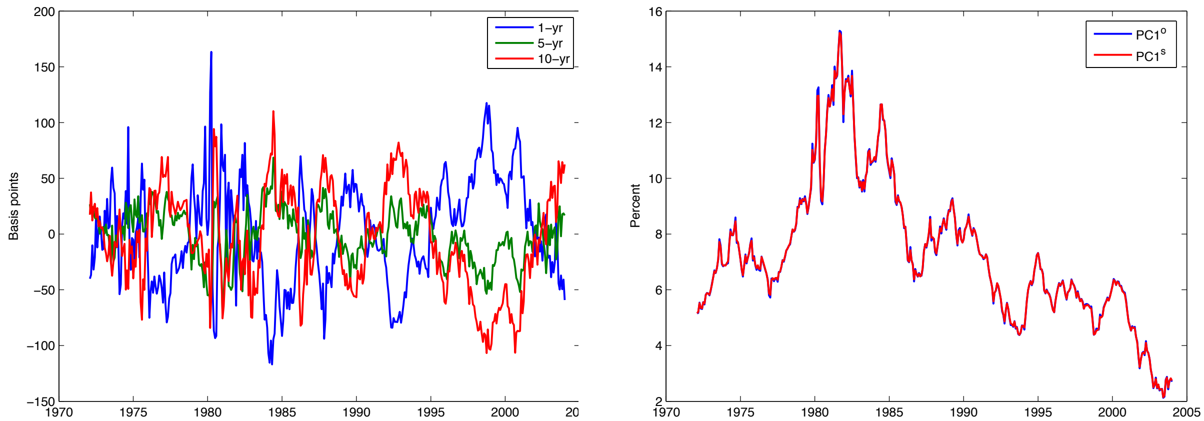
We focus on a *MTSM*– model $GM_3(g,\pi)$– with $\mathcal{N}=3$, $\mathcal{M}=2$, and $M_t=(g_t,\pi_t)'$, where g_t is a measure of real output growth and π_t is a measure of inflation as in, for example, Ang, Dong, and Piazzesi (2007) and Smith and Taylor (2009). We follow Ang and Piazzesi (2003) and use the first PC of the help wanted index, unemployment, the growth rate of employment, and the growth rate of industrial production ($REALPC$) as our measure of g, and the first PC of measures of inflation based on the CPI, the PPI of finished goods, and the spot market commodity prices ($INFPC$) for π.[26] The monthly zero yields are the unsmoothed Fama-Bliss series for maturities three- and six-months, and one through ten years ($J=12$) over the sample period 1972 through 2003.[27] The weighting matrix W is

[24] JSZ prove an analogous irrelevancy result for conditional means within *YTSM*s.

[25] This is the counterpart within a *MTSM* of Duffee (2011a)'s observation that *YTSM*s are just identified when $J=\mathcal{N}+1$.

[26] All of our results are qualitatively the same if we replace these measures of (g,π) by the help wanted index and CPI inflation used by Bikbov and Chernov (2010).

[27] We use Fama-Bliss data from the Center on Research on Security Prices, and the data for ten-year yields ends in 2003. We have experimented with an extended sample through 2007 (shortly before the Conference Board discontinued publication of the Help Wanted Index) and the subsequent results on irrelevance are qualitatively unchanged. We started our sample in 1972, instead of in 1970 as in Bikbov and Chernov (2010), because data on yields for the maturities between five and ten years are sparse before 1972.



(A) Comparison of individual yields (B) Comparison of $PC1^o$ and $PC1^s$

Figure 2: This figure compares observed yields with smoothed yields estimated from model $GM_3(g, \pi)$. Panel (A) plots the difference between observed yields and the smoothed versions from the model $(y_t^m)^{\text{s}}$. Panel (B) plots the observed $PC1^o$ and its smoothed version $PC1^{\text{s}}$.

chosen to be the PC loadings so that the state yield portfolio is the level of interest rates (PC1). The measurement errors e_t in (7) and (8) are *i.i.d.* Normal$(0, \sigma_y^2 I_{12})$.

4.1 On the Need For Filtering PCs Within Canonical $MTSM$s

A key part of our derivation of conditions under which the filtered versions of the state yield portfolios agree with their observed counterparts was the "diversification" effect of averaging the errors across maturities. Even when individual yields have large measurement errors, the yield portfolios can be measured precisely. Panel (a) of Figure 2 plots the time series of the differences between observed (annualized) yields, y_t^{mo}, and their smoothed counterparts $(y_t^m)^{\text{s}}$, for m=12, 60, and 120 months. These errors are large, occasionally exceeding 100 basis points and $\hat{\sigma}_y$ is 43.1 basis points, so variant TS^f clearly has difficulty matching *individual yields*. The reason for this poor fit is that the macro variables (g_t, π_t) replicate only a small portion of the variation in the slope and curvature of the yield curve.

Although the individual yields are poorly fit by the model, model TS^f provides an excellent fit to $PC1$. Panel (b) of Figure 2 plots $PC1_t^o$ against the smoothed $PC1_t^s$. The sample standard deviation of the difference $\{PC1_t^o - PC1_t^s\}$ is only 1.7 basis points; the standard deviation of $\{PC1_t^o - PC1_t^f\}$ is only 4.3 basis points.

These numbers are fully anticipated by our theory in Section 3.1. Consider again the filtered observation error (10) and the associated filtering root mean squared error $RMSFE$. For model $GM_3(g, \pi)$ the standard deviation $\sqrt{S} = \sqrt{Var(PC1_t^o | \mathcal{I}_t)}$ is 40.7 basis points.[28]

[28] Note that the sample standard deviation of the first difference $\Delta PC1_t^o$ is 42.5 basis points. Comparing 40.7 to 42.5 it follows that very little of $\Delta PC1_t^o$ is predictable based on the information structure of $GM_3(g, \pi)$.

The estimated standard deviation of the measurement error on $PC1^o$ is 12.5 basis points, which approximately equals $\hat{\sigma}_y$ (43.1 basis points) divided by the square root of the number of yields used in estimation ($J = 12$). Using again the expression $RMSFE = (\sigma_y/J) \times (\sigma_y/\sqrt{S})$, the estimated standard deviation of $\{PC1_t^o - PC1_t^f\}$ is 3.8 basis points, close to the sample value of 4.3 basis points.

4.2 ML Estimation of the Conditional Distribution

Section 3.2 gives conditions for the Kalman filter estimator of model TS^f and the ML estimator of the factor-VAR FV^n to produce (nearly) identical fitted distributions of $(M_t^o, \mathcal{P}_t^{\mathcal{L}o})$. They are that (i) $\mathcal{P}_t^{\mathcal{L}s}$ tracks $\mathcal{P}_t^{\mathcal{L}o}$ closely; (ii) there is a low amount of uncertainty about the (unobserved) theoretical $\mathcal{P}_t^{\mathcal{L}}$; and (iii) the time series average of the measurement errors, relative to their variances, should be small for the higher order portfolios $\mathcal{P}^{-\mathcal{L}o}$.

We have just seen that the first two of these conditions are satisfied at the Kalman filter estimates of model $GM_3(g, \pi)$. Intuitively, the second condition follows from the first and, indeed, the estimates indicate that the square root of $\text{Var}(\mathcal{P}_t^1|\mathcal{F}_t)$ is only 11.4 basis points. The final condition for equivalence is that the time series average of the measurement errors (relative to their variances) are small. Although Panel (a) of Figure 2 indicates that at times the errors for individual yields can be very large, visually we can see that the time series averages are small. In fact, for $GM_3(g, \pi)$ they are only 0.6, -1.4, and -4.6 basis points for the one-, five-, and ten-year yields, respectively!

Given that all three conditions are (approximately) satisfied, the ML estimates of $(K_0^{\mathbb{P}}, K_1^{\mathbb{P}}, \Sigma)$ should agree for all three models TS^f, FV^f, and FV^n. Table 2 displays the ratios of the estimated parameters from $GM_3(g, \pi)$ and its associated factor-VAR, with and without filtering. Consistent with our theory, they are all virtually identical.

4.3 Statistics of the Distribution of (M_t^o, y_t^o)

It follows that the distributions of the risk factors are virtually the same across these different factor models. This, in turn, implies that all statistics of the distribution, such as the IRs, will be nearly identical as well. These results underlie Figure 1, where the IRs of $PC1$ to a shock to inflation in model $GM_3(g, \pi)$ and the associated model FV^n (nearly) coincide. Neither the no-arbitrage restrictions nor filtering in the presence of sizable measurement errors for the individual bond yields impact estimates of these responses.

4.4 Invariance of the Distribution of (M_t^o, y_t^o)

For models TS^f and FV^f these empirical irrelevancy results extend to any full rank portfolio matrix W (Section 2.4). In particular, had we chosen to normalize the model so that $\mathcal{P}_t^1$ was any of the individual twelve yields instead of $PC1$, all of the results in Figures 1 and 2 would be exactly the same. The results in Table 2 would have been identical after rotation. The

This is consistent with the near-random walk behavior of the level of interest rates.

	$K_0^{\mathbb{P}}$	$I+K_1^{\mathbb{P}}$			Σ		
	1	1	1	0.999	1.01	-	-
$\frac{TS^f}{FV^f}$	1	1	1	1	0.987	1	-
	1	0.999	1	1	0.998	1.01	1
	1.12	0.998	1.04	1.07	1.07	-	-
$\frac{TS^f}{TS^n}$	0.999	0.999	1	1	0.90	1	-
	0.988	0.93	1	1	0.885	1.11	1.01
	1.12	0.998	1.04	1.02	1.07	-	-
$\frac{FV^f}{FV^n}$	0.999	0.999	1	1	0.898	1	-
	0.989	0.929	1	1	0.885	1.11	1.01

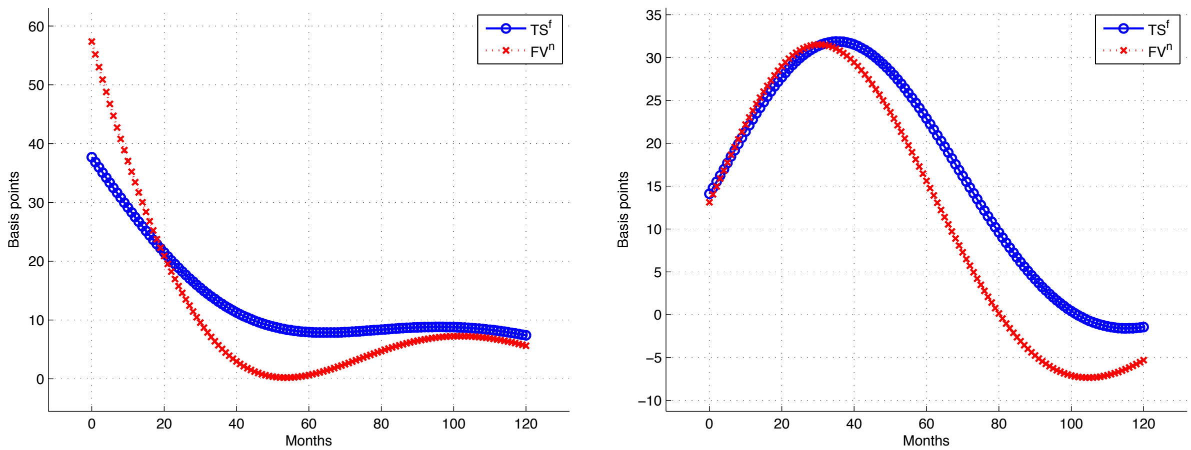
Table 2: Ratios of estimated $K_0^{\mathbb{P}}$, $I+K_1^{\mathbb{P}}$, and Σ for model $GM_3(g,\pi)$. The first block compares the estimates for models TS^f and FV^f, the second block compares models TS^f and TS^n, and the third compares models FV^f and FV^n.

parameters governing the conditional distribution of Z_t^o would change, of course, since any such reweighting leads to different risk factors. Such renormalizations do not, however, affect the implied relationships among any given set of yields and macro variables.

As was discussed in Section 2.4, this invariance does not extend to comparisons across models constructed with different W and in which $\mathcal{P}_t^{\mathcal{L}}$ is assumed to be measured without error (models TS^n or FV^n). To illustrate this rotation sensitivity consider first the case where W is chosen so that y_t^3, the yield on three-month Treasury bills, is the state yield factor $\mathcal{P}_t^1$. This yield is one of the state yield factors in the models of Ang, Piazzesi, and Wei (2006) and Jardet, Monfort, and Pegoraro (2010), and in both studies y_t^3 is presumed to be measured without error. We compare results from $GM_3(g,\pi)$ (i.e., model TS^f) which has all bonds priced imperfectly and $\mathcal{P}_t^1 = y_t^3$, to those from its factor-VAR counterpart FV^n in which y_t^3 is presumed to be measured without error. Figure 3(A) displays the IRs of y_t^3 to its own innovation (in basis points) for these two models. Because of rotation invariance, the response for model TS^f is identical to what we would have obtained from estimation of this *MTSM* normalized so that $\mathcal{P}_t^1 = PC1_t$. However the IR from model FV^n is very different: it is nearly fifty percent larger over very short horizons, decays much faster, and troughs at a lower value than the IR from model TS^f. The reason for these differences is that model FV^n captures the dynamic responses of the observed data, while model TS^f presumes that a portion of these responses are attributable to measurement error.

The IRs of y_t^3 to a shock in output growth $REALPC$ implied by models TS^f and FV^n follow similar patterns (Figure 3(B)), and the gap in responses is not as large as with the own responses. Yet the *MTSM* implies a more persistent response that peaks later and dies out more slowly than what emerges from the factor-VAR.

The differences in attribution of dynamic responses to economic forces across a *MTSM* and it factor-VAR counterpart can be extreme. Consider, for example, the version of $GM_3(g,\pi)$



(A) Impulse Response of y_t^3 to y_t^3 (B) Impulse Response of y_t^3 to g_t

Figure 3: Impulse responses of y_t^3 to its own innovation (Panel (a)) and an innovation in g_t (Panel (b)) within models TS^f and FV^n for the family $GM_3(g, \pi)$.

in which $\mathcal{P}_t^1$ is normalized to be the third PC of bond yields ($PC3$).[29] Again, we stress that under the assumption that all bonds are measured with error, the Kalman-filter/ML estimates of the joint distribution of (M_t^o, y_t^o) under the rotations with $\mathcal{P}_t^1 = PC1$ or $\mathcal{P}_t^1 = PC3$ are *identical*. However, as can be seen from Figure 4, the model-implied IRs of $PC3$ to its own innovation are very different across models TS^f and FV^n. The $MTSM$ that enforces no arbitrage implies that there is essentially no response at all, whereas the factor-VAR shows a large (though short-lived) response. This difference arises because, within $GM_3(g, \pi)$, the sufficient conditions for $PC3_t^o \approx PC3_t^f$ derived in Section 3.1 are not satisfied even though the differences $\{PC1_t^o - PC1_t^f\}$ are small (Figure 2). Essentially, $GM_3(g, \pi)$ does a poor job of replicating the historical time-series properties of $PC3^o$ owing to the presence of (g_t, π_t) as two of the three risk factors.

5 Macro Factors with Measurement Errors

The common presumption that bond yields are measured less accurately than such macro variables as output growth and inflation seems implausible. Indeed, the more natural presumption is that the reverse is true. By filtering the macro factors we may extract more pure economic factors and this in turn may give more reliable inferences for forecasts, impulse response functions, and other objects of interest.

Our canonical framework is well suited to relaxation of the assumption that $M_t^o = M_t$ to allow for measurement errors on all $\mathcal{N}$ of the risk factors Z_t. To assess the implications of

[29]For computing the impulse IRs for $PC3$ displayed in Figure 4 we scale its loadings so that $PC3$ has the same sample standard deviation as curvature measured as $y_t^{120} + y_t^3 - 2y_t^{24}$.

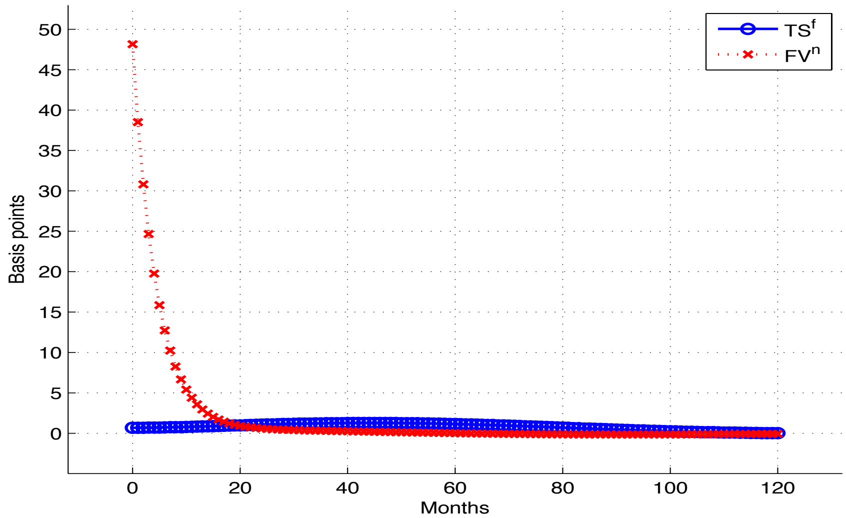


Figure 4: Impulse responses of $PC3$ to its own innovation within models TS^f and FV^n for the family $GM_3(g, \pi)$.

errors in measuring M_t for term structure modeling we replace $M_t^o = M_t$ with

$$M_t^o = M_t + \eta_t, \quad \eta_t \sim N(0, \Sigma_M), \tag{19}$$

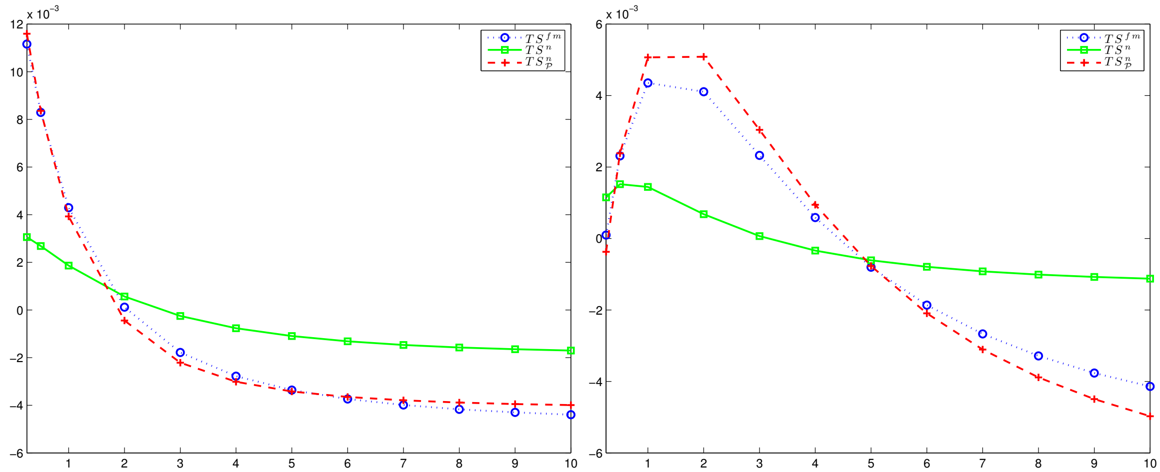
where Σ_M is a $\mathcal{M} \times \mathcal{M}$ covariance matrix. For simplicity, we assume that the errors in measuring y_t and M_t are mutually independent and that η_t is serially uncorrelated.

As an illustration of a *MTSM* with measurement errors on the entire vector (y_t, M_t) we examine model $GM_3(g, \pi)$ from Section 4 under the measurement assumptions (7) and (19) for (y_t, M_t), with $\Sigma_e = \sigma_y^2 I_{12}$ and diagonal Σ_M. (Results are qualitatively similar for an unconstrained Σ_M.) For this *MTSM*, denoted TS^{fm}, the state equation is (6) and the observation equations are (7) and (19). The corresponding macro-filtered FVAR model, denoted FV^{fm}, replaces the observation equation (7) with (8).

Allowing for $M_t^o \neq M_t$ leads to strikingly different relationships among the macro factors and bond yields. Figure 5 plots the loadings of the bond yields on g and π for model TS^{fm}, as functions of maturity. We also plot the loadings for model TS^n in which $PC1_t = PC1_t^o$ and $M_t = M_t^o$, as well as those for model TS$^n_{\mathcal{P}}$ in which the first three PCs are measured without error.[30] Clearly the yield curve responds to shocks to the macro factors shocks very differently in models with and without filtering on M_t.

One might be tempted to conclude that filtering on the macro variables is leading to more informative estimates of responses of bond yields to macro shocks. However, the fact that the estimates for the loadings in model TS^{fm} nearly coincide with those from model TS$^n_{\mathcal{P}}$

[30] For the TS$^n_{\mathcal{P}}$ model, the loadings on $(PC1, g, \pi)$ are computed by transforming the loadings on $(PC1, PC2, PC3)$ through (5). For each specification, the matching results for models TS and FV are nearly identical, so we plot only the TS specification.



(A) Loading on π_t (B) Loadings on g_t

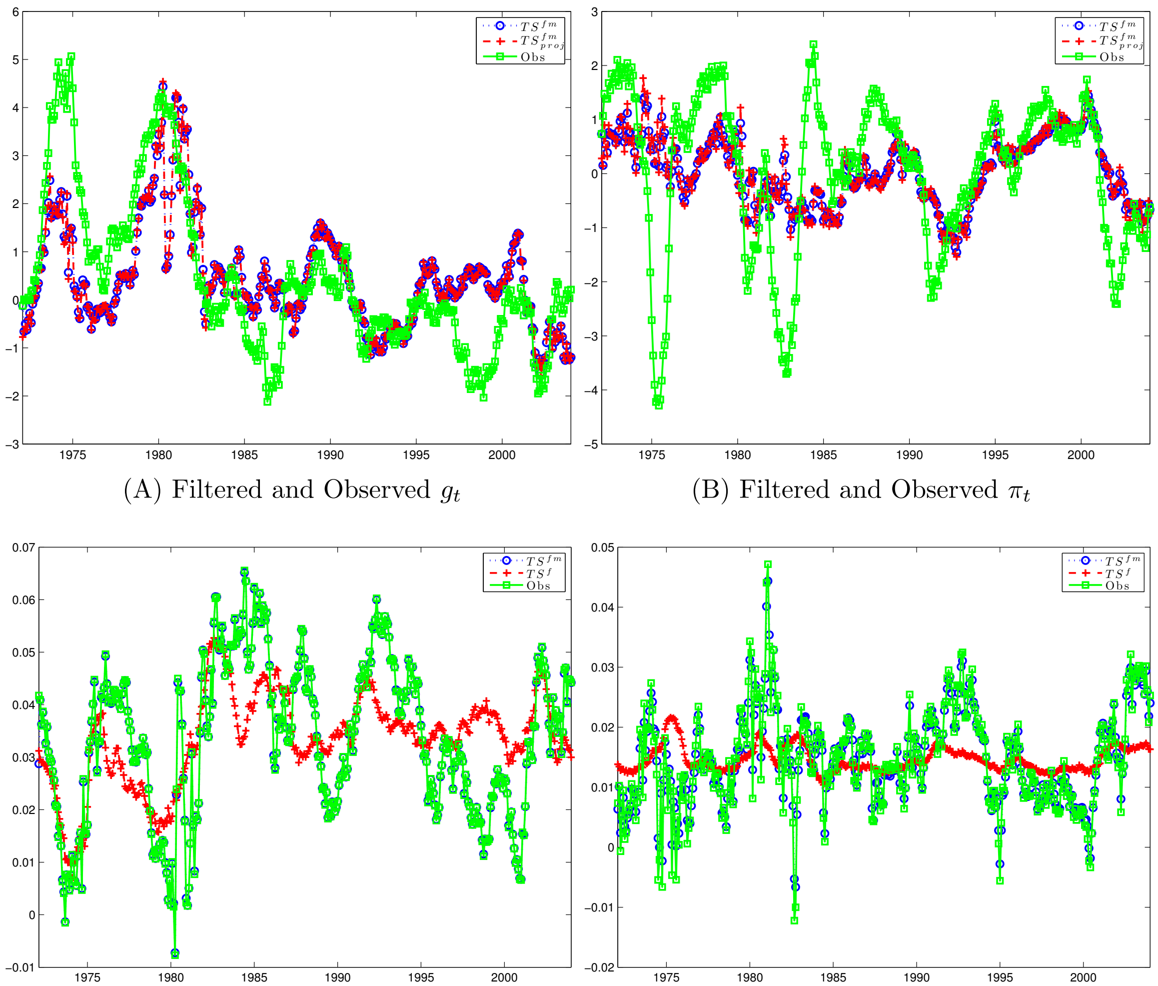
Figure 5: This figure plots the loadings for response of yields to shocks in the macro-variables. Panel A plots the response for the INFPC growth variable. Panel B plots the response for REALPC inflation variable.

lead us to a different conclusion. Namely, allowing for filtering on M_t gives the likelihood function the flexibility to focus on matching the distribution of yields; that is, to substantially reduce the pricing errors for individual bonds in model TS^{fm} relative to those displayed in Figure 2(A) for model TS^{f}.

These improved fits are achieved at the "cost" of substantial deterioration in the fits to the macro factors: their observed and filtered counterparts in Figure 6 are very different, particularly for π_t. Moreover, the filtered g_t^f and π_t^f agree quite closely with their within-model projections given by $\hat{\gamma}_0 + \hat{\gamma}_1 \mathcal{P}_t^o$ and denoted by TS_{proj}^{fm}. Thus model TS^{fm} is essentially selecting risk factors that mimic the first three PCs of bond yields and, thereby, leave substantial components of M_t^o unexplained.

To see this another way, consider panels C and D of Figure 6 which plot the observed and filtered counterparts of $PC2_t$ and $PC3_t$, respectively, for models TS^{fm} and TS^{f}. These series are virtually indistinguishable within model TS^{fm}. On the other hand in model TS^{f}, in which the fit to the macro factors is exact ($M_t^o = M_t$), the filtered PCs differ substantially from their observed counterparts.

It is these calculations and comparisons that underlie the cautionary assessment in our introduction about what can be learned about the joint distribution of (y_t, M_t) from $MTSM$s in which M_t is included among the risk factors that price bonds. Since three-factor versions of model TS^{fm} essentially use the factors to match the cross-section of yields, one is naturally led to consider increasing the number of factors in order to more reliably model the impact of macro factors on bond-market risk premiums. Yet the evidence from estimated $YTSM$s suggests that models with $\mathcal{N} > 3$ are over-parameterized and, at least for some fixed-income portfolios, imply implausibly high Sharpe ratios (JSZ, Duffee (2010)). In the light of this evidence, larger $\mathcal{N}$ may not resolve the mispecification of the joint distribution of (y_t, M_t) in



(A) Filtered and Observed g_t

(B) Filtered and Observed π_t

(C) Filtered and Observed $PC2_t$

(D) Filtered and Observed $PC3_t$

Figure 6: This figure plots observed and filtered version of macro and yield variables. Panel A plots the REALPC growth variable. Also plotted are the filtered counterpart from the model with yield and macro-variables filtering, TS^{fm}. Additionally, the figure plots the within-model projection counterpart given by the linear combination of the filtered yields as given by the model, $\gamma_0 + \gamma_1 \mathcal{P}_t^o$, which we denote by TS^{fm}_{proj}. Panel B plots the corresponding quantities for INFPC. Panels C and D plot the observed and filtered versions of $PC2$ and $PC3$, respectively.

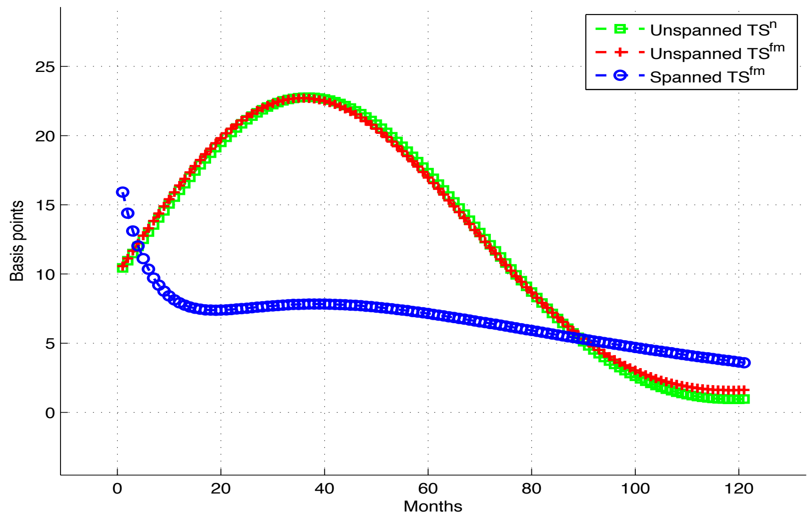


Figure 7: Impulse response of PC1 to a one standard deviation shock to REALPC. The states are ordered as $(INFPC, REALPC, PC1, PC2, PC3)$ for the $MTSM$ with spanned macro factors and $(INFPC, REALPC, PC1)$ for the $MTSM$ with spanning.

$MTSM$s with M_t included in the risk factors Z_t.

Left open by this analysis is whether an important role for M_t reemerges once the macro spanning condition (4) is relaxed– so that unspanned macro risks are allowed– in the presence of measurement errors on the entire vector (y_t, M_t). To answer this question we follow Joslin, Priebsch, and Singleton (2011) and examine a $MTSM$ in which the three pricing factors $\mathcal{P}_t^3$ are normalized to the first three PCs of bond yields and forecasts of $\mathcal{P}_t^3$ under the $\mathbb{P}$ distribution are based on a VAR model for $(\mathcal{P}_t^3, M_t)$, where as above $M_t' = (g_t, \pi_t)$. This formulation ensures that M_t is not spanned by $\mathcal{P}_t^3$, and it allows for M_t to have incremental forecasting power for $\mathcal{P}^3$ after conditioning on $\mathcal{P}_t^3$. Estimation is by the Kalman filter and, for comparability, we preserve the error specification with $\Sigma_e = \sigma_y^2 I_{12}$ and diagonal Σ_M.

Figure 7 displays the IRs of $PC1$ to a shock to $REALPC$ for four different models. Considering first the case of $MTSM$s with unspanned M, the IRs with (Unspanned TS^{fm}) and without (Unspanned TS^n) filtering on (y_t, M_t) are virtually identical.[31] There are two key ingredients to this rather striking (near) equivalence: (i) for the reasons given in Section 3, when we rotate to the first three PCs as risk factors the filtered and observed $\mathcal{P}_t^3$ are nearly identical; and (ii) once we allow for unspanned macro risks, the filtered and observed macro factors are also very similar $((g_t^o, \pi_t^o) \approx (g_t^f, \pi_t^f))$.

For comparison, we also display the IR from the $MTSM$ that enforces macro spanning and both yields and macro factors are measured with errors (Spanned TS^{fm}). Clearly, enforcing spanning of M in this setting leads to a substantially distorted picture of the impact of macro risks on the level of Treasury yields.

[31] This is equally true for comparisons of all other IRs for these two models.

6 No-Arbitrage and Filtering Under Restrictions

When the no-arbitrage structure of a *MTSM* is combined with over-identifying restrictions on the parameters governing the physical distribution of (y_t, M_t), their dynamic properties within a *MTSM* and its factor-VAR counterpart may differ. It then becomes an empirical question as to whether any such differences are economically significant. The most commonly imposed restrictions are zero restrictions on the feedback matrix $K_1^{\mathbb{P}}$ in the Markov $\mathbb{P}$-representation of Z_t. For example, Diebold and Li (2006) find that constraining $K_1^{\mathbb{P}}$ to be diagonal within their family of *YTSM*s improves the out-of-sample forecasts of bond yields. Within *MTSM*s Ang, Piazzesi, and Wei (2006), among others, imposed weaker sets of restrictions on $K_1^{\mathbb{P}}$ based on the asymptotic standard errors from less constrained models.

To explore whether the imposition of constraints on the $\mathbb{P}$ distribution Z_t lead to economically significant differences between models FV^n and TS^f within the family $GM_3(g, \pi)$ we enforce the constraint that $K_1^{\mathbb{P}}$ is diagonal. This constraint is strongly rejected by a likelihood ratio test and, thus, there is at least the possibility that the properties of the no-arbitrage model TS_D^f that enforces diagonality of $K_1^{\mathbb{P}}$ are different than those of model FV_D^n.[32] Consistent with the test results, the properties of the $\mathbb{P}$ distribution of Z_t within the models TS^f and TS_D^f are very different. Nevertheless, with the diagonality constraint in place, adding the no-arbitrage constraint (going from model FV_D^f to model TS_D^f) has a negligible effect on the conditional distribution of (y_t, M_t). This holds for the parameters as well as conditional forecasts, variances, and impulse response functions.

Consider, for example, the root mean-squared differences of within-sample forecasts of yields and *PC*s one- and six-months ahead displayed in Table 3. The column "$FV^n \, vs \, TS^f$" provides a baseline for the size of the differences in forecasts when no arbitrage and filtering are used without the diagonal constraint. These models are canonical so the economically small differences (ranging from one to seven basis points) are implications of our irrelevancy propositions. The large differences for models TS^f and TS_D^f in the next column are a manifestation of the binding nature of the diagonality constraint on $K_1^{\mathbb{P}}$.

Most important for the theme of this section are the small root mean-square differences in the forecasts from models TS_D^f and FV_D^n, at both the one- and six-month horizons. The magnitudes of these differences are nearly identical to those from the canonical models (TS^f, FV^f). We conclude that, even in the presence of the constraint that $K_1^{\mathbb{P}}$ is diagonal, no-arbitrage restrictions shed no incremental light on the $\mathbb{P}$ distribution of yields and macro factors, once the low-dimensional factor structure of model FV_D^f has been imposed.

These findings are complementary to those for *YTSM*s reported by JSZ, who found that the constraints on the feedback matrix $K_1^{\mathbb{P}}$ imposed by Christensen, Diebold, and Rudebusch (2009) had small effects on out-of-sample forecasts. Further, Ang, Dong, and Piazzesi (2007) found that impulse response functions implied by their three-factor ($\mathcal{M} = 2, \mathcal{L} = 1$) *MTSM* that imposed zero restrictions on lag coefficients and the parameters governing the market prices of risk were nearly identical to those computed from their corresponding unrestricted

[32] The failure to reject this constraint would suggest (by transitivity) that these models are (nearly) identical, since we found that models TS^f and FV^n imply near identical $\mathbb{P}$ distributions of Z_t.

		FV^n vs TS^f	TS^f vs TS^f_D	TS^f_D vs FV^n_D
	PC1	4.14	7.92	4.13
	PC2	1.33	4.26	1.36
1-month	Y12	4.61	8.86	4.67
	Y60	4.53	7.60	4.60
	Y120	6.75	7.11	6.99
	PC1	4.13	42.30	4.54
	PC2	1.51	23.81	1.62
6-month	Y12	5.04	48.34	5.46
	Y60	4.84	40.51	5.16
	Y120	7.16	37.25	7.07

Table 3: The table presents the root mean square differences between forecasts across different model specifications for one month and six month horizons.

VAR. All of these results illustrate cases where our propositions on the near irrelevance of no-arbitrage restrictions in *MTSM*s (and *YTSM*s) carry over to non-canonical models.

Of course this finding does not imply that *YTSM*s or *MTSM*s are of little value for understanding the risk profiles of portfolios of bonds. Restrictions on risk premiums in bond markets typically amount to constraints across the $\mathbb{P}$ and $\mathbb{Q}$ distributions of Z_t, and such constraints cannot be explored outside of a term structure model that (implicitly or explicitly) links these distributions. Moreover, the presence of constraints on risk premiums will in general imply that ML estimates of the $\mathbb{P}$ distribution of yields within a *MTSM* are more efficient than those from its corresponding factor-VAR.

Illustrative of this point are the findings in Joslin, Priebsch, and Singleton (2011) and Jardet, Monfort, and Pegoraro (2010) that, within their versions of models TS^n and FV^n, enforcing near cointegration of Z_t under $\mathbb{P}$ leads to very different dynamic properties of their risk factors. The constraint that expected excess returns lie in a lower than $\mathcal{N}$-dimensional space (Cochrane and Piazzesi (2005), JSZ) might also have material effects on the efficiency of ML/Kalman filter estimates. Both of these constraints are qualitatively different from the zero restrictions on $K_1^{\mathbb{P}}$ and $K_1^{\mathbb{Q}}$ that are often imposed in the literature on *MTSM*s.

Summarizing this evidence, our results on the irrelevance of no-arbitrage restrictions for the analysis of the distribution of bond yields and macro factors appear robust to a widely applied set of restrictions on canonical *MTSM*s. At the same time, the extant evidence suggests that constraints inducing greater persistence of the risk factors under $\mathbb{P}$ (thereby mitigating small sample bias in ML estimators) may drive an economically large wedge between the dynamic properties of a *MTSM* and its factor-VAR counterpart. The extent to which any such wedge impacts impulse response functions or conditional forecasts is an informative diagnostic about the economic content of a *MTSM* relative to its less constrained factor-VAR.

Appendices

A A Canonical Form for *MTSM*s

Our objective is to show that each *MTSM* where

$$r_t = \rho_0^{\mathcal{L}} + \rho_1^{\mathcal{L}} \cdot Z_t^{\mathcal{L}} \tag{A1}$$

with the risk factors $Z_t^{\mathcal{L}} \equiv (M_t', L_t')'$ following the Gaussian processes

$$\Delta Z_t^{\mathcal{L}} = \kappa_0^{\mathbb{Q}} + \kappa_1^{\mathbb{Q}} Z_{t-1}^{\mathcal{L}} + \sqrt{\Omega}\epsilon_t^{\mathbb{Q}} \text{ under } \mathbb{Q} \text{ and} \tag{A2}$$

$$\Delta Z_t^{\mathcal{L}} = \kappa_0^{\mathbb{P}} + \kappa_1^{\mathbb{P}} Z_{t-1}^{\mathcal{L}} + \sqrt{\Omega}\epsilon_t^{\mathbb{P}} \text{ under } \mathbb{P}, \tag{A3}$$

is observationally equivalent to a *unique* member of *MTSM* in which $Z_t = (M_t', \mathcal{P}_t^{\mathcal{L}\prime})'$ with $\mathcal{L}$ yield portfolios $\mathcal{P}_t^{\mathcal{L}}$:

$$r_t = \rho_0 + \rho_1 \cdot Z_t, \tag{A4}$$

$$\Delta Z_t = K_0^{\mathbb{Q}} + K_1^{\mathbb{Q}} Z_{t-1} + \sqrt{\Sigma}\epsilon_t^{\mathbb{Q}} \text{ under } \mathbb{Q} \text{ and} \tag{A5}$$

$$\Delta Z_t = K_0^{\mathbb{P}} + K_1^{\mathbb{P}} Z_{t-1} + \sqrt{\Sigma}\epsilon_t^{\mathbb{P}} \text{ under } \mathbb{P} \tag{A6}$$

where $(\rho_0, \rho_1, K_0^{\mathbb{Q}}, K_1^{\mathbb{Q}})$ are explicit functions of some underlying parameter set $\Theta_{TS}^{\mathbb{Q}} = (r_\infty^{\mathbb{Q}}, \lambda^{\mathbb{Q}}, \gamma_0, \gamma_1, \Sigma)$ to be described. We will make precise the sense in which $\Theta_Z = (\Theta_{TS}^{\mathbb{Q}}, K_0^{\mathbb{P}}, K_1^{\mathbb{P}})$ uniquely characterizes the latter *MTSM*.

Observational Equivalence

Assuming, for ease of exposition, that $\kappa_1^{\mathbb{Q}}$ has nonzero, real and distinct eigenvalues with the standard eigendecomposition $\kappa_1^{\mathbb{Q}} = A^{\mathbb{Q}} diag(\lambda^{\mathbb{Q}}) A^{\mathbb{Q}^{-1}}$, we follow Joslin (2006) and JSZ by adopting the rotation:[33]

$$X_t = \mathcal{V}^{-1}\left(Z_t^{\mathcal{L}} + (\kappa_1^{\mathbb{Q}})^{-1}\kappa_0^{\mathbb{Q}}\right) \;\text{ where }\; \mathcal{V} = A^{\mathbb{Q}} diag((\rho_1^{\mathcal{L}})' A^{\mathbb{Q}})^{-1} \tag{A7}$$

to arrive at the following $\mathbb{Q}$ specification:

$$r_t = r_\infty^{\mathbb{Q}} + \iota \cdot X_t, \;\text{ and }\; \Delta X_t = diag(\lambda^{\mathbb{Q}}) X_{t-1} + \sqrt{\Sigma_X}\epsilon_t^{\mathbb{Q}} \tag{A8}$$

where $\lambda^{\mathbb{Q}}$ is ordered, ι denotes a vector of ones, and

$$r_\infty^{\mathbb{Q}} = \rho_0^{\mathcal{L}} + (\rho_1^{\mathcal{L}})'(\kappa_1^{\mathbb{Q}})^{-1}\kappa_0^{\mathbb{Q}} \;\text{ and }\; \Omega = \mathcal{V}\Sigma_X\mathcal{V}'.$$

From (A8), the $J \times 1$ vector of yields y_t is affine in X_t:

$$y_t = A_X(r_\infty^{\mathbb{Q}}, \lambda^{\mathbb{Q}}, \Sigma_X) + B_X(\lambda^{\mathbb{Q}}) X_t \tag{A9}$$

[33] See JSZ for detailed treatments of cases with complex, repeated or zero eigenvalues.

with A_X, B_X obtained from standard recursions. Following JSZ we fix a full-rank loadings matrix $W \in \mathbb{R}^{J\times J}$ and let $\mathcal{P}_t = Wy_t$. Focusing on the first $\mathcal{N}$ portfolios $\mathcal{P}_t^{\mathcal{N}}$, we have:

$$\mathcal{P}_t^{\mathcal{N}} = W^{\mathcal{N}}A_X + W^{\mathcal{N}}B_X X_t. \tag{A10}$$

Based on (A7) and (A10), there is a linear mapping between M_t and $\mathcal{P}_t^{\mathcal{N}}$:

$$M_t = \gamma_0 + \gamma_1 \mathcal{P}_t^{\mathcal{N}} \tag{A11}$$

where

$$\gamma_1 = \mathcal{V}_{\mathcal{M}}(W^{\mathcal{N}}B_X)^{-1} \text{ and } \gamma_0 = -\gamma_1 W^{\mathcal{N}}A_X - A^{\mathbb{Q}}_{\mathcal{M}}(\lambda^{\mathbb{Q}})^{-1}A^{\mathbb{Q}^{-1}}\kappa_0^{\mathbb{Q}}, \tag{A12}$$

and $\mathcal{V}_{\mathcal{M}}$, $A^{\mathbb{Q}}_{\mathcal{M}}$ denote the first $\mathcal{M}$ rows of $\mathcal{V}$, $A^{\mathbb{Q}}$, respectively. This allows us to write:

$$Z_t = \Gamma_0 + \Gamma_1\mathcal{P}_t^{\mathcal{N}} = \Gamma_0 + \Gamma_1(W^{\mathcal{N}}A_X + W^{\mathcal{N}}B_X X_t) = \mathcal{U}_0 + \mathcal{U}_1^{-1}X_t \tag{A13}$$

where

$$\Gamma_0 = (\gamma_0{}', 0_{\mathcal{L}}')', \;\; \Gamma_1 = \begin{pmatrix} \gamma_1 \\ I_{\mathcal{L}}, 0_{\mathcal{L}\times\mathcal{M}} \end{pmatrix}, \;\; \mathcal{U}_0 = \Gamma_0 + \Gamma_1 W^{\mathcal{N}}A_X, \;\text{ and }\; \mathcal{U}_1 = (\Gamma_1 W^{\mathcal{N}}B_X)^{-1}.$$

Combining (A8) and (A13), the $\mathbb{Q}$-specification of Z_t is:

$$r_t = \rho_0 + \rho_1 \cdot Z_t \text{ and } \Delta Z_t = K_0^{\mathbb{Q}} + K_1^{\mathbb{Q}}Z_{t-1} + \sqrt{\Sigma}\epsilon_t^{\mathbb{Q}}, \tag{A14}$$

where

$$\rho_1 = (\mathcal{U}_1)'\iota \text{ and } \rho_0 = r_\infty^{\mathbb{Q}} - \rho_1 \cdot \mathcal{U}_0,$$
$$K_1^{\mathbb{Q}} = \mathcal{U}_1^{-1}\lambda^{\mathbb{Q}}\mathcal{U}_1, \;\; K_0^{\mathbb{Q}} = -K_1^{\mathbb{Q}}\mathcal{U}_0 \;\text{ (and } \Sigma_X = \mathcal{U}_1\Sigma\mathcal{U}_1').$$

Based on (A7) and (A13), there must be a linear mapping between Z_t and $Z_t^{\mathcal{L}}$. It follows that the $\mathbb{P}$-dynamics of Z_t must be Gaussian as in (A6).

To summarize, the *MTSM* with mixed macro-latent risk factors $Z_t^{\mathcal{L}}$, described by (A1), (A2), and (A3), is observationally equivalent to one with observable mixed macro-yield-portfolio risk factors Z_t, characterized by (A4), (A5), and (A6). The *primitive* parameter set is $\Theta_Z = (r_\infty^{\mathbb{Q}}, \lambda^{\mathbb{Q}}, \gamma_0, \gamma_1, \Sigma, K_0^{\mathbb{P}}, K_1^{\mathbb{P}})$. The mappings between $(\rho_0, \rho_1, K_0^{\mathbb{Q}}, K_1^{\mathbb{Q}})$ and $\Theta_{TS}^{\mathbb{Q}} = (r_\infty^{\mathbb{Q}}, \lambda^{\mathbb{Q}}, \gamma_0, \gamma_1, \Sigma)$ are:

$$\rho_1 = (\mathcal{U}_1)'\iota, \;\; \rho_0 = r_\infty^{\mathbb{Q}} - \rho_{1Z}\cdot\mathcal{U}_0, \;\; K_1^{\mathbb{Q}} = \mathcal{U}_1^{-1}\lambda^{\mathbb{Q}}\mathcal{U}_1, \;\; K_0^{\mathbb{Q}} = -K_1^{\mathbb{Q}}\mathcal{U}_0, \tag{A15}$$

where

$$\mathcal{U}_1 = (\Gamma_1 W^{\mathcal{N}}B_X(\lambda^{\mathbb{Q}}))^{-1}, \;\; \mathcal{U}_0 = \Gamma_0 + \Gamma_1 W^{\mathcal{N}}A_X(r_\infty^{\mathbb{Q}}, \lambda^{\mathbb{Q}}, \mathcal{U}_1\Sigma\mathcal{U}_1'), \;\text{ and}$$
$$\Gamma_0 = (\gamma_0{}', 0_{\mathcal{L}}')', \;\; \Gamma_1 = \begin{pmatrix} \gamma_1 \\ I_{\mathcal{L}}, 0_{\mathcal{L}\times\mathcal{M}} \end{pmatrix}.$$

Uniqueness

Consider two parameter sets, Θ_Z and $\tilde{\Theta}_Z$, that give rise to two observationally equivalent *MTSM*'s with risk factors Z_t. Since Z_t is observable, the parameters, $\Sigma, K_0^{\mathbb{P}}, K_1^{\mathbb{P}}$, describing the $\mathbb{P}$-dynamics of Z_t must be identical. Additionally, based on (A11), the following identity must hold state by state:

$$M_t \equiv \gamma_0 + \gamma_1 \mathcal{P}_t^{\mathcal{N}} \equiv \tilde{\gamma}_0 + \tilde{\gamma}_1 \mathcal{P}_t^{\mathcal{N}}. \tag{A16}$$

Since W is full rank and the $\mathcal{P}_t^{\mathcal{N}}$ are linearly independent, it follows that:

$$\gamma_0 = \tilde{\gamma}_0 \text{ and } \gamma_1 = \tilde{\gamma}_1. \tag{A17}$$

Finally, writing the term structure with $\mathcal{P}_t^{\mathcal{N}}$ as risk factors:

$$y_t = A_X + B_X(W^{\mathcal{N}} B_X)^{-1}(P_t^{\mathcal{N}} - W^{\mathcal{N}} A_X), \tag{A18}$$

it follows that

$$B_X(W^{\mathcal{N}} B_X)^{-1} = \tilde{B}_X(W^{\mathcal{N}} \tilde{B}_X)^{-1}, and \tag{A19}$$

$$(I_J - B_X(W^{\mathcal{N}} B_X)^{-1} W^{\mathcal{N}})A_X = (I_J - \tilde{B}_X(W^{\mathcal{N}} \tilde{B}_X)^{-1} W^{\mathcal{N}})\tilde{A}_X. \tag{A20}$$

Now (A19) is equivalent to:

$$diag(\frac{1-\lambda_i^n}{1-\lambda_i})(W^{\mathcal{N}} B_X)^{-1} = diag(\frac{1-\tilde{\lambda}_i^n}{1-\tilde{\lambda}_i})(W^{\mathcal{N}} \tilde{B}_X)^{-1} \tag{A21}$$

for every horizon n. As long as both $W^{\mathcal{N}} B_X$ and $W^{\mathcal{N}} \tilde{B}_X$ are full rank, it must follow that $\lambda_i^{\mathbb{Q}} \equiv \tilde{\lambda}_i^{\mathbb{Q}}$ for all i's.

Turning to (A20), we note that

$$A_X = \iota\, r_\infty^{\mathbb{Q}} + \beta_X vec(\Sigma_X) \tag{A22}$$

where β_X is a function of $\lambda^{\mathbb{Q}}$, and thus must be the same for both Θ_Z and $\tilde{\Theta}_Z$. Likewise, $\Sigma_X = \mathcal{U}_1 \Sigma \mathcal{U}_1'$, dependent only on $(\gamma_1, \lambda^{\mathbb{Q}}, \Sigma)$, must be the same for both parameter sets. It follows that $r_\infty^{\mathbb{Q}} = \tilde{r}_\infty^{\mathbb{Q}}$. Therefore, $\Theta_Z \equiv \tilde{\Theta}_Z$.

Regularity Conditions

First, we assume that the diagonal elements of $\lambda^{\mathbb{Q}}$ are non-zero, real and distinct. These assumptions can be easily relaxed - see JSZ for detailed treatments. Second, we assume that the *MTSM*'s are non-degenerate in the sense that there is no transformation such that the effective number of risk factors is less than $\mathcal{N}$. For this, the requirement is that all elements of $(\rho_1^{\mathcal{L}})' A^{\mathbb{Q}}$ are non-zero. In terms of the parameters of our canonical form, we require that none of the eigenvectors of the risk-neutral feedback matrix $K_1^{\mathbb{Q}}$ is orthogonal to the loadings vector ρ_1 of the short rate. Finally, to maintain valid transformations between alternative choices of risk factors, we require that the matrices $W^{\mathcal{N}} B_X$ and Γ_1 be full rank. These are conditions on $(\lambda^{\mathbb{Q}}, W)$ and γ_1, respectively.

The following theorem summarizes the above derivations:

Theorem A1. *Fix a full-rank portfolio matrix* $W \in \mathbb{R}^{J \times J}$*, and let* $\mathcal{P}_t = W y_t$*. Any canonical form for the family of* $\mathcal{N}$*-factor models* MTSM *is observationally equivalent to a unique* MTSM *in which the first* $\mathcal{M}$ *components of the pricing factors* Z_t *are the macro variables* M_t*, and the remaining* $\mathcal{L}$ *components of* Z_t *are* $\mathcal{P}_t^{\mathcal{L}}$*;* r_t *is given by (A4);* M_t *is related to* $\mathcal{P}_t$ *through*

$$M_t = \gamma_0 + \gamma_1 \mathcal{P}_t^{\mathcal{N}}, \tag{A23}$$

for $\mathcal{M} \times 1$ *vector* γ_0 *and* $\mathcal{M} \times \mathcal{N}$ *matrix* γ_1*; and* Z_t *follows the Gaussian* $\mathbb{Q}$ *and* $\mathbb{P}$ *processes (A5), and (A6), where* $K_0^{\mathbb{Q}}$*,* $K_1^{\mathbb{Q}}$*,* ρ_0*, and* ρ_1 *are explicit functions of* $\Theta_{TS}^{\mathbb{Q}} = (r_\infty^{\mathbb{Q}}, \lambda^{\mathbb{Q}}, \gamma_0, \gamma_1, \Sigma)$*, given by (A15). For given* W*, our canonical form is parametrized by* $\Theta_{TS} = (\Theta_{TS}^{\mathbb{Q}}, K_0^{\mathbb{P}}, K_1^{\mathbb{P}})$*.*

B Bond Pricing in *MTSM*s

Under (A4–A6), the price of an m-year zero-coupon bond is given by

$$D_{t,m} = E_t^{\mathbb{Q}}[e^{-\sum_{i=0}^{m-1} r_{t+i}}] = e^{\mathcal{A}_m + \mathcal{B}_m \cdot Z_t}, \tag{A24}$$

where $(\mathcal{A}_m, \mathcal{B}_m)$ solve the first-order difference equations

$$\mathcal{A}_{m+1} - \mathcal{A}_m = K_0^{\mathbb{Q}'} \mathcal{B}_m + \frac{1}{2} \mathcal{B}_m' H_0 \mathcal{B}_m - \rho_0 \tag{A25}$$

$$\mathcal{B}_{m+1} - \mathcal{B}_m = K_1^{\mathbb{Q}'} \mathcal{B}_m - \rho_1 \tag{A26}$$

subject to the initial conditions $\mathcal{A}_0 = 0, \mathcal{B}_0 = 0$. See, for example, Dai and Singleton (2003). The loadings for the corresponding bond yield are $A_m = -\mathcal{A}_m/m$ and $B_m = -\mathcal{B}_m/m$.

C Invariant Transformations of *MTSM*s

As in Dai and Singleton (2000), given a *MTSM* with parameters as in (A4–A6) and state Z_t, application of the invariant transformation $\hat{Z}_t = C + D Z_t$ gives an observationally equivalent term structure model with state $\hat{Z}_t$ and parameters

$$K_{0\hat{Z}}^{\mathbb{Q}} = D K_0^{\mathbb{Q}} - D K_1^{\mathbb{Q}} D^{-1} C, \tag{A27}$$

$$K_{1\hat{Z}}^{\mathbb{Q}} = D K_1^{\mathbb{Q}} D^{-1}, \tag{A28}$$

$$\rho_{0\hat{Z}} = \rho_0 - \rho_1' D^{-1} C \tag{A29}$$

$$\rho_{1\hat{Z}} = (D^{-1})' \rho_1, \tag{A30}$$

$$K_{0\hat{Z}}^{\mathbb{P}} = D K_0^{\mathbb{P}} - D K_1^{\mathbb{P}} D^{-1} C, \tag{A31}$$

$$K_{1\hat{Z}}^{\mathbb{P}} = D K_1^{\mathbb{P}} D^{-1}, \tag{A32}$$

$$\Sigma_{\hat{Z}} = D \Sigma D'. \tag{A33}$$

D Filtering Invariance of the Mean Parameters

This appendix shows that when $\Sigma_{e\mathcal{L}}S_t^{-1}$ is small the filtered version of equation (15),

$$[\hat{K}_0^{\mathbb{P}}, I + \hat{K}_1^{\mathbb{P}}]' = \left(\frac{1}{T}\sum_t Z_{t+1}^f, \frac{1}{T}\sum_t (Z_{t+1}Z_t')^f\right)\left(\begin{array}{cc} 1 & \frac{1}{T}\sum_t Z_t^{f\prime} \\ \frac{1}{T}\sum_t Z_t^f & \frac{1}{T}\sum_t (Z_t Z_t')^f \end{array}\right)^{-1}, \quad \text{(A34)}$$

gives (under mild assumptions) estimates that are close to the OLS estimates. Assuming further that $(Z_tZ_t')^s$ and $(Z_{t+1}Z_t')^s$ are close to their filtered counterparts, it follows that the smoothed version of (15) will also give approximately the OLS estimates of $K_0^{\mathbb{P}}$ and $K_1^{\mathbb{P}}$.

As shown in Section D.1, when $\Sigma_{e\mathcal{L}}S_t^{-1}$ is small, convergence to the steady-state distribution will be fast. As such we can treat $\Omega_t = Var(Z_t^o|\mathcal{P}_t^{-\mathcal{L}o}, \mathcal{F}_{t-1})$ as a constant matrix, with $\mathcal{P}_t^{-\mathcal{L}o}$ being the $J - \mathcal{L}$ higher order PCs. Post-multiplying both terms on the right-hand side of (A34) by $\left(\begin{array}{cc} 1 & 0 \\ 0 & \Omega_t^{-1} \end{array}\right)$ leads to:

$$\left(\frac{1}{T}\sum_t Z_{t+1}^f, \frac{1}{T}\sum_t (Z_{t+1}Z_t')^f\Omega_t^{-1}\right)\left(\begin{array}{cc} 1 & \frac{1}{T}\sum_t Z_t^{f\prime}\Omega_t^{-1} \\ \frac{1}{T}\sum_t Z_t^f & \frac{1}{T}\sum_t (Z_t Z_t')^f\Omega_t^{-1} \end{array}\right)^{-1}. \quad \text{(A35)}$$

Now,

$$\begin{aligned}(Z_tZ_t')^f\Omega_t^{-1} =& Var(Z_t|\mathcal{F}_t)\Omega_t^{-1} + Z_t^f(Z_t^f)'\Omega_t^{-1} \\ =& Var(Z_t|\mathcal{F}_t)\Omega_t^{-1} + Z_t^o(Z_t^o)'\Omega_t^{-1}, \end{aligned} \quad \text{(A36)}$$

where the second line follows from results in Section 3.1. Using block inversion, the non-zero block of the first term is:

$$\Sigma_{e\mathcal{L}}S_t^{-1} - \Sigma_{e\mathcal{L}}S_t^{-1}\Sigma_{e\mathcal{L}}S_t^{-1}$$

which under our assumption must be close to zero. Therefore we can replace the term $\frac{1}{T}\sum_t (Z_tZ_t')^f\Omega_t^{-1}$ in (A35) by $\frac{1}{T}\sum_t Z_t^o Z_t^{o\prime}\Omega_t^{-1}$. Using a similar argument, $\frac{1}{T}\sum_t (Z_{t+1}Z_t')^f\Omega_t^{-1}$ can also be replaced by $\frac{1}{T}\sum_t Z_{t+1}^o Z_t^{o\prime}\Omega_t^{-1}$. Furthermore, results in Section 3.1 allow us to replace Z_t^f in (A35) by its observed counter-part:

$$\left(\frac{1}{T}\sum_t Z_{t+1}^o, \frac{1}{T}\sum_t Z_{t+1}^o Z_t^{o\prime}\Omega_t^{-1}\right)\left(\begin{array}{cc} 1 & \frac{1}{T}\sum_t Z_t^{o\prime}\Omega_t^{-1} \\ \frac{1}{T}\sum_t Z_t^o & \frac{1}{T}\sum_t Z_t^o Z_t^{o\prime}\Omega_t^{-1} \end{array}\right)^{-1}. \quad \text{(A37)}$$

Finally, if $Var_T(Z_t^o)Var(Z_t^o|\mathcal{P}_t^{-\mathcal{L}o}, \mathcal{F}_{t-1})^{-1}$ is non-degenerate relative to $\Sigma_{e\mathcal{L}}S_t^{-1}$, then all Ω_t's cancel out and (A37) reduces to the the familiar OLS estimates.

D.1 Speed of Convergence to Steady States

Consider the following *generic* state space system:

$$Z_{t+1} = K_0 + K_1 Z_t + \sqrt{\Sigma}\epsilon_{t+1}, \quad \text{(A38)}$$

$$Z_{t+1}^o = Z_{t+1} + e_{Z,t+1}, \quad \text{(A39)}$$

$$Y_{t+1}^o = A + BZ_{t+1} + e_{Y,t+1} \quad \text{(A40)}$$

where $e_{Z,t}$ and $e_{Y,t}$ are independent and $e_{Z,t} \sim N(0, \Sigma_{Ze})$ and $e_{Y,t} \sim N(0, \Sigma_{Ye})$. Let Σ_{t+1}, Ω_{t+1} denote $Var(Z_{t+1}|\mathcal{F}_t)$ and $Var(Z^o_{t+1}|Y^o_{t+1}, \mathcal{F}_t)$ respectively. It is standard to show that Σ_{t+1} follows the recursion:

$$\Sigma_{t+1} = \Sigma + K_1(\Sigma_t - \Sigma_t\tilde{B}'(\tilde{B}\Sigma_t\tilde{B}' + \Sigma_e)^{-1}\tilde{B}\Sigma_t)K_1' \tag{A41}$$

where Σ_e is the variance matrix of $(e'_{Z,t}, e'_{Y,t})'$ and $\tilde{B}' = (I, B')$. We first show that when $\Sigma_e\Omega_t^{-1}$ is small then Σ_t, and therefore the Kalman gain matrix, will approach their steady-state values rapidly. Then we specialize this condition to our pricing framework.

Standard linear algebra allows us to express the term between K_1 and K_1' in (A41) as:

$$\Sigma_{Ze} - (\Sigma_{Ze}, 0)\left(\tilde{B}\Sigma_t\tilde{B}' + \Sigma_e\right)^{-1}\begin{pmatrix} \Sigma_{Ze} \\ 0 \end{pmatrix}. \tag{A42}$$

Now consider a small variation in Σ_t of $\partial\Sigma_t$, the corresponding change in Σ_{t+1} (the Fréchet derivative) will be:

$$\partial\Sigma_{t+1} = \Phi\partial\Sigma_t\Phi' \;\text{ with }\; \Phi = K_1(\Sigma_{Ze}, 0)\left(\tilde{B}\Sigma_t\tilde{B}' + \Sigma_e\right)^{-1}\begin{pmatrix} I \\ B \end{pmatrix}. \tag{A43}$$

Replacing $\left(\tilde{B}\Sigma_t\tilde{B}' + \Sigma_e\right)$ by $Var\left(\begin{matrix} Z^o_t \\ Y^o_t \end{matrix} |\mathcal{F}_{t-1}\right)$ and applying block-wise inversion to this matrix, gives:

$$\Phi = K_1\Sigma_{Ze}\Omega_t^{-1}(I - \Sigma_t B'(B\Sigma_t B' + \Sigma_{Ye})^{-1}B). \tag{A44}$$

As a result, as $\Sigma_{Ze}\Omega_t^{-1}$ approaches zeros, so do the eigenvalues of Φ. Since the recursion (A41) can be written approximately as:

$$vec(\Sigma_{t+1} - \bar{\Sigma}) \approx (\Phi \otimes \Phi)vec(\Sigma_t - \bar{\Sigma}), \tag{A45}$$

where $\bar{\Sigma}$ denotes the steady state value of Σ_t, small eigenvalues of Φ (and hence $\Phi \otimes \Phi$) induce fast convergence to the steady state.

For *MTSM*s we assume that M_t is perfectly observed, and the $\mathcal{M}$ rows and columns of Σ_e corresponding to M_t are zeros. Applying block inversion to Ω_t and collecting the $\mathcal{L} \times \mathcal{L}$ block corresponding to the yield portfolios $\mathcal{P}^{\mathcal{L}}_t$, it can be seen that we need $\Sigma_{e\mathcal{L}}S_t^{-1}$ to be small.

E Filtering Invariance of the Variance Parameters

The term structure corresponding to our canonical form with the observable risk factors Z_t can be obtained by substituting (A13) into (A18):

$$y_t = A_X + B_X(W^{\mathcal{N}}B_X)^{-1}(\Gamma_1^{-1}(Z_t - \Gamma_0) - W^{\mathcal{N}}A_X). \tag{A46}$$

From this we can write $\mathcal{P}_t = A_{TS} + B_{TS} Z_t$, where

$$A_{TS} = \mathcal{G}\gamma_r + \beta_Z vec(\Sigma), \tag{A47}$$
$$B_{TS} = W B_X \mathcal{U}_1, \tag{A48}$$
$$\mathcal{G} = W\left((I_J - B_X(W^{\mathcal{N}}B_X)^{-1}W^{\mathcal{N}})\iota, B_X\mathcal{U}_{1,\mathcal{M}}\right), \tag{A49}$$
$$\beta_Z = W(I_J - B_X(W^{\mathcal{N}}B_X)^{-1}W^{\mathcal{N}})\beta_X(\mathcal{U}_1 \otimes \mathcal{U}_1), \tag{A50}$$

$\gamma_r' = (r_\infty^{\mathbb{Q}}, \gamma_0{}')$, and $\mathcal{U}_{1,\mathcal{M}}$ denotes the first $\mathcal{M}$ columns of $\mathcal{U}_1$. Importantly, $\mathcal{G}$ and $\mathcal{T}$ are only dependent on $\lambda^{\mathbb{Q}}$ and γ_1. Therefore, from (7), the errors in pricing $\mathcal{P}_t$ are given by

$$e_t = \mathcal{P}_t^o - \mathcal{G}\gamma_r - \beta_Z vec(\Sigma) - B_{TS} Z_t. \tag{A51}$$

Since

$$f(\mathcal{P}_t^o | Z_t; \Theta^{\mathbb{Q}}, \Sigma_e) = (2\pi)^{-J/2}|\Sigma_e|^{-1/2} \exp\left(-\frac{1}{2} e_t' \, \Sigma_e^{-1} \, e_t\right), \tag{A52}$$

it follows that

$$\sum_t \partial \log f(\mathcal{P}_t^o | Z_t; \Theta^{\mathbb{Q}}, \Sigma_e) / \partial vec(\Sigma) = \hat{\beta}_Z' (\hat{\Sigma}_e)^{-1} \sum_t \hat{e}_t^u, \tag{A53}$$

where the unobserved pricing errors $\hat{e}_t^u$ from (7) are evaluated at the ML estimators and depend on the partially observed $\vec{Z}$.

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