1. (20 points)
a. (10 points) Set up an equation of the form $Ku = f$ for finding the equilibrium states of a system of 2 masses of sizes $m_1 = 1, m_2 = 1$ attached to three springs with constants $c_1 = 2, c_2 = 1, c_3 = 2$ with fixed-fixed boundary conditions experiencing the external force caused by the downward pull of gravity.

b. (10 points) Find the general solution to the matrix ODE given by

$$u' = Ku - f.$$
Problem 1 Cont.
2. (20 points)
a. (10 points) Given the Midpoint Rule for approximate integration,

\[ \int_a^b g(s)ds \approx (b - a)g\left(\frac{a + b}{2}\right), \]

design an implicit numerical scheme for approximately solving the 1d ODE

\[ u' = f(t, u). \]

b. (10 points) Is your scheme linearly stable (i.e. given \( f(t, u) = \lambda u, \) \( \text{Re}(\lambda) \leq 0, \) is the approximation \( U_n \) bounded for each \( n)? \)
Problem 2 Cont.
3. (20 points) For the matrix

\[ A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \]

find:

a. (10 points) The QR decomposition.
b. (10 points) Is \( A \) positive definite? If not, why not? If so, explain why and write an energy functional, \( P(u) \), such that the solution, \( \hat{u} \), to

\[ A\hat{u} = f \]

is such that \( P(\hat{u}) = P_{\text{min}} \).
Problem 3 Cont.
4. (20 points) Fit the best linear plot \( x = ct + d \) to the following data points:

\[
t = -1, 0, 1; \ x = -1, 2, 4
\]

respectively.
Problem 4 Cont.
5. (20 points)
Given a smooth function \( v \) defined on \((0, 1)\), define an energy functional

\[
P(v) = \frac{1}{2} \int_0^1 (v_x)^2 \, dx - \int_0^1 fv \, dx.
\]

Show that the minimum of the function \( P \) occurs at a function \( u \) if and only if

\[
\int_0^1 u_x v_x \, dx - \int_0^1 fv \, dx = 0
\]

for all sufficiently smooth \( v \). (Hint: Look at the function \( q(t) = P(u + tv) \). Treat \( q \) as a smooth function of one variable \( t \) that you can differentiate, ...).
Problem 5 Cont.