

Portfolio Selection with Estimation Risk: a Test Based Approach*

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Abstract

An important challenge of portfolio allocation arises when the (true) characteristics of returns distribution are replaced by some estimates. This introduces estimation risk, which is crucial for portfolio management just like traditional financial risk. This paper contrasts with existing literature by focusing on a different measure of performance. We borrow from practitioners and evaluate different funds allocations through their likelihood of beating a benchmark. Then, the optimal portfolio which accounts for estimation risk is known in closed-form and does not depend on any nuisance parameter. This investment rule corresponds to a mean-variance investor with a corrected, sample-dependent risk aversion parameter.

JEL Classification: C4, D8, G0.

Key words: Portfolio theory; Estimation risk; Benchmark performance; Mean-variance efficiency; Test.

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1 Introduction

An optimal portfolio is the *best* allocation of funds across available assets. Optimality depends on the selected performance measure. Markowitz (1959) offers the classic definition of portfolio efficiency: a portfolio is efficient if it has the largest expected return for a given target of risk measured by the variance. This mean-variance efficiency provides a single-period framework that still remains among the most important benchmark models used by practitioners (Michaud (1998); Meucci (2005)). In practice, however, Markowitz' associated optimal investment rule depends on unknown parameters, the mean and the variance of returns distribution. To get a feasible version of this optimal rule, he simply replaces the unknown parameters by sample estimates. This substitution gives rise to several issues. First, the estimation risk is overlooked: in practice samples are finite, hence estimates are different from their respective true (unknown) values. This new source of risk even appears in well-specified parametric models and adds to the traditional financial risk¹. Second, is this feasible rule optimal? Markowitz' approach can only be motivated when one believes that the estimated rule is *not too far* from the true optimal one.

In response to these limitations, we develop an alternate framework that relies on a more conservative definition of optimality. We borrow from practitioners and evaluate funds allocations through their likelihood of beating a chosen benchmark. Several industries are actually interested in such a goal: for instance, institutional money managers, and among others the defined benefits pension plans and the endowment plans are devoted to guarantee a (chosen) minimal performance. For the chosen benchmark, we deduce the associated optimal investment rule. It naturally incorporates the estimation risk and is directly applicable without requiring any additional (suboptimal) substitution step. More precisely, our portfolio selection method is based on a one-sided test ensuring that the portfolio performance is above a given threshold; then we obtain the optimal allocation from the maximization of the associated p-value. This investment rule is *optimal* in the sense that it is associated with the highest probability of defeating

¹Kan and Zhou (2007) provide an extensive study of the financial consequences of ignoring estimation risk.

the chosen benchmark. It also offers two main advantages. First, testing is the natural and valid statistical tool to compare random quantities (here estimated portfolio performances). Hence the uncertainty of the problem is directly accounted for: we will see that this is crucial to get a feasible (true) optimal investment rule that does not require any additional substitution step. Second, maximizing the p-value increases the likelihood of the event of interest (here to beat the chosen benchmark).

Our P-value selection method is very general and flexible, as it allows for any performance measure and any reference benchmark. When the performance is measured by the Markowitz' mean-variance criterion and the benchmark is chosen as a fixed target, the P-value investment rule (after ignoring the estimation risk of the variance²) belongs to the class of two-fund investment rules³. Two-fund rules invest in the sample tangency portfolio and in the riskless asset: only the share of wealth invested in the risky assets (vs in the riskless asset) varies among 2-fund rules, and not the repartition of wealth between risky assets which is controlled by the sample tangency portfolio. The (feasible) Markowitz' optimal mean-variance rule is the 2-fund rule where the share of wealth invested in the (sample) tangency portfolio is controlled by the risk-aversion parameter. Our optimal P-value investment rule can be reinterpreted as a (feasible) mean-variance optimal rule associated with a *corrected* risk-aversion parameter. While existing literature usually recommends increasing the risk-aversion parameter to account for estimation risk, our P-value investment rule is more flexible as it is sample-dependent: the corrected risk-aversion parameter tends to be higher in profitable financial environments, and lower in bad times.

The issue of estimation risk in portfolio allocation is not new⁴. One of the earliest and maybe natural solution proposed in the literature is Bayesian. Since the parameters are now random variables, it provides a general framework where estimation risk is

²When the number of assets is moderate compared to the number of observations (and the time-span is fixed) mean asset returns are harder to estimate: see Merton (1980) and Kan and Zhou (2007).

³This (restricted) class of investment rules has already been considered in the literature: see ter Horst, de Roon and Werker (2006) and Kan and Zhou (2007). However, here, it directly follows from our portfolio selection method and not from a simplifying assumption.

⁴Brandt (2004) provides a broad survey on general issues related to portfolio choice.

naturally accounted for. The posterior distribution captures the possible outcomes of the parameters and is combined to a prior model to derive the predictive distribution (Zellner and Chetty (1965)) under which expectations are now considered. The study by Bawa, Brown and Klein (1979) surveys the early literature; it has been followed by many others including Jorion (1986), Black and Litterman (1992), Pastor and Stambaugh (2000)... However, it is not clear how the prior model should be chosen, even though based on the investor's knowledge and experience: different priors may lead to very contrastive investment strategies. We only consider non-informative prior models in our comparative study.

Recently, interest has grown to develop procedures focusing directly on the expected financial loss when replacing the optimal investment rule by some feasible version of it. These approaches are intuitively very appealing as the emphasis is set on the financial cost of implementing infeasible optimal investment rules. However, simplifying assumptions are required to tackle the associated optimization problem. ter Horst, de Roon and Werker (2006) and Kan and Zhou (2007) restrict their attentions to the class of two-fund investment rules. While ter Horst *et al.* (2006) ignore the estimation risk of the variance, Kan and Zhou (2007) (under the normality assumption of the returns) provide a closed-form optimal investment rule. However, this rule depends on nuisance parameters. So, in order to implement it, an additional suboptimal plug-in step is required⁵. More generally, this issue arises when one maximizes some expected quantity: the associated optimal rule always depends on some of the (unknown) characteristics of the underlying distribution of the returns. This motivated us to depart from the traditional optimization of an expected financial loss function to rather maximize the likelihood of some desirable event⁶.

⁵Of course, by construction, the (infeasible) optimal mean-variance investment rule outperforms any two-fund rule, especially Kan and Zhou's; by construction, the latter also outperforms any P-value investment rule. However nothing is guaranteed when one considers feasible versions of the optimal mean-variance and Kan and Zhou's rules as shown in our comparative study.

⁶Others have also departed from the classical mean-variance framework: Garlappi, Uppal and Wang (2007) propose a sequential max-min method where the worst performance (when the unknown parameters fall into a confidence interval) is maximized with respect to the portfolio weights; Harvey, Liechty, Liechty and Muller (2004) adopt a Bayesian setting under the assumption that the returns

Finally, previous studies have already focused on defeating a benchmark: see Stutzer (2003) and references therein. However, to our knowledge, this has not yet been related to estimation risk. Moreover, these studies work in a continuous time framework which is not our primary interest here.

To conclude, we compare eleven investment strategies on simulated and empirical data. These are compared with respect to their out-of-sample performances, as measured by the Sharpe-ratio and the certainty equivalence, their turnover and stability over time. The P-value selection method performs surprisingly well considering it is not specifically designed to maximize any performance measure. Moreover, it avoids extreme positions in the assets and remains relatively stable over time.

The remainder of the paper is organized as follows. Section 2 solves the classical mean-variance problem. The P-value selection method is introduced in Section 3. Section 4 reviews some competing investment strategies. Section 5 presents the results of our comparative study on simulated and empirical data. Section 6 concludes.

Calculations details, tables and graphs are gathered in the Appendix.

2 Classical Mean-Variance problem

This section discusses the mean-variance problem and introduces estimation risk. Consider an investor who chooses a portfolio among N financial risky assets and the riskless asset. At time t , $R_t \equiv (r_{1t} \cdots r_{Nt})'$ and R_{ft} denote respectively the rates of returns on the N risky assets and the riskless asset. The vector of excess returns is defined as $\tilde{R}_t \equiv R_t - R_{ft}\iota$ where ι is the conformable vector of ones. The following standard assumption is maintained on the probability distribution of excess returns \tilde{R}_t :

Markowitz' maintained assumption:

The vector of excess returns \tilde{R}_t is independent and identically distributed over time. In addition, \tilde{R}_t is normally distributed with mean $\tilde{\mu}_0$ and variance Σ_0 .

follow a skew-normal distribution.

At time t , the portfolio is built after investing a vector θ into the risky assets and $(1 - \theta' \iota)$ in the riskless asset. The portfolio excess return is $r_t^P(\theta) \equiv \theta' \tilde{R}_t$, and its associated mean and variance are then respectively, $\mu_P = \theta' \tilde{\mu}_0$ and $\sigma_P^2 = \theta' \Sigma_0 \theta$.

Each vector of weights θ defines a different investment rule. Markowitz' optimal investment rule maximizes the following mean-variance objective function:

$$\max_{\theta \in \mathbb{R}^N} \left\{ E [r_t^P(\theta)] - \frac{\eta}{2} \text{Var} [r_t^P(\theta)] \right\} \iff \max_{\theta \in \mathbb{R}^N} \left\{ \theta' \tilde{\mu}_0 - \frac{\eta}{2} \theta' \Sigma_0 \theta \right\}$$

where η is the coefficient of relative risk-aversion. This leads to the following optimal vector of weights and maximal performance:

$$\theta_{MV}^0 = \frac{1}{\eta} \Sigma_0^{-1} \tilde{\mu}_0 \quad \text{and} \quad Q_{MV}^0 = \frac{1}{2\eta} \tilde{\mu}_0' \Sigma_0^{-1} \tilde{\mu}_0 \quad (2.1)$$

In practice, parameters $\tilde{\mu}_0$ and Σ_0 are unknown: the optimal mean-variance investment rule θ_{MV^0} is therefore *infeasible* and cannot be calculated. Markowitz (1959) simply replaces the unknown parameters by some estimates. This provides a convenient *feasible* version of the above optimal rule. More precisely, for some estimates $\hat{\mu}$ and $\hat{\Sigma}$ of the unknown parameters $\tilde{\mu}_0$ and Σ_0 , one defines the feasible (random) investment rule and its associated (random) performance as:

$$\theta_{MV} = \frac{1}{\eta} \hat{\Sigma}^{-1} \hat{\mu} \quad \text{and} \quad Q_{MV} = \frac{1}{2\eta} \hat{\mu}' \hat{\Sigma}^{-1} \hat{\mu} \quad (2.2)$$

where $\hat{\mu}$ and $\hat{\Sigma}$ are, for instance, the maximum likelihood estimators,

$$\hat{\mu} = \frac{1}{T} \sum_{t=1}^T \tilde{R}_t \quad \text{and} \quad \hat{\Sigma} = \frac{1}{T} \sum_{t=1}^T (\tilde{R}_t - \hat{\mu})(\tilde{R}_t - \hat{\mu})' \quad (2.3)$$

Applying this *plug-in* method comes at a price. First, estimation risk is overlooked. In practice, the sample size is only T (finite), hence $\hat{\mu}$ and $\hat{\Sigma}$ are different from their respective true values. Second, precisely because the feasible rule θ_{MV} is numerically different from the true optimal one, its optimality cannot be guaranteed. In the next section, we propose a portfolio selection method that incorporates estimation risk and does not require any additional (suboptimal) step.

3 P-value investment rule

This section introduces the P-value selection method and derives the associated optimal investment rule for a given benchmark c . The existence of an optimal benchmark is also discussed.

3.1 Definition and Optimal investment rule

As emphasized earlier, this paper departs from the classical mean-variance framework and the popular minimization of some (expected) financial risk function. More precisely, in sharp contrast with existing literature, we do not maximize (minimize) any usual portfolio performance measure (loss function). We rather compare available funds allocations through their likelihood of beating the chosen benchmark. Of course, our portfolio selection method crucially depends on this benchmark. *Reasonable* benchmark choices yield to more *conservative* objective functions than the classic maximization of the (mean-variance) performance. Our investor is more conservative in the sense that she is not interested in achieving the maximal performance at every period; she rather selects the investment rule that maximizes the likelihood of defeating the benchmark. By maximizing the p-value, this selection method directly accounts for the random nature of the problem while being of primary concern for several industries, like institutional money managers.

Our portfolio selection method is based on a one-sided test that the chosen measure of portfolio performance is above the given threshold. Obviously, two unknowns remain here: first the choice of the performance measure and second the threshold. As pointed out earlier, Markowitz' mean-variance efficiency is a convenient framework privileged by practitioners. Accordingly, we consider the following measure of portfolio performance:

$$Q(\mu_P, \sigma_P^2) = \mu_P - \frac{\eta}{2}\sigma_P^2 \tag{3.1}$$

where (μ_P, σ_P^2) are respectively the first two moments of the probability distribution of the portfolio. Our P-value selection method works with any other performance mea-

sure⁷: the above measure of performance (3.1) has mainly been chosen for comparison and tractability purposes. Not only the test is the natural statistical tool to compare random quantities and incorporate estimation risk, but it also focuses directly on the well-defined objective for a portfolio manager, to beat the performance of a benchmark index. Formally, the null hypothesis of interest is stated as:

$$H_0 : Q(\mu_P, \sigma_P^2) > c \quad (3.2)$$

where c is the (deterministic) performance of the (chosen) benchmark index. To construct the associated test statistic, some assumptions are needed on the probability distribution of the returns. Consider an investor at time T who has observed the N risky asset returns from time $t = 1$ to T .

ASSUMPTION 1 *The vectors of the N financial excess returns of interest at time t , $\tilde{R}_t = [\tilde{r}_{1t} \cdots \tilde{r}_{Nt}]'$ for $t=1$ to T , are stationary and Central Limit Theorem applies. More formally,*

(i) $\tilde{R}_t \sim \mathcal{F}(\tilde{\mu}_0, \Sigma_0)$ for any $t = 1, \dots, T$ where \mathcal{F} is some smooth distribution function whose first two moments exist.

(ii) $\sqrt{T} \sum_{t=1}^T \tilde{R}_t$ is asymptotically normally distributed with mean $\tilde{\mu}_0$ and variance Σ_0 .

We consider from now on the portfolio excess return $\tilde{r}_t^P(\theta) = \theta' \tilde{R}_t$. This only shifts the deterministic benchmark c , so that only strictly positive benchmarks c are now considered. Note that a null benchmark corresponds to the minimal acceptable performance, guaranteed when always investing in the riskless asset. The measure of portfolio performance is then written as:

$$Q_P(\theta) = E\tilde{r}_t^P(\theta) - \frac{\eta}{2} \text{Var}(\tilde{r}_t^P(\theta))$$

and is estimated by⁸:

⁷Any performance measure works under regularity assumption like Assumption 1. In particular, we could think of incorporating higher moments to account for effects of skewness, kurtosis... This only affects the tractability of the optimal investment rule. See also discussion p10.

⁸The procedure remains similar for any other set of consistent estimates. We could even think of the selection problem as starting right here, with a set of estimates given by a practitioner.

$$\hat{Q}_P(\theta) = \theta' \hat{\mu} - \frac{\eta}{2} \theta' \hat{\Sigma} \theta \quad (3.3)$$

$$\text{with } \hat{\mu} = \frac{1}{T} \sum_{t=1}^T \tilde{R}_t \text{ and } \hat{\Sigma} = \frac{1}{T} \sum_{t=1}^T (\tilde{R}_t - \hat{\mu})(\tilde{R}_t - \hat{\mu})' \quad (3.4)$$

The application of the vectorial central limit theorem yields the asymptotic distribution of the estimated performance: $\sqrt{T} [\hat{Q}_P(\theta) - Q_P(\theta)]$ is asymptotically normally distributed with mean 0 and variance $Var(\hat{Q}_P(\theta))$. Then, for an estimator \hat{S} of its standard deviation, the test statistic and associated p-value are defined as follows:

$$St(\theta) = \frac{\hat{Q}_P(\theta) - c}{\hat{S}/\sqrt{T}} \text{ and } \text{p-value}(\theta) = \int_{-\infty}^{St(\theta)} f_{\mathcal{T}}(u) du$$

with $f_{\mathcal{T}}$ the density function of a student random variable with $(T - 1)$ degrees of freedom. Hence the maximization problem is finally stated as:

$$\max_{\theta \in \mathbb{R}^N} [\text{p-value}(\theta)] \iff \max_{\theta \in \mathbb{R}^N} [St(\theta)]$$

The P-value selection method can be linked to the well-known financial risk measure, the Value-at-Risk (VaR hereafter). Briefly the VaR at level α represents an estimate of the level of loss on a portfolio which is expected to be equaled or exceeded with the given, small probability α : risk regulations usually dictates the choice of this level of confidence. Our selection method rather guarantees the chosen minimal level of performance with the highest level of confidence. We think that choosing the benchmark is more inline with institutional money managers concerns.

Obviously, estimation risk is related to the estimation of both the mean and the variance of the portfolio. It is commonly accepted that the estimation error on the sample mean is much larger than on the sample variance; however, recent studies suggest that it might not always be the case: see Cho (2007) and Kan and Zhou (2007). According to the latter study, the above claim is only acceptable when the ratio of the number of assets and the sample size (that is N/T) is *small*: in particular, there is an interactive effect between both estimation errors. Here, to simplify the problem and get an interpretable

(closed-form) investment rule, we ignore the estimation risk of the variance⁹. The simplified maximization problem is now:

$$\theta_p(c) = \arg \max_{\theta \in \mathbb{R}^N} \left[\frac{\theta' \hat{\mu} - \eta/2\theta' \hat{\Sigma} \theta - c}{(\theta' \hat{\Sigma} \theta)^{1/2} / \sqrt{T}} \right]$$

where $\hat{\mu}$ and $\hat{\Sigma}$ have been defined in equation (3.4).

DEFINITION 1 *Under Assumption 1, let $\hat{\mu}$ and $\hat{\Sigma}$ respectively be estimators of the first two moments of the distribution of the excess returns as in (3.4). Then, for a given (deterministic) benchmark c , the optimal P-value investment rule is defined as:*

$$\theta_p(c) = \sqrt{\frac{2\eta c}{\hat{\mu}' \hat{\Sigma}^{-1} \hat{\mu} \eta}} \frac{1}{\hat{\Sigma}^{-1} \hat{\mu}} \quad (3.5)$$

Several comments are worth mentioning.

First, the optimal P-value rule $\theta_p(c)$ is random and depends on the (chosen) estimates of the mean and variance of the excess returns distribution. However, this random rule (3.5) is the genuine rule that solves our optimization problem. In other words, it does not come from an additional (suboptimal) plug-in step (see also Section 4). The deep reason for this exactness lies in the definition of our P-value selection method: the randomness of the problem precisely defines our selection procedure. Without uncertainty, there would not be any purpose to run a test and therefore no p-value maximization.

Second, the rule (3.5) is a two-fund investment rule, just like the (feasible) mean-variance optimization problem θ_{MV} (see equation (2.2)): both rules yield to the same repartition of wealth among the different financial risky assets. This allows us to reinterpret the P-value investor in terms of mean-variance behavior with a corrected risk-aversion parameter in Section 4. This result obviously depends on the choice of the performance measure Q : it is unlikely to hold with a different Q .

Third, under Assumption 1, our selection method amounts to maximizing the asymptotic p-value of the one-sided test (3.2). Associated with the choice of Q , this leads to

⁹In our simulation and empirical studies in Section 5, the ratio N/T is kept small. In this sense, our methodology applies more to pension funds than mutual funds.

a closed-form investment rule. However, our methodology is much more general. In particular, one may want to relax Assumption 1 and maximize instead the stationary p-value, or even the bootstrap p-value (depending on how much one is ready to assume on the asset returns). This would only affect the tractability of the associated investment rule, and not the validity of our procedure.

Finally, note that the optimal P-value investment rule works for a given c . The next section naturally asks whether there exists an *optimal* benchmark.

3.2 An optimal choice for the benchmark?

The above selection method depends on the choice of the benchmark c : it represents the minimal level of portfolio performance the investor wants to guarantee with the highest possible level of confidence. As already discussed, this benchmark is not really a choice variable as it reflects the degree of conservatism of the financial institution. However, it is still helpful to exhibit the *optimal* benchmark for comparison purposes.

We define the optimal benchmark c^* as the maximizer of the expected performance of the portfolio:

$$c^* = \arg \max_{c \geq 0} E [Q_P(\theta_p(c))]$$

$$c^* = \frac{1}{2\eta} \times \frac{\left[E \left(\frac{\hat{\mu}' \hat{\Sigma}^{-1} \tilde{\mu}_0}{\sqrt{\hat{\gamma}^2}} \right) \right]^2}{\left[E \left(\frac{\hat{\mu}' \hat{\Sigma}^{-1} \Sigma_0 \hat{\Sigma}^{-1} \hat{\mu}}{\hat{\gamma}^2} \right) \right]^2} \quad \text{where } \hat{\gamma}^2 \equiv \hat{\mu}' \hat{\Sigma}^{-1} \hat{\mu} \quad (3.6)$$

The optimal benchmark c^* is clearly infeasible since it depends on the unknown parameters $\tilde{\mu}_0$ and Σ_0 ¹⁰. Interestingly enough, without estimation risk (or assuming $\tilde{\mu}_0$ and Σ_0 are known), we can check that the associated investment rule is numerically equal to the true mean-variance rule, which is also the optimal rule in absence of estimation risk. See also Section 4.2.

¹⁰This is not really surprising since we maximize the expected performance for a given c .

4 Theoretical comparison with existing literature

This section compares competing investment strategies after introducing the useful concept of *corrected* risk-aversion parameter, already considered in ter Horst *et al.* (2006).

4.1 Overview of some competing selection methods

First, we briefly introduce some competing investment rules. See also Appendix B.

- Mean-variance (Markowitz (1959)) (see Section 2): this rule selects the portfolio with the maximal mean-variance performance. The optimal allocation is infeasible: it depends on the first two unknown moments of the excess returns distribution. When some estimates (see equation (2.3)) of the unknowns are plugged into the formula, it becomes feasible and the estimation risk is ignored. This rule is given by:

$$\theta_{MV} = \frac{1}{\eta} \hat{\Sigma}^{-1} \hat{\mu}$$

- Bayesian (Bawa, Brown and Klein (1979)): the Bayesian approach maximizes the expected performance of the portfolio where the expectation is computed according to the predictive distribution of the market. In turn, this predictive distribution is built from a combination of historical observations and the prior. Estimation risk is made explicit by considering the unknown parameters as random variables, described by the posterior distribution. However, it is not always clear how the prior can be chosen. Under the standard assumption of diffuse priors on both the mean and the variance of the excess returns, it can be shown that the Bayesian optimal portfolio weights are:

$$\theta_B = \frac{1}{\eta} \left(\frac{T - N - 2}{T + 1} \right) \hat{\Sigma}^{-1} \hat{\mu}$$

- ter Horst, de Roon and Werker (2006): the portfolio weights are chosen to minimize the risk function based on the loss of replacing the true (unknown) mean of the portfolio

by its sample estimate. They restrict their attention to the class of two-fund rules and ignore the estimation risk of the variance:

$$\theta_{HRW}^0 = \frac{1}{\eta} \left(\frac{\gamma^2}{\gamma^2 + N/T} \right) \hat{\Sigma}^{-1} \hat{\mu} \quad \text{with} \quad \gamma^2 = \tilde{\mu}'_0 \Sigma_0^{-1} \tilde{\mu}_0$$

The resulting optimal rule θ_{HRW}^0 is infeasible: γ^2 is then replaced by its sample counterpart $\hat{\gamma}^2 = \hat{\mu}' \hat{\Sigma}^{-1} \hat{\mu}$. Optimality of θ_{HRW} is not guaranteed anymore.

- Kan and Zhou (2007) extend the previous selection method to incorporate the estimation risk of the variance:

$$\theta_{KZ}^0 = \frac{1}{\eta} \left(\frac{(T - N - 4)(T - N - 1)}{T(T - 2)} \times \frac{\gamma^2}{\gamma^2 + N/T} \right) \hat{\Sigma}^{-1} \hat{\mu}$$

Just like θ_{HRW}^0 , the resulting optimal rule θ_{KZ}^0 is infeasible: see Appendix B for its feasible version θ_{KZ} . They also explore the class of three-fund investment rules when considering in addition the sample global mean-variance portfolio. The associated optimal rule θ_{KZ3}^0 is infeasible as well: see also Appendix B for additional details.

- Garlappi, Uppal and Wang (2006) consider a model that allows for multi priors and where the investor is averse to ambiguity. The standard mean-variance framework is modified by adding a preliminary minimization step. A constraint restricts the expected return to fall into a confidence interval around its estimated value and recognizes the existence of estimation risk. The minimization over the possible expected returns subject to this constraint reflects the investor's aversion to ambiguity. While this approach has a solid axiomatic foundation, its sequentiality cannot be directly linked to an optimality criterion. The optimal rule θ_{GUW} is not exactly a two-fund rule and is precisely defined in Appendix B.

The following theoretical rankings have been derived by Kan and Zhou (2007):

$$MV^0 \gg KZ3^0 \gg KZ^0 \gg B \gg MV, \quad KZ^0 \gg HRW^0 \quad \text{and} \quad KZ^0 \gg GUW$$

where \gg stands for "outperforms in terms of mean-variance performance". We argue that this ranking might not be guaranteed in practice (even in simple simulation frameworks where the returns are normally distributed) when strategies HRW^0 , KZ^0

and KZ^0 are replaced by their feasible counterparts. Kan and Zhou (2007) already mentioned this issue when comparing their (feasible) optimal two-fund rule to the one of Garlappi *et al.*. See also Section 5.

4.2 Comparison of the reinterpreted investment rules

Despite their differences, most of the selection methods described above yield an optimal rule within the class of two-fund rules, just like the (feasible) Markowitz' mean-variance approach¹¹. Accordingly, the same repartition of wealth among the different risky financial assets is recommended: only the shares of wealth invested in risky assets relative to the riskless asset are different. The (feasible) mean-variance rule can be reinterpreted as a function of the risk-aversion parameter η :

$$\theta_{MV}(\eta) = \frac{1}{\eta} \hat{\Sigma}^{-1} \hat{\mu}$$

$\left[\hat{\Sigma}^{-1} \hat{\mu} \right]$ defines how wealth is allocated among risky assets while η weights the share of wealth assigned to the risky assets: the greater η , the lower the (global) share to the risky assets. Each two-fund rule can then be written as a mean-variance rule with a *corrected* risk aversion parameter. In fact, any two-fund rule vector of weights θ_r can be rewritten as follows:

$$\theta_r = \theta_{MV}(\tilde{\eta}) \quad \text{for some } \tilde{\eta} > 0 \quad (4.1)$$

Therefore, the behavior of any two-fund investor can be characterized in terms of a mean-variance associated to a new (corrected) risk-aversion parameter $\tilde{\eta}$. The following corrected risk-aversion parameters can be deduced for the two-fund rules discussed

¹¹This is especially surprising for our P-value selection method since it does not come from any simplifying assumption (as for θ_{KZ}^0 and θ_{HRW}^0).

above¹²:

$$\begin{aligned}
\tilde{\eta}_{HRW} &= \eta \times \frac{\gamma^2 + N/T}{\gamma^2} \\
\tilde{\eta}_{KZ} &= \eta \times \frac{\gamma^2 + N/T}{\gamma^2} \times \frac{T(T-2)}{(T-N-4)(T-N-1)} \\
\tilde{\eta}_B &= \eta \times \frac{T+1}{T-N-2} \\
\tilde{\eta}_p(c) &= \eta \times \sqrt{\frac{Q_{MV}}{c}}
\end{aligned}$$

where Q_{MV} is the performance associated to the feasible mean-variance investment rule (see equation (2.2)).

It is easy to see that $\tilde{\eta}_B$, $\tilde{\eta}_{HRW}$, and $\tilde{\eta}_{KZ}$ are always larger than η ,

$$\tilde{\eta}_{KZ} > \tilde{\eta}_{HRW} > \eta \quad \text{and} \quad \tilde{\eta}_B > \eta$$

Hence, the investors respectively associated with the three competing rules θ_B , θ_{HRW} and θ_{KZ} are always more risk-averse than the mean-variance investor. Recall now that θ_{KZ} is nothing but θ_{HRW} where the additional estimation risk coming from the variance is accounted for. So one could be tempted to conclude that increasing the risk-aversion parameter is a sensible way to account for estimation risk.

On the other hand, the P-value corrected risk-aversion linearly depends on the original risk-aversion parameter: hence, the P-value investor might be characterized as a mean-variance investor either by increasing or decreasing the risk-aversion η . Depending on the choice of the benchmark c , one falls into one of the following cases:

- (i) if $c = Q_{MV}$ then $\tilde{\eta}_p = \eta$
- (ii) if $c > Q_{MV}$ then $\tilde{\eta}_p < \eta$
- (iii) if $c < Q_{MV}$ then $\tilde{\eta}_p > \eta$

Intuitively, this additional flexibility might be profitable, especially because it can be linked to the actual sample realizations. Consider an investor who chooses a moderate

¹²We could also consider θ_{GUW} as a two-fund rule with a corrected risk-aversion parameter that can be infinite with non-zero probability.

benchmark c . Assume now that, *by chance*, she faces a profitable financial environment (or a sample associated to a relatively high performance): likely $c < Q_{MV}$ and so $\tilde{\eta}_p > \eta$. Overall, the part invested in the risky assets is going to be lower. The profitable financial conditions offer additional safety to the P-value investor: it is more likely to beat the target. On the contrary, with a *not so good* financial environment, one may expect the investor to become less risk-averse, still hoping to defeat the benchmark. Intuitively, it makes sense to incorporate the sample-information into the decision process. The P-value selection method might also overcome the well-known problem of the mean-variance investment rule which takes extreme positions. The next section further investigates this.

5 Comparative study

5.1 Comparison procedure

Our analysis relies on a *rolling-window* approach. Specifically, for a given T -month long dataset of asset returns, we choose an estimation window of length T_w months. In each month t , starting from $t = T_w$, we use the data in the previous T_w months to estimate the parameters needed to implement a particular strategy. These estimated parameters are then used to determine the relative portfolio weights in the portfolio of only-risky assets. We then use these weights to compute the return in month $t + 1$. This process is continued by adding the return for the next period in the dataset and dropping the earliest return, until the end of the dataset is reached. The outcome of the rolling-window approach is a series of $(T - T_w)$ monthly *out-of-sample* returns for each portfolio strategy k denoted $\hat{r}_{k,t}$ for $t = T_w + 1, \dots, T$.

Given the series of monthly out-of-sample returns, we compare the portfolio strategies by computing the following quantities.

(1) The out-of-sample Sharpe-ratio (SR hereafter). For a strategy k , its out-of-sample SR, denoted \widehat{SR}_k , is defined as the ratio of the mean of (out-of-sample) returns

(over the risk-free asset), $\hat{\mu}_k$, and the standard deviation of the (out-of-sample) returns, $\hat{\sigma}_k$:

$$\widehat{SR}_k = \frac{\hat{\mu}_k}{\hat{\sigma}_k}$$

where $\hat{\mu}_k = \frac{1}{T - T_w} \sum_{t=T_w+1}^T \hat{r}_{k,t}$ and $\hat{\sigma}_k = \frac{1}{T - T_w} \sum_{t=T_w}^T \left(\hat{r}_{k,t} - \hat{\mu}_k \right) \left(\hat{r}_{k,t} - \hat{\mu}_k \right)'$

(2) The out-of-sample certainty-equivalent return (CE hereafter). For a strategy k , its out-of-sample CE, denoted \widehat{CE}_k is defined as the risk-free rate that an investor is willing to accept rather than adopting a particular risky portfolio strategy. Following the common practice, we calculate CE as the level of expected utility of a mean-variance investor, that is:

$$\widehat{CE}_k = \hat{\mu}_k - \frac{\gamma}{2} \hat{\sigma}_k^2$$

It can be shown that this corresponds to the CE of an investor with quadratic utility.

(3) The portfolio turnover. For a strategy k , its portfolio turnover is defined as the average sum of the absolute value of the trades across the N available assets. It provides information about the stability of a specific portfolio strategy, as it measures the transaction costs incurred to reallocate the portfolio at each period. Here, we arbitrarily assume that the cost is the same for each risky asset.

$$\text{Turnover}_k = \frac{1}{T - T_w - 1} \sum_{t=T_w+1}^{T-1} (|\theta_{k,t+1} - \theta_{k,t}|)' \iota$$

where ι is the column vector of ones of size N .

The results discussed in the following subsections and presented in the Appendix have been obtained with a rolling window of size $T_w = 120$: this corresponds to 10 years of monthly data. We have also considered rolling window of sizes 60 and 180 (respectively 5 and 15 years of monthly data): the conclusions were very similar. The risk-aversion parameter η is set equal to 1, though we also considered other values for robustness.

All the portfolio rules considered in our simulated and empirical studies are listed in Table 1. They have been introduced in Section 4. We also consider three P-value rules $P1$, $P2$ and $P3$ with respective benchmarks c_1 , c_2 , and c_3 . For the simulated datasets,

these benchmarks are selected with respect to the maximal mean-variance performance Q_{MV}^0 (see equation (2.1)): $c_1 = .1Q_{MV}^0$, $c_2 = .5Q_{MV}^0$ and $c_3 = .9Q_{MV}^0$. For the empirical datasets, Q_{MV}^0 is unknown. More generally, in practice, one can think of at least two ways to get a convenient benchmark: c might be a numerical target that has been set by the board of directors; c can also be based on the historical performance of some reference index. For the empirical datasets, we select targets based on historical performance.

5.2 Monte-Carlo results

In this section, we first describe our simulation experiments and then discuss the performance and stability of the different portfolio strategies on simulated data.

5.2.1 Simulated dataset

We generate asset returns according to a distribution F that deviates slightly from the normal distribution. More precisely, F is a mixture of a joint normal distribution N , and a *deviation* distribution D . The parameter h captures the proportion of the data that deviates from the normal,

$$F = (1 - h)N + hD$$

We consider here two different procedures to generate the part of the data that follows the joint normal distribution N , three different proportions of the data that deviates from N , and four different deviation distributions D . This provides us with 24 simulated datasets.

We consider two procedures to generate the part of the data that follows the joint normal distribution. First, we generate five risky assets and the riskless asset according to a multivariate normal distribution calibrated from monthly unhedged returns of stock indices for the G5 countries over the period January 1974 to December 1998. The G5 stock indices are the MSCI indices for France, Germany, Japan, the UK and the US¹³.

¹³See for instance ter Horst, de Roon and Werker (2006).

See Table 2 in Appendix.

Second, we generate a factor model¹⁴ with 4 risky assets including one factor, and one riskless asset. The excess returns of the factor $R_{f,t}$ follow a normal distribution and the excess returns of the remaining risky assets $R_{r,t}$ are generated according to:

$$R_{r,t} = \alpha + BR_{f,t} + e_t$$

α is the vector of mispricing coefficients: it is set to 0. B is the matrix of factor loadings: its coefficients are drawn from a uniform distribution between 0.5 and 1.5. The excess returns of the factor follow a normal distribution with mean 8% and standard deviation 16%. The error process e_t (uncorrelated with $R_{f,t}$) follows a multivariate normal distribution with mean 0 and diagonal covariance matrix with coefficients that are drawn from a uniform distribution between 0.15 and 0.25. These parameters are set according to the simulated experiment run by DeMiguel and Nogales (2008).

We consider three different proportions of the data that deviates from the joint normal distribution: $h = 0$, no deviation from the joint normal distribution; $h = 0.025$, and $h = 0.05$, respectively 2.5% and 5% of the data deviate from the joint normal distribution¹⁵.

To conclude the description of the simulated datasets, we consider four different deviations D from the joint normal distribution. Here again, we follow the set-up of DeMiguel and Nogales (2008).

(i) D_1 is deterministic, equal to the expected return of the asset plus five standard deviations;

(ii) D_2 is deterministic, equal to the expected return of the asset plus three standard deviations;

(iii) D_3 is binomial, equal to the expected return of the asset plus five standard deviations with probability 0.5 and to the expected return of the asset minus five standard deviations with probability 0.5;

(iv) D_4 is a normal distribution with mean equal to the expected return of the asset

¹⁴The factor model is similar to the one used in MacKinlay and Pastor (2000).

¹⁵Das and Uppal (2004) calibrate a jump diffusion process to historical returns on the indexes for six countries. They found that on average a jump occurs every 20 months. This corresponds to 5% of the data that deviate from the joint normal distribution.

plus five standard deviations and same covariance matrix as N .

In the following subsection, we only discuss the results obtained with the second dataset (generated from the factor model and its deviations). Results for the other dataset are very similar to the ones presented here.

5.2.2 Comparison of the strategies

We now discuss the performance and the stability of the different portfolio rules listed in Table 1. First we discuss the P-value rules, then the feasible rules, and finally the cost of feasibility.

P-value rules

Table 3 reports the out-of-sample mean, variance, SR, CE and turnover for the three P-value rules P1, P2 and P3 with respective benchmarks $c_1 = 0.1Q_{MV}^0$, $c_2 = 0.5Q_{MV}^0$, and $c_3 = 0.9Q_{MV}^0$ ¹⁶. Numerical values of these targets are provided, as well as the out-of-sample probability of providing a corrected risk-aversion parameter larger than 1, $P(\tilde{\eta} > \eta)$.

Targets c_1 and c_2 are always well defeated by $P1$ and $P2$ respectively and the associated corrected risk-aversion parameters are almost always larger than η . This means that P-value investors 1 and 2 can be reinterpreted as *more conservative* MV-investors: more conservative in the sense that what is reinterpreted as a corrected risk-aversion parameter is larger than the actual risk-aversion parameter.

Target c_3 is defeated most of the time but not as easily: in particular, without deviation, investor $P3$ does not defeat c_3 despite a highly aggressive investment strategy: the corrected risk-aversion parameter is smaller than η with out-of-sample probability 1/3. In addition, it is informative to compare investors $P2$ and $P3$: their CE performances are really close to each other, however $P3$ always invests a larger share in the risky assets (as can be seen from the probability of corrected risk-aversion parameter being larger than η). This supports the intuition that setting a target *too high* can be detrimental (see also the results for the empirical dataset with sector portfolios).

¹⁶ Q_{MV}^0 is the maximum mean-variance performance. See equation (2.1).

Figure 1 displays the out-of-sample CE for the above three P-value rules as a function of the size of the rolling window when the deviation is D_3 . $P1$ and $P2$ provide very stable CE performances, well-above their respective targets. $P3$ requires a rolling window of 150 monthly data to defeat its benchmark.

In terms of stability and affordability as measured by the turnover, the P-value rule associated with the lowest target ($P1$) is the most stable and affordable.

Feasible rules

Table 4 reports the out-of-sample mean, variance, SR, CE and turnover for 8 feasible portfolio rules. Irrespective of the deviation considered, the following ranking can be observed in terms of SR and CE performances:

$$P2, KZ3 \gg B, KZ, HRW \gg MV \gg G UW \gg EQ$$

where \gg stands for "outperforms in terms of SR and CE".

Investors $P2$ and $KZ3$ perform comparably and outperform investors B , HRW and KZ who perform similarly. Then follow investors MV , $G UW$ and EQ . This is reasonably consistent with the partial theoretical ranking¹⁷ provided page 13. The major difference involves the performance of KZ : KZ^0 is expected to outperform B , HRW^0 and any P-value rule. In our simulations, $P2$ outperforms KZ , that performs comparably to B and HRW .

When the size of the rolling window increases, the observed ranking should get closer to the theoretical one. The simulated samples can be separated into two groups: group 1 generated with deviation distributions D_1 , D_2 , and D_4 ; group 2 generated without deviation and with deviation distribution D_3 . The following rankings are observed:

¹⁷This partial ranking is provided by Kan and Zhou (2007) in terms of expected performances under the assumption that the returns are normally distributed for the rules B , $G UW$, HRW^0 , KZ^0 , $KZ3^0$ and MV .

	Group 1 (D_1, D_2, D_4)	Group 2 (no dev., D_3)
Small rolling window sizes CE	$P2 \gg KZ3 \gg KZ$	$KZ3 \gg P2 \gg KZ$
SR	$P2 \gg KZ3, KZ$	$KZ3, P2 \gg KZ$
Large rolling window sizes CE	$KZ3, KZ \gg P2$	$KZ3 \gg KZ \gg P2$
SR	$P2 \gg KZ3, KZ$	$KZ3, P2 \gg KZ$

To conclude, the performance of $P2$ is very satisfactory especially with smaller rolling window sizes. Figures 2 and 3 display the out-of-sample SR for the above feasible rules as a function of the size of the rolling window when the deviations are respectively D_1 and D_3 .

Infeasible rules

Table 5 reports the out-of-sample mean, variance, SR, CE and turnover for the 4 infeasible portfolio rules and their feasible counterparts. Tables 6 to 9 report the out-of-sample SR and CE as a function of the size of the rolling window without deviation and with deviation D_1 , as well as percentage losses from using a specific rule instead of the optimal MV^0 . The feasibility cost decreases with the sample size and the 4 feasible rules perform similarly when the size of the rolling window is larger than 360 (30 years of monthly data). In addition, comparing rules HRW and KZ confirm previous findings that the variance is easier to accurately estimate than the mean: even for small sample sizes (still with N/T small), the cost of ignoring the estimation risk of the variance is low.

5.3 Empirical results

In this section, we first describe the empirical datasets and then discuss the performance and stability of the different portfolio strategies on empirical data.

5.3.1 Empirical datasets

We consider two empirical datasets. Both contain excess monthly returns over the 90-day T-Bill.

The first dataset consists of monthly excess returns on ten industry portfolios in the United States. The ten industries considered are: Consumer-Discretionary (non-durable), Consumer-Staples (durable), Manufacturing, Energy, High-Tech, Telecommunications, Wholesale and Retail, Health, Utilities and Other. The dataset spans from January 1981 to July 2008 and is available on Kenneth French's web site.

The second dataset consists of monthly excess returns on ten value-weighted industry portfolios formed by using the Global Industry Classification Standard (GICS). The dataset has been created by Roberto Wessels. It has been used in several empirical studies (including DeMiguel and Nogales (2008)) and is available on DeMiguel's web site. The ten industries considered are: Energy, Material, Industrials, Consumer-Discretionary, Consumer-Staples, Healthcare, Financials, Information-Technology, Telecommunications and Utilities. It has been augmented by adding as a factor the excess return on the US equity market portfolio, defined as the value-weighted return on all NYSE, AMEX, and NASDAQ stocks minus the one-month Treasury bill rate. The dataset spans from January 1981 to December 2002.

5.3.2 Comparison of the strategies

P-value rules

Except for $P1$, the two other P-value rules $P2$ and $P3$ cannot beat their target despite an aggressive investment strategy: for dataset Industry, the corrected risk-aversion parameter of $P3$ is smaller than η with out-of-sample probability 0.13, and it goes down to 0.03 for dataset Sector. Both datasets tend to favor the conservative P-value rule $P1$, which provides the most stable and reliable performance while being the cheapest. See also figure 4.

We now discuss the performance and the stability of the different portfolio rules listed

in Table 1: Table 10 reports the out-of-sample mean, variance, SR, CE and turnover for these rules.

Industry portfolios

The following ranking can be observed in terms of SR and CE performances:

$$KZ3, P1, P2, P3 \gg B, MV \gg HRW, KZ \gg G UW \gg EQ$$

where \gg stands for "outperforms in terms of SR". See also figure 5.

$$P1 \gg KZ3, G UW \gg KZ \gg P2 \gg B \gg HRW \gg MV \gg EQ \gg P3$$

where \gg stands for "outperforms in terms of CE".

Here again the P-value rules perform very well. In terms of CE, the most conservative of the three rules $P1$ outperforms every other rule while also being the overall second cheapest one, after the equi-weighted portfolio rule which performs poorly.

Sector portfolios

The following ranking can be observed in terms of SR and CE performances:

$$EQ \gg P1, P2, P3 \gg B, HRW, KZ3, MV \gg KZ \gg G UW$$

where \gg stands for "outperforms in terms of SR". See also figure 6.

$$EQ \gg P1 \gg G UW \gg KZ, KZ3 \gg HRW \gg P2 \gg B \gg MV \gg P3$$

where \gg stands for "outperforms in terms of CE".

The main difference with the previous empirical dataset is the good performance of the equi-weighted portfolio rule EQ (also the cheapest rule). $P1$ performs very well too.

6 Conclusion

In this paper we propose a new way to account for estimation risk when selecting the optimal portfolio. In sharp contrast with existing literature, the optimal portfolio is not

defined as the one maximizing some expected mean-variance performance: we consider here a more conservative definition of optimality which focuses on guaranteeing some minimal performance. More precisely, our portfolio selection method is based on a one-sided test ensuring that the portfolio performance is above a given threshold. The optimal weights are then obtained from the maximization of the associated p-value. The test provides an integrated method to account for estimation risk. Moreover, after neglecting the estimation risk of the sample variance, it leads to a closed-form investment rule which can be used without requiring any additional (suboptimal) step.

Of course the performance of the P-value investment rule (which is not designed or meant to achieve the maximal performance) depends on the chosen benchmark c . However, as illustrated in our comparative study where we consider a wide range of benchmarks, the overall performance is quite satisfactory. In particular, it performs pretty well for relatively small samples (we believe mainly because it does not require an additional suboptimal plug-in step) and outperforms reasonable choices of targets. We find these preliminary results really encouraging.

The great advantage of the simple framework we consider here consists in providing closed-form optimal investment rules, interpretable in terms of mean-variance behavior. Compared to competing two-fund rules (e.g. Kan and Zhou (2007) and ter Horst, de Roon and Werker (2006)), we have shown that this is not always an increase of the original risk-aversion parameter that works to account for estimation risk.

For future research, several directions might be worth examining. First, one could extend our selection method to random targets. This would permit to track the performance of benchmark indices, rather than deterministic targets that may not always be inline with the financial environment. Second, considering that we generally do better than the feasible optimal two-fund rule and the very good results of the feasible three-fund rule, we may wonder how the P-value selection method, adapted to consider three-fund investment rules, would perform: as suggested by Kan and Zhou (2007), even more than three assets may help. Finally, recent papers have considered the related issue of model uncertainty. In particular, Cavadini, Sbuelz and Trojani (2001) extend the study of ter Horst, de Roon and Werker (2006) to incorporate model risk:

they use robust inference methods *à la Huber*, or local deviations to the chosen initial distribution. Of course, the interpretability of the investment rules is likely the price to pay to consider such extensions.

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A Proofs of the main results

- Proof of equation (3.5) $\theta_p(c)$:

The first order conditions can be reinterpreted as a function of the (feasible) vector of the mean-variance weights θ_{MV} defined in equation (2.2) as follows:

$$\begin{aligned}
 (\hat{\mu} - \eta \hat{\Sigma} \theta_p) \sqrt{\theta_p' \hat{\Sigma} \theta_p} - \frac{\theta_p' \hat{\mu} - \eta/2 \theta_p' \hat{\Sigma} \theta_p - c}{\sqrt{\theta_p' \hat{\Sigma} \theta_p}} \hat{\Sigma} \theta_p = 0 &\Leftrightarrow \hat{\mu} - \eta \hat{\Sigma} \theta_p - \frac{\theta_p' \hat{\mu} - c}{\theta_p' \hat{\Sigma} \theta_p} \hat{\Sigma} \theta_p + \frac{\eta}{2} \hat{\Sigma} \theta_p = 0 \\
 &\Leftrightarrow \hat{\mu} - \frac{\eta}{2} \hat{\Sigma} \theta_p - \frac{\theta_p' \hat{\mu} - c}{\theta_p' \hat{\Sigma} \theta_p} \hat{\Sigma} \theta_p = 0 \\
 &\Leftrightarrow \hat{\Sigma}^{-1} \hat{\mu} - \frac{\eta}{2} \theta_p - \frac{\theta_p' \hat{\mu} - c}{\theta_p' \hat{\Sigma} \theta_p} \theta_p = 0 \\
 &\Leftrightarrow \eta \times \theta_{MV} = \left[\frac{\theta_p' \hat{\mu} - c}{\theta_p' \hat{\Sigma} \theta_p} + \frac{\eta}{2} \right] \theta_p \quad (\text{A.1})
 \end{aligned}$$

Now for a given threshold c , we can always define a constant real number k_c such that:

$$k_c \times \eta = \frac{\theta_p' \hat{\mu} - c}{\theta_p' \hat{\Sigma} \theta_p} \quad (\text{A.2})$$

Then, substituting (A.2) into (A.1) yields to:

$$\eta \times \theta_{MV} = \left(k_c + \frac{1}{2} \right) \times \eta \times \theta_p \Leftrightarrow \theta_p = \frac{1}{k_c + 1/2} \frac{1}{\eta} \hat{\Sigma}^{-1} \hat{\mu} \quad (\text{A.3})$$

If I consider θ_{MV} as a function of η such that $\theta_{MV} = (\hat{\Sigma}^{-1} \tilde{R})/\eta$ and θ_p as a function of η (where θ_p is the weighting vector maximizing the p-value of the test with a parameter η of risk-aversion), then I get:

$$\theta_p(\eta) = \theta_{MV}(\tilde{\eta}) \quad \text{with} \quad \tilde{\eta} = \eta \times \left(k_c + \frac{1}{2} \right) \quad (\text{A.4})$$

The interpretation of $\tilde{\eta}$ as a *corrected* parameter of risk aversion is valid if and only if $k_c + 1/2 > 0$. This result may appear a bit *ad hoc* at first because k_c depends on the

unknown vector of weights θ . But from equation (A.2), we are actually able to deduce its explicit expression as a function of known quantities only:

$$(A.2) \iff \theta'_p \hat{\mu} - c = \theta'_p \hat{\Sigma} \theta_p \times k_c \times \eta$$

Then after replacing θ_p by its expression (A.3), we get:

$$\begin{aligned} \frac{1}{k_c + 1/2} \frac{1}{\eta} \hat{\mu}' \hat{\Sigma}^{-1} \hat{\mu} - c &= \frac{k_c}{(k_c + 1/2)^2} \frac{1}{\eta} \hat{\mu}' \hat{\Sigma}^{-1} \hat{\mu} \Rightarrow c = \frac{\hat{\mu}' \hat{\Sigma}^{-1} \hat{\mu}}{2\eta(k_c + 1/2)^2} \\ &\Rightarrow (k_c + 1/2) = \sqrt{\frac{\hat{\mu}' \hat{\Sigma}^{-1} \hat{\mu}}{2\eta}} \times \frac{1}{\sqrt{c}} \end{aligned}$$

- Proof of equation (3.6) c^* :

$$Q_P(\theta_p(c)) = \theta'_p(c) \tilde{\mu}_0 - \frac{\eta}{2} \theta'_p(c) \Sigma_0 \theta_p(c) = \frac{\sqrt{2c}}{\sqrt{\eta \hat{\gamma}^2}} \hat{\mu}' \hat{\Sigma}^{-1} \tilde{\mu}_0 - \frac{c}{\hat{\gamma}^2} \hat{\mu}' \hat{\Sigma}^{-1} \Sigma_0 \hat{\Sigma}^{-1} \hat{\mu}$$

where $\hat{\gamma}^2 = \hat{\mu}' \hat{\Sigma}^{-1} \hat{\mu}$. We then maximize it with respect to c : $\max_{c \geq 0} E[Q_P(\theta_p(c))]$.

The associated first order conditions are:

$$\frac{1}{\sqrt{2c^* \eta}} E \left(\frac{\hat{\mu}' \hat{\Sigma}^{-1} \tilde{\mu}_0}{\sqrt{\hat{\gamma}^2}} \right) = E \left(\frac{\hat{\mu}' \hat{\Sigma}^{-1} \Sigma_0 \hat{\Sigma}^{-1} \hat{\mu}}{\hat{\gamma}^2} \right) \implies c^* = \frac{1}{2\eta} \times \frac{\left[E \left(\frac{\hat{\mu}' \hat{\Sigma}^{-1} \tilde{\mu}_0}{\sqrt{\hat{\gamma}^2}} \right) \right]^2}{\left[E \left(\frac{\hat{\mu}' \hat{\Sigma}^{-1} \Sigma_0 \hat{\Sigma}^{-1} \hat{\mu}}{\hat{\gamma}^2} \right) \right]^2}$$

The associated optimal vector of weights is the following:

$$\theta_p(c^*) = \frac{\left| E \left(\frac{\hat{\mu}' \hat{\Sigma}^{-1} \tilde{\mu}_0}{\sqrt{\hat{\gamma}^2}} \right) \right|}{\left| E \left(\frac{\hat{\mu}' \hat{\Sigma}^{-1} \Sigma_0 \hat{\Sigma}^{-1} \hat{\mu}}{\hat{\gamma}^2} \right) \right|} \frac{1}{\sqrt{\hat{\gamma}^2} \eta} \hat{\Sigma}^{-1} \hat{\mu}$$

Note in particular that if $\tilde{\mu}_0$ and Σ_0 were known, we would get $\theta_p(c^*) = \theta_{MV}$, which corresponds to the best portfolio rule in absence of estimation risk.

B Results of the simulation and empirical studies

We now provide additional description of the investment rules considered in this paper.

- Two-fund rule of Kan and Zhou:

$$\theta_{KZ}^0 = \frac{1}{\eta} \left[\left(\frac{(T-N-1)(T-N-4)}{T(T-2)} \right) \left(\frac{\gamma^2}{\gamma^2 + N/T} \right) \right] \hat{\Sigma}^{-1} \hat{\mu} \quad \text{with } \gamma^2 = \tilde{\mu}' \Sigma^{-1} \tilde{\mu}$$

Kan and Zhou (2007) recommend the following feasible rule θ_{KZ} where γ^2 is replaced by

$$\hat{\gamma}_a^2 = \frac{(T-N-2)\hat{\gamma}^2 - N}{T} + \frac{2(\hat{\gamma}^2)^{N/2}(1+\hat{\gamma}^2)^{-(T-2)/2}}{TB_{\hat{\gamma}^2/(1+\hat{\gamma}^2)}(N/2, (T-N)/2)}$$

with $\hat{\gamma}^2 = \hat{\mu}' \hat{\Sigma}^{-1} \hat{\mu}$ and $B_x(a, b)$ is the incomplete beta function

$$B_x(a, b) = \int_0^x y^{a-1}(1-y)^{b-1} dy$$

- Three-fund rule of Kan and Zhou:

$$\theta_{KZ3}^0 = \frac{c_3}{\eta} \left[\left(\frac{\psi^2}{\psi^2 + N/T} \right) \hat{\Sigma}^{-1} \hat{\mu} + \left(\frac{N/T}{\psi^2 + N/T} \right) \mu_g \hat{\Sigma}^{-1} \iota \right]$$

with $\mu_g = \frac{\iota' \Sigma^{-1} \tilde{\mu}}{\iota' \Sigma^{-1} \iota}$, $c_3 = \left(\frac{T-N-4}{T} \right) \left(\frac{T-N-1}{T-2} \right)$, $\psi^2 = (\mu - \mu_g \iota)' \Sigma^{-1} (\mu - \mu_g \iota)$

Kan and Zhou (2007) recommend the following feasible rule θ_{KZ3} where μ_g and ψ^2 are respectively replaced by:

$$\begin{aligned} \hat{\mu}_g &= \frac{\hat{\mu}' \hat{\Sigma}^{-1} \iota}{\iota' \hat{\Sigma}^{-1} \iota} \\ \hat{\psi}_a^2 &= \frac{(T-N-1)\hat{\psi}^2 - (N-1)}{T} + \frac{2(\hat{\psi}^2)^{(N-1)/2}(1+\hat{\psi}^2)^{-(T-2)/2}}{TB_{\hat{\psi}^2/(1+\hat{\psi}^2)}((N-1)/2, (T-N+1)/2)} \end{aligned}$$

- Sequential min-max of Garlappi, Uppal and Wang:

$$\theta_{GUW} = \frac{1}{\eta} d \frac{T-1}{T} \hat{\Sigma}^{-1} \hat{\mu}$$

$$\text{where } d = \begin{cases} 1 - (\epsilon/\hat{\gamma}^2)^{1/2} & \text{if } \hat{\gamma}^2 > \epsilon \\ 0 & \text{if } \hat{\gamma}^2 \leq \epsilon \end{cases} \quad \text{with } \epsilon = N\mathcal{F}_{N, T-N}^{-1}(p)/(T-N)$$

where $\mathcal{F}_{N, T-N}^{-1}$ is the inverse of the cumulative distribution function of a central F-distribution with $(N, T-N)$ degrees of freedom and p is a probability. We use $p = .99$ as suggested in Garlappi *et al.* (2007).

Infeasible rules (only for simulated data)	
MV^0	(infeasible) Mean-variance
KZ^0	Kan and Zhou (infeasible) 2-fund
HRW^0	ter Horst, de Roon and Werker (infeasible) 2-fund
$KZ3^0$	Kan and Zhou (infeasible) 3-fund
Feasible rules (both for simulated and empirical data)	
P1	P-value with target 10% Optimum (with simulated datasets) P-value with target 10% estimated MV-CE (with empirical datasets)
P2	P-value with target 50% Optimum (with simulated datasets) P-value with target 50% estimated MV-CE (with empirical datasets)
P3	P-value with target 90% Optimum (with simulated datasets) P-value with target 90% estimated MV-CE (with empirical datasets)
EQ	Equi-weighted portfolio
MV	Feasible counterpart of MV^0
B	Bayesian with diffuse priors
KZ	Feasible counterpart of KZ^0
HRW	Feasible counterpart of HRW^0
KZ3	Feasible counterpart of $KZ3^0$
G UW	Garlappi, Uppal and Wang

Table 1: Summary Table of all the portfolio rules considered in the simulated and empirical experiments.

	Mean	Standard deviation
France	0.014	0.069
Germany	0.013	0.059
Japan	0.011	0.067
UK	0.015	0.073
USA	0.012	0.044

$$\text{and } \rho_0 = \begin{pmatrix} 1 & .590 & .390 & .541 & .456 \\ & 1 & .338 & .424 & .347 \\ & & 1 & .342 & .221 \\ & & & 1 & .506 \\ & & & & 1 \end{pmatrix}$$

Table 2: Summary statistics and matrix of correlations for the MSCI of G5 countries over the period January 1974 to December 1998.

	No Deviation			Deviation D_1		
Rule	P1	P2	P3	P1	P2	P3
Mean	0.0741	0.1657	0.2223	0.0933	0.2087	0.2800
Std dev.	0.1678	0.3752	0.5033	0.1772	0.3963	0.5316
SR	0.4416	0.4416	0.4416	0.5267	0.5267	0.5267
CE	0.0600	0.0953	0.0956	0.0776	0.1302	0.1387
Target	0.0127	0.0633	0.1139	0.0141	0.0705	0.1268
$P(\tilde{\eta} > \eta)$	1.0000	0.9731	0.6630	1.0000	1.0000	0.8889
Turnover	0.0400	0.0894	0.1199	0.0361	0.0808	0.1084
	Deviation D_3			Deviation D_4		
Rule	P1	P2	P3	P1	P2	P3
Mean	0.0540	0.1208	0.1621	0.0902	0.2016	0.2705
Std dev.	0.1373	0.3070	0.4119	0.1756	0.3925	0.5267
SR	0.3935	0.3935	0.3935	0.5135	0.5135	0.5135
CE	0.0446	0.0737	0.0773	0.0747	0.1245	0.1318
Target	0.0084	0.0420	0.0755	0.0135	0.0676	0.1217
$P(\tilde{\eta} > \eta)$	1.0000	1.0000	0.8694	1.0000	1.0000	0.9019
Turnover	0.0298	0.0667	0.0894	0.0343	0.0767	0.1029

Table 3: Out-of-sample mean, standard deviation, SR, CE and Turnover for 3 P-Value rules listed in Table 1. We also provide the associated targets (in terms of CE) and the out-of-sample probability of getting a corrected risk-aversion parameter larger than the actual one. Results are for simulated factor model with no deviation, and 5% deviation according to deviation distributions D_1 , D_3 and D_4 ; a window of size 120 months and risk-aversion parameter of 1.

Rule	EQ	MV	P2	B	KZ	HRW	KZ3	GUW
No Deviation								
Mean	0.0930	0.2419	0.1657	0.2279	0.1903	0.2148	0.2130	0.0814
Std dev.	0.2737	0.5645	0.3752	0.5318	0.4520	0.5075	0.4859	0.2243
SR	0.3396	0.4286	0.4416	0.4286	0.4210	0.4232	0.4384	0.3630
CE	0.0555	0.0826	0.0953	0.0865	0.0881	0.0860	0.0949	0.0563
Turnover	0	0.1539	0.0894	0.1450	0.1322	0.1453	0.1121	0.0874
Deviation D_1								
Mean	0.1948	0.3340	0.2087	0.3146	0.2737	0.3054	0.2864	0.1424
Std dev.	0.5212	0.6490	0.3963	0.6115	0.5361	0.5972	0.5578	0.2957
SR	0.3736	0.5146	0.5267	0.5146	0.5105	0.5114	0.5135	0.4816
CE	0.0589	0.1233	0.1302	0.1277	0.1300	0.1271	0.1308	0.0987
Turnover	0	0.1679	0.0808	0.1582	0.1465	0.1612	0.1399	0.1042
Deviation D_3								
Mean	0.0691	0.2020	0.1208	0.1903	0.1555	0.1765	0.1825	0.0581
Std dev.	0.5315	0.5341	0.3070	0.5032	0.4199	0.4736	0.4556	0.1868
SR	0.1301	0.3782	0.3935	0.3782	0.3703	0.3728	0.4005	0.3111
CE	-0.0721	0.0593	0.0737	0.0637	0.0673	0.0644	0.0787	0.0407
Turnover	0	0.1577	0.0667	0.1486	0.1411	0.1531	0.1212	0.1028
Deviation D_4								
Mean	0.1957	0.3227	0.2016	0.3040	0.2633	0.2941	0.2765	0.1338
Std dev.	0.5270	0.6425	0.3925	0.6053	0.5283	0.5890	0.5519	0.2841
SR	0.3713	0.5023	0.5135	0.5023	0.4984	0.4993	0.5011	0.4709
CE	0.0568	0.1163	0.1245	0.1208	0.1238	0.1206	0.1242	0.0934
Turnover	0	0.1569	0.0767	0.1478	0.1363	0.1501	0.1300	0.0948

Table 4: Out-of-sample mean, standard deviation, SR, CE and Turnover for 8 feasible rules listed in Table 1. Results are for simulated factor model with no deviation, and 5% deviation according to deviation distributions D_1 , D_3 and D_4 ; a window of size 120 months and risk-aversion parameter of 1.

Rule	Infeasible				Feasible			
	MV^0	KZ^0	HRW^0	$KZ3^0$	MV	KZ	HRW	KZ3
	No Deviation							
Mean	0.2530	0.1944	0.2137	0.2232	0.2419	0.1903	0.2148	0.2130
Std dev.	0.5030	0.4537	0.4988	0.4861	0.5645	0.4520	0.5075	0.4859
SR	0.5030	0.4286	0.4286	0.4592	0.4286	0.4210	0.4232	0.4384
CE	0.1265	0.0915	0.0894	0.1051	0.0826	0.0881	0.0860	0.0949
Turnover	0	0.1237	0.1360	0.0648	0.1539	0.1322	0.1453	0.1121
	Deviation D_1							
Mean	0.2818	0.2716	0.2986	0.2786	0.3340	0.2737	0.3054	0.2864
Std dev.	0.5309	0.5279	0.5804	0.5308	0.6490	0.5361	0.5972	0.5578
SR	0.5309	0.5146	0.5146	0.5248	0.5146	0.5105	0.5114	0.5135
CE	0.1409	0.1323	0.1302	0.1377	0.1233	0.1300	0.1271	0.1308
Turnover	0	0.1366	0.1502	0.1032	0.1679	0.1465	0.1612	0.1399
	Deviation D_3							
Mean	0.1679	0.1533	0.1685	0.1697	0.2020	0.1555	0.1765	0.1825
Std dev.	0.4097	0.4053	0.4456	0.4027	0.5341	0.4199	0.4736	0.4556
SR	0.4097	0.3782	0.3782	0.4214	0.3782	0.3703	0.3728	0.4005
CE	0.0839	0.0711	0.0692	0.0886	0.0593	0.0673	0.0644	0.0787
Turnover	0	0.1197	0.1316	0.0584	0.1577	0.1411	0.1531	0.1212
	Deviation D_4							
Mean	0.2705	0.2613	0.2873	0.2698	0.3227	0.2633	0.2941	0.2765
Std dev.	0.5201	0.5203	0.5720	0.5279	0.6425	0.5283	0.5890	0.5519
SR	0.5201	0.5023	0.5023	0.5111	0.5023	0.4984	0.4993	0.5011
CE	0.1352	0.1260	0.1237	0.1305	0.1163	0.1238	0.1206	0.1242
Turnover	0	0.1270	0.1397	0.0965	0.1569	0.1363	0.1501	0.1300

Table 5: Out-of-sample mean, standard deviation, SR, CE and Turnover for 4 infeasible rules and their feasible counterpart listed in Table 1. Results are for simulated factor model with no deviation, and 5% deviation according to deviation distributions D_1 , D_3 and D_4 ; a window of size 120 months and risk-aversion parameter of 1.

SR with No Deviation								
T_w	Infeasible				Feasible			
	MV^0	KZ^0	HRW^0	$KZ3^0$	MV	KZ	HRW	KZ3
60	0.5030	0.4175	0.4175	0.4687	0.4175	0.3997	0.4074	0.4451
		(17%)	(17%)	(7%)	(17%)	(21%)	(19%)	(12%)
120	0.5030	0.4526	0.4526	0.4794	0.4526	0.4461	0.4480	0.4657
		(10%)	(10%)	(5%)	(10%)	(11%)	(11%)	(7%)
180	0.5030	0.4600	0.4600	0.4790	0.4600	0.4574	0.4579	0.4700
		(9%)	(9%)	(5%)	(9%)	(9%)	(9%)	(7%)
240	0.5030	0.4731	0.4731	0.4840	0.4731	0.4715	0.4717	0.4775
		(6%)	(6%)	(4%)	(6%)	(6%)	(6%)	(5%)
300	0.5030	0.4769	0.4769	0.4849	0.4769	0.4760	0.4761	0.4805
		(5%)	(5%)	(4%)	(5%)	(5%)	(5%)	(4%)
360	0.5030	0.4787	0.4787	0.4843	0.4787	0.4780	0.4781	0.4809
		(5%)	(5%)	(4%)	(5%)	(5%)	(5%)	(4%)
420	0.5030	0.4811	0.4811	0.4867	0.4811	0.4806	0.4806	0.4838
		(4%)	(4%)	(3%)	(4%)	(4%)	(4%)	(4%)

Table 6: SR as a function of the size of the rolling window for 4 infeasible rules and their feasible counterpart listed in Table 1. The numbers in parentheses represent the loss in percentage from using a specific rule instead of the optimal (infeasible) MV^0 . Results are for simulated factor model with no deviation.

CE with No Deviation								
T_w	Infeasible				Feasible			
	MV^0	KZ^0	HRW^0	$KZ3^0$	MV	KZ	HRW	KZ3
60	12.6503	8.7027	8.1220	10.9835	5.6515	7.8882	6.9774	9.7162
		(31%)	(36%)	(13%)	(55%)	(38%)	(45%)	(23%)
120	12.6503	10.2377	10.0975	11.4891	9.5185	9.9384	9.7911	10.7893
		(19%)	(20%)	(9%)	(25%)	(21%)	(23%)	(15%)
180	12.6503	10.5758	10.5008	11.4661	10.2290	10.4510	10.3738	11.0143
		(16%)	(17%)	(9%)	(19%)	(17%)	(18%)	(13%)
240	12.6503	11.1891	11.1651	11.7112	11.0405	11.1158	11.0947	11.3921
		(12%)	(12%)	(7%)	(13%)	(12%)	(12%)	(10%)
300	12.6503	11.3730	11.3600	11.7550	11.2840	11.3267	11.3158	11.5382
		(10%)	(10%)	(7%)	(11%)	(10%)	(11%)	(9%)
360	12.6503	11.4560	11.4473	11.7238	11.3953	11.4250	11.4178	11.5617
		(9%)	(10%)	(7%)	(10%)	(10%)	(10%)	(9%)
420	12.6503	11.5708	11.5648	11.8440	11.5271	11.5465	11.5418	11.7006
		(9%)	(9%)	(6%)	(9%)	(9%)	(9%)	(8%)

Table 7: CE as a function of the size of the rolling window for 4 infeasible rules and their feasible counterpart listed in Table 1. The numbers in parentheses represent the loss in percentage from using a specific rule instead of the optimal (infeasible) MV^0 . Results are for simulated factor model with no deviation.

SR with Deviation D_1								
T_w	Infeasible				Feasible			
	MV^0	KZ^0	HRW^0	$KZ3^0$	MV	KZ	HRW	KZ3
60	0.5309	0.4615	0.4615	0.4843	0.4615	0.4465	0.4529	0.4679
		(13%)	(13%)	(9%)	(13%)	(16%)	(15%)	(12%)
120	0.5309	0.4919	0.4919	0.4997	0.4919	0.4871	0.4884	0.4895
		(7%)	(7%)	(6%)	(7%)	(8%)	(8%)	(8%)
180	0.5309	0.5031	0.5031	0.5076	0.5031	0.5011	0.5015	0.5011
		(5%)	(5%)	(4%)	(5%)	(6%)	(6%)	(6%)
240	0.5309	0.5109	0.5109	0.5133	0.5109	0.5100	0.5101	0.5086
		(4%)	(4%)	(3%)	(4%)	(4%)	(4%)	(4%)
300	0.5309	0.5129	0.5129	0.5146	0.5129	0.5123	0.5124	0.5106
		(3%)	(3%)	(3%)	(3%)	(3%)	(3%)	(4%)
360	0.5309	0.5115	0.5115	0.5127	0.5115	0.5110	0.5110	0.5094
		(4%)	(4%)	(3%)	(4%)	(4%)	(4%)	(4%)
420	0.5309	0.5139	0.5139	0.5151	0.5139	0.5136	0.5136	0.5127
		(3%)	(3%)	(3%)	(3%)	(3%)	(3%)	(3%)

Table 8: SR as a function of the size of the rolling window for 4 infeasible rules and their feasible counterpart listed in Table 1. The numbers in parentheses represent the loss in percentage from using a specific rule instead of the optimal (infeasible) MV^0 . Results are for simulated factor model with no deviation.

CE with Deviation D_1								
T_w	Infeasible				Feasible			
	MV^0	KZ^0	HRW^0	$KZ3^0$	MV	KZ	HRW	KZ3
60	14.0913	10.6475	10.1978	11.7118	8.0819	9.9587	9.3020	10.8068
		(24%)	(28%)	(17%)	(43%)	(29%)	(34%)	(23%)
120	14.0913	12.0968	11.9793	12.4800	11.4652	11.8639	11.7386	11.9419
		(14%)	(15%)	(11%)	(19%)	(16%)	(17%)	(15%)
180	14.0913	12.6555	12.6154	12.8828	12.4077	12.5570	12.5168	12.5475
		(10%)	(10%)	(9%)	(12%)	(11%)	(11%)	(11%)
240	14.0913	13.0501	13.0280	13.1738	12.9117	13.0035	12.9807	12.9305
		(7%)	(8%)	(7%)	(8%)	(8%)	(8%)	(8%)
300	14.0913	13.1552	13.1377	13.2429	13.0588	13.1223	13.1043	13.0310
		(7%)	(7%)	(6%)	(7%)	(7%)	(7%)	(8%)
360	14.0913	13.0793	13.0592	13.1418	12.9941	13.0542	13.0337	12.9712
		(7%)	(7%)	(7%)	(8%)	(7%)	(8%)	(8%)
420	14.0913	13.2033	13.1887	13.2672	13.1413	13.1862	13.1714	13.1417
		(6%)	(6%)	(6%)	(7%)	(6%)	(7%)	(7%)

Table 9: CE as a function of the size of the rolling window for 4 infeasible rules and their feasible counterpart listed in Table 1. The numbers in parentheses represent the loss in percentage from using a specific rule instead of the optimal (infeasible) MV^0 . Results are for simulated factor model with deviation distribution D_1 .

Rule	EQ	MV	P1	P2	P3	B	KZ	HRW	KZ3	GUW
	Industry Portfolios									
Mean	-2.8608	1.1391	0.4589	1.0262	1.3767	1.0167	0.8746	1.0823	0.9351	0.7231
Std dev.	3.7475	1.4950	0.5831	1.3038	1.7492	1.3344	1.1580	1.4297	1.1848	0.9844
SR	-0.7634	0.7619	0.7871	0.7871	0.7871	0.7619	0.7553	0.7570	0.7893	0.7345
CE	-9.8825	0.0216	0.2889	0.1763	-0.1531	0.1264	0.2041	0.0603	0.2332	0.2386
Turnover	0	0.0698	0.0267	0.0597	0.0801	0.0623	0.0541	0.0668	0.0476	0.0461
Target			0.1336	0.6681	1.2026					
$P(\tilde{\eta} > \eta)$			1.0000	0.7014	0.1280					
	Sector Portfolios									
Mean	0.0079	0.0570	0.0244	0.0546	0.0732	0.0509	0.0198	0.0368	0.0262	-0.0023
Std dev.	0.0417	0.5818	0.2210	0.4941	0.6629	0.5193	0.2566	0.4085	0.2842	0.0181
SR	0.1906	0.0980	0.1105	0.1105	0.1105	0.0980	0.0771	0.0900	0.0922	-0.1244
CE	0.0071	-0.1123	-0.0000	-0.0675	-0.1465	-0.0840	-0.0132	-0.0467	-0.0142	-0.0024
Turnover	0	4.4594	1.6652	3.7235	4.9957	3.9803	2.1724	3.2371	2.1464	0.1288
Target			0.0139	0.0697	0.1254					
$P(\tilde{\eta} > \eta)$			1.0000	0.9306	0.0278					

Table 10: Out-of-sample mean, standard deviation, SR, CE and Turnover for 10 feasible rules listed in Table 1. For the P-value rules, we also provide the associated targets (in terms of CE) and the out-of-sample probability of getting a corrected risk-aversion parameter larger than the actual one. Results are for the 2 empirical datasets, industry portfolios and sector portfolios, with a window of size 120 months and risk-aversion parameter of 1.

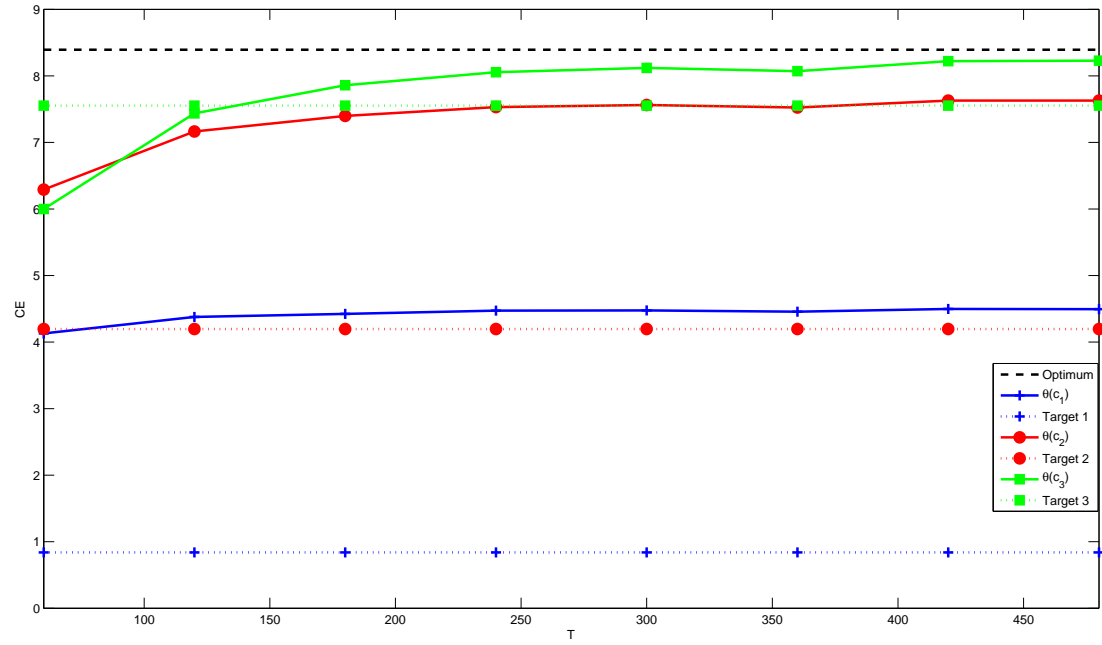


Figure 1: Out-of-sample CE for 3 P-value rules as a function of the size of the rolling window for the simulated factor model dataset with 5% deviation D_3 .

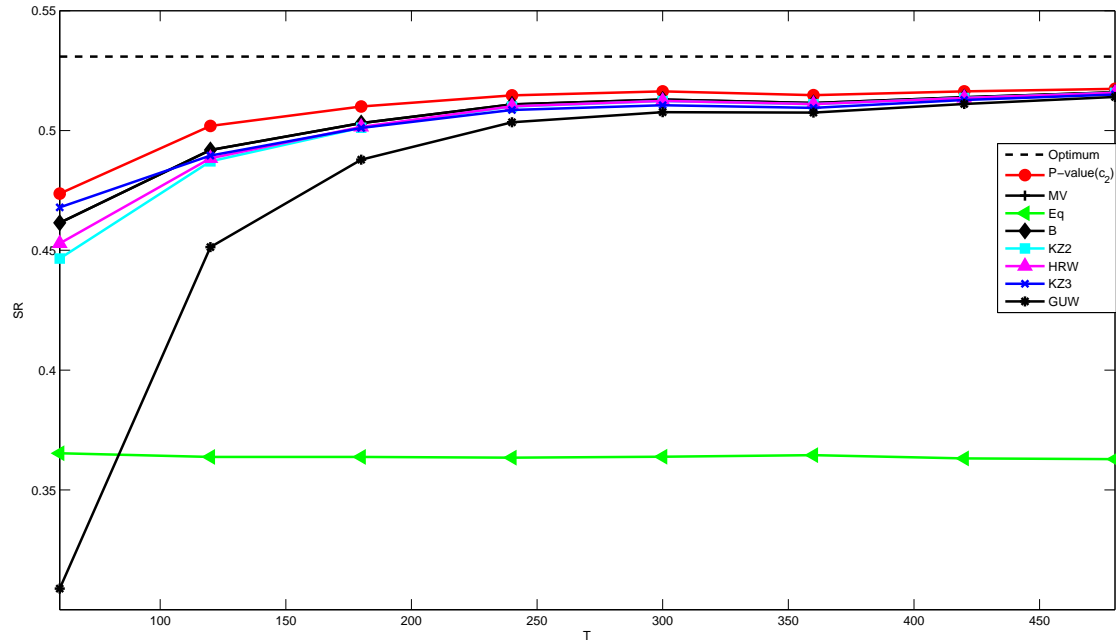


Figure 2: Out-of-sample SR for feasible rules as a function of the size of the rolling window for the simulated factor model dataset with 5% deviation D_1 .

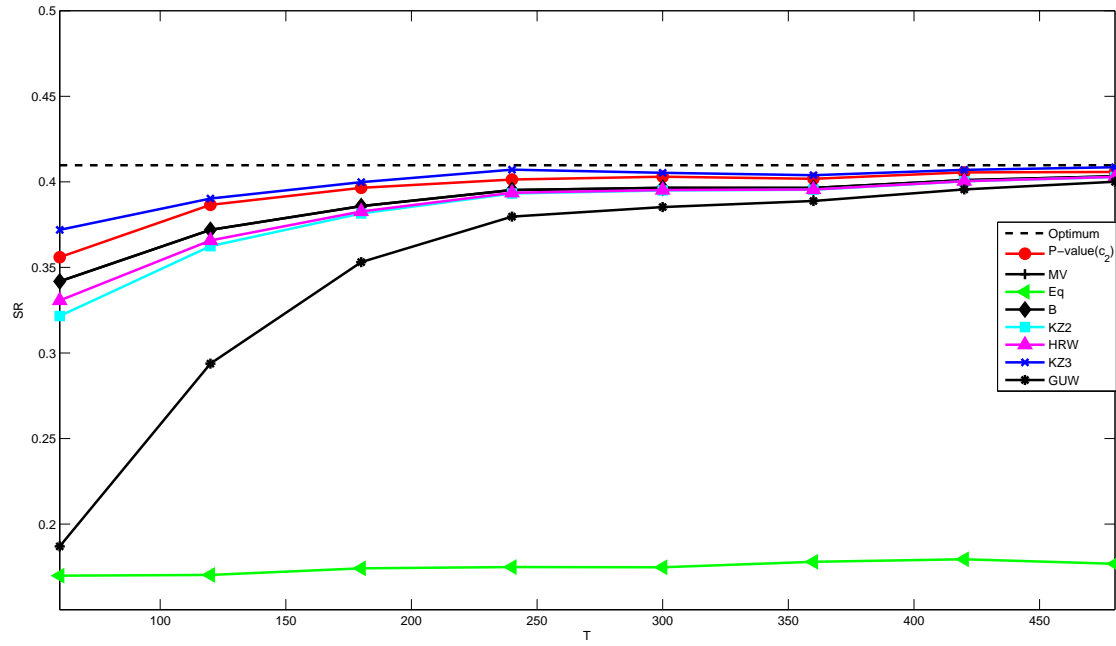


Figure 3: Out-of-sample SR for feasible rules as a function of the size of the rolling window for the simulated factor model dataset with 5% deviation D_3 .

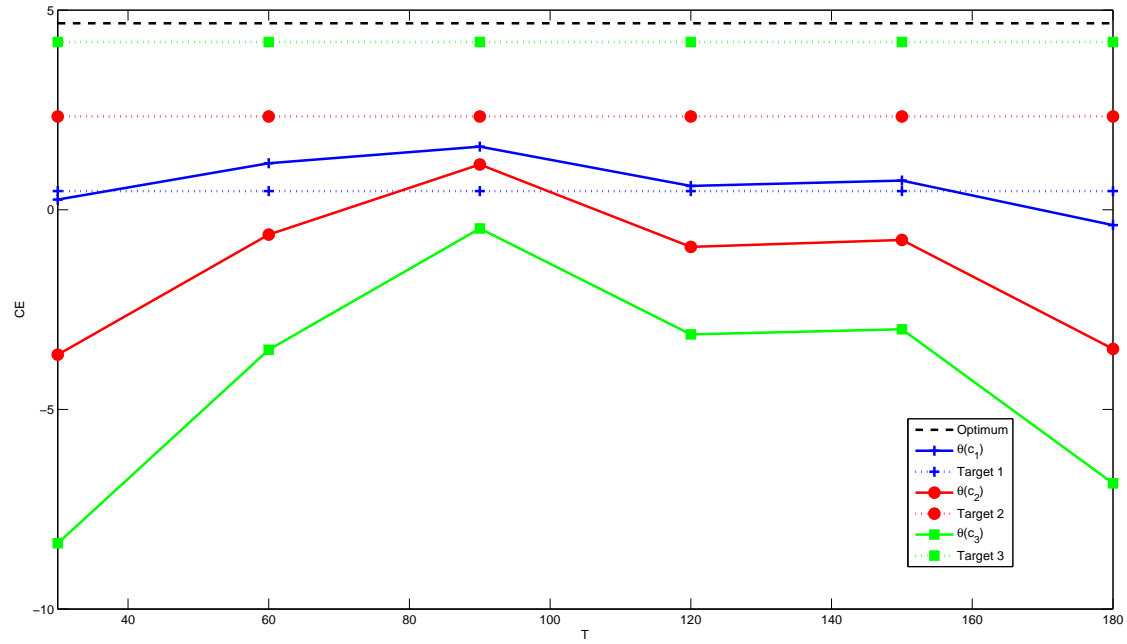


Figure 4: Out-of-sample CE for 3 P-value rules as a function of the size of the rolling window for the empirical dataset Sector Portfolios.

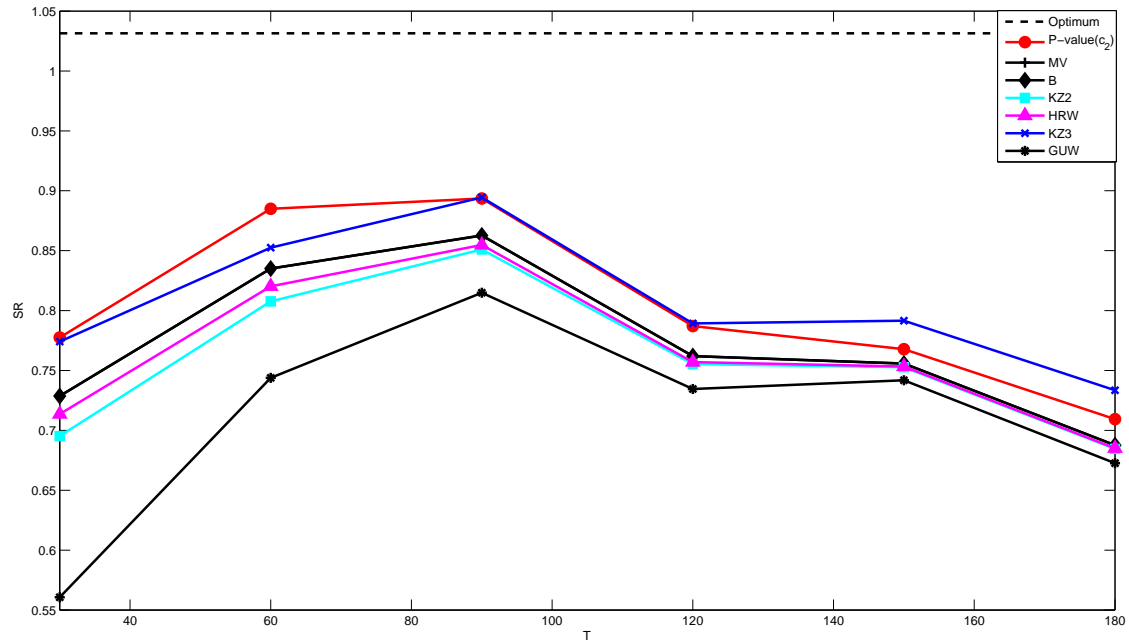


Figure 5: Out-of-sample SR for feasible rules as a function of the size of the rolling window for the empirical dataset Industry Portfolios.

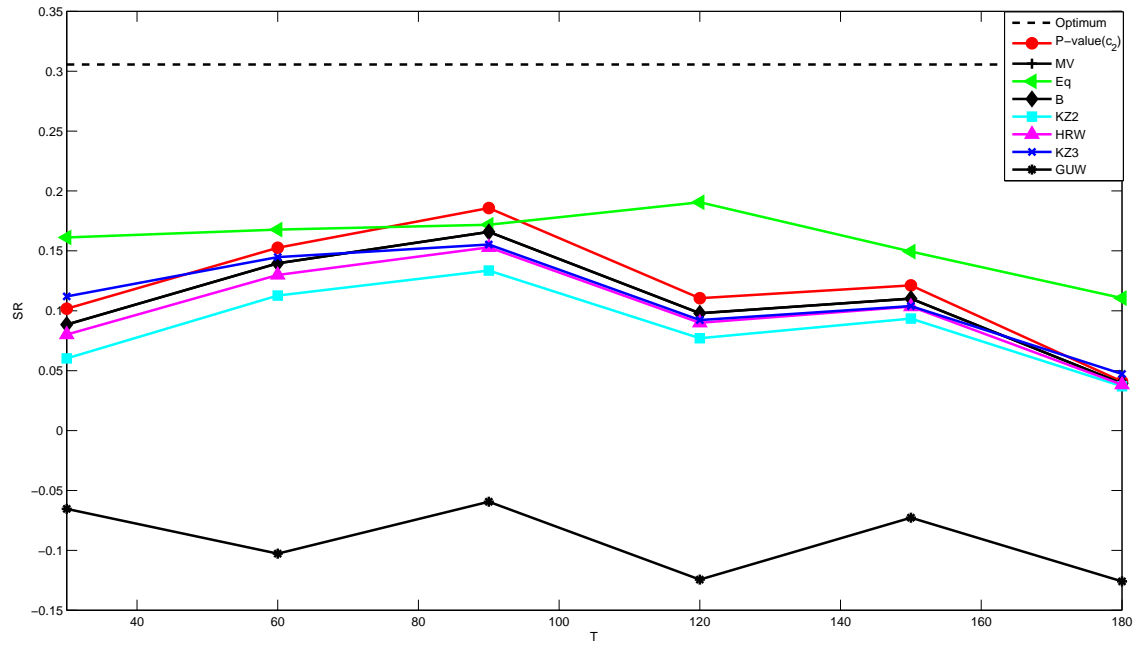


Figure 6: Out-of-sample SR for feasible rules as a function of the size of the rolling window for the empirical dataset Sector Portfolios.